

## **The Dynamics of Thin Liquid Film**

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### **Abstract**

The dynamics of the thin layer which flows steadily between two vertical guide wires was investigated but with zero shear stress at their bounding surfaces where the gravity has no significant effect on the liquid film. We apply the Navier-Stokes equations in two dimensional steady flows for incompressible fluid to a falling liquid curtain and we present the derivation of the differential equation that governs such flow and we obtain a solution for these equations which is valid for this liquid curtain , where we restrict our works to the case where the domain under consideration is long and thin, the solution of the governing equation is obtained by analytical method, and in this case there is a critical solution  $g_c(\xi) = (\frac{2}{\beta})^{1/2}$  for large  $\xi$  when the parameter  $\alpha$  is equal to zero, where  $\alpha = \frac{P_0}{\rho g^2 R^2} H^2$  and which is identical to the case when the normalized pressure  $p_0$  is equal to zero. Generally, we solve the equation when  $p_0$  is not equal to zero, and the thickness of the film increases as  $\beta$  increases where  $\beta = \frac{\gamma}{\rho g^2} H$ .

**Keywords:** Navier-Stokes equations, continuity equation, Non-linear differential equations.

### **Introduction**

The dynamic of a thin liquid film flowing steadily between two vertical guide wires, where the effect of surface curvature is taken into account, is investigated. The Navier –stokes equations integrated over the film thickness and an approximate non-linear differential equation is obtained by neglecting the higher order terms with respect the thickness of the thin liquid films and the results are compared with Cyrus (Cyrus,1987) works who neglects the effect of surface curvature and also results are compared with the experimental measurements of Brown (Brown, 1961).

The objective of the present analysis is to apply the Navier-Stokes equation to a falling liquid curtain, and present the derivation of the differential equations that governs the flow of the liquid curtain and to obtain a solution of these equations which is valid for thin liquid film.

In addition to engineering applications, a solution will serve as a first order approximation of the velocity profile. The domain of validity for each equation is established by comparing the numerical solution with the experimental results of Brown (Brown, 1961)

**Equations of steady motion in films with zero shear-stress at their bounding surface.**

To describe the flow of a viscous fluid within a symmetric film in two dimensions, the Cartesian coordinates  $x$  and  $y$  are taken in which the  $x$ -axis is the axis of symmetry, and the flow is predominantly in the  $x$ -direction. In figure (1) the transverse dimensions are greatly enlarged by comparison with the longitudinal dimensions for clarity of presentation.

Let  $u(x,y)$ ,  $v(x,y)$  be the corresponding velocity component in  $x$  and  $y$  directions respectively and  $p(x,y)$  the pressure , obtain the non-linear equation with rupture time (Leshansky & Rubinstein ,2004). Let the equation of the free surface of the liquid film with be

$$y = h(x) \tag{1}$$

where the liquid film flows steadily between two vertical wires as shown in figure (1)

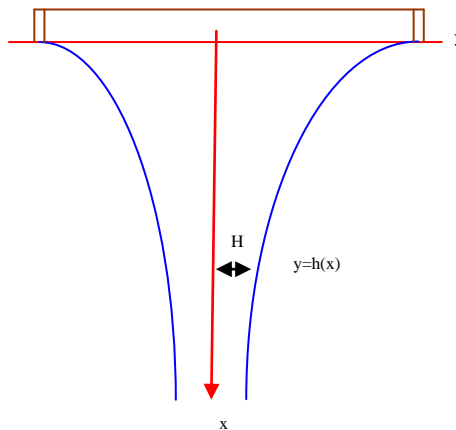


Figure (1): Schematic diagram of a free surface- liquid film flow

Normally in thin liquid films, the film thickness is much smaller than the width as (Rutayna, 2005), and therefore we assume two-dimensional incompressible flow.

The steady two dimensional incompressible fluid flows governed by the following equation of motion:

**1-Continuity equation:**

We can express the connection between area and velocity in an equation, called the equation of continuity. The continuity equation in a differential form for two dimensional incompressible flows has the form (Stokes,1945)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \dots(2)$$

For steady flow from Stoke’s (Stokes,1945) the momentum equation has the form:

x-momentum

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\rho} \left( \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} \right) \quad \dots(3)$$

y-momentum

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \frac{1}{\rho} \left( \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} \right) \quad \dots(4)$$

where  $\sigma_{xx}$ ,  $\sigma_{xy}$  and  $\sigma_{yy}$  are the components of the total stress tensor given in the standard notation, where the stress tensor and then for incompressible flow, these stresses has the following forms:

$$\left. \begin{aligned} \sigma_{xx} &= p - 2\mu \frac{\partial u}{\partial x} \\ \sigma_{xy} &= \sigma_{yx} = -\mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \end{aligned} \right\} \quad \dots (5)$$

and

$$\sigma_{yy} = p - 2\mu \frac{\partial v}{\partial y}$$

where  $\mu$  is the coefficient of viscosity of liquid, and  $u$  is component velocity, the density  $\rho$  is assumed to be constant throughout the process.

Let  $y = h(x)$  represent the thickness of the liquid film at a point  $x$  (see equation (1)). We define the equation of the free surface of the film by the function  $F(x, y)$  as follow

Since from equation (1) we have

$$y - h(x) = 0 \text{ and then } F(x, y) = y - h(x) = 0 \quad \dots (6)$$

represents the free surface equation.

Eventually, we restrict attention to cases in which

$$\frac{dh}{dx} < 1 \quad \dots (7)$$

over the domain  $x$  under consideration. However we do not impose any

geometrical constraint on the total variation of  $h(x)$  over this domain. In these respects, the theory retains many of the features of standard theories of fluid flow in thin domains: namely mathematical hydraulics in an experiments, Lubrication theory and boundary layer theory.

The appropriate mode of incorporating the condition eq.(7) into the theory appears to be as follows.

We consider the class of steady flows in which the asymptotic conditions as  $x \rightarrow \infty$  are uniform:

$$\left. \begin{aligned} h(x) &\rightarrow H \\ u(x, y) &\rightarrow U \\ v(x, y) &\rightarrow 0 \end{aligned} \right\} \dots (8)$$

where  $H$  and  $U > 0$  are constants .

The boundary of the film is a streamline, and therefore the substantial (material) derivative of  $F(x, y)$ , that is  $\frac{DF}{Dt}$  must vanish at  $F(x, y) = 0$ .

Thus results in the following boundary conditions:

From eq. (2), we have

$$\frac{DF}{Dt} = u \frac{\partial F}{\partial x} + v \frac{\partial F}{\partial y} = 0 \dots(9)$$

$$\text{which gives } v = u \frac{dh}{dx} \quad \text{at } y = h(x) \dots(10)$$

For the balance of the surface forces on the boundary, the Cartesian components of the unit normal vector  $n$  are needed which are given by:

$$\begin{aligned} \vec{n} &= in_x + jn_y \\ \text{and } \left. \begin{aligned} n_x &= -h'(x) \{1 + (h'(x))^2\}^{-1/2} \\ n_y &= \{1 + (h'(x))^2\}^{-1/2} \end{aligned} \right\} \dots(11) \end{aligned}$$

The curvature of the liquid film is given by (Rutayna, 2005)

$$K = \frac{h''(x)}{[1 + h'^2(x)]^{3/2}} , \dots(12)$$

and since we restrict attention to the case given by eq.(7), the curvature eq.(12) can be simplified by since  $\frac{dh}{dx} < 1$  then the term  $h'^2(x)$  is very small to give

$$K = h''(x) \dots (13)$$

The surface tension  $\gamma$ , creates a stress on the free surface of the liquid film, following (Stokes,1945), the balance of the surface forces on the free surface, is given by

$$\sigma_{xy}n_y + \sigma_{xx}n_x = \gamma K n_x \quad \dots (14)$$

and

$$\sigma_{yx}n_x + \sigma_{yy}n_y = \gamma K n_y \quad \dots(15)$$

where  $\gamma$  is the surface tension, using equations (6) for incompressible flow ( $\nabla \cdot q = 0$ ), equations (14) and (15) are gives respectively as follows:

$$-\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)n_y + \left(p - 2\mu\frac{\partial u}{\partial x}\right)n_x = \gamma K n_y \quad \dots (16)$$

and

$$-\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)n_x + \left(p - 2\mu\frac{\partial v}{\partial y}\right)n_y = \gamma K n_x \quad \dots (17)$$

Now we decompose the velocity  $u$  and the normal stress  $\sigma_{xx}$  as

$$u(x, y) = u_0(x) + \varepsilon u_1(x, y) + \varepsilon^2 u_2(x, y) + \dots \quad \dots (18)$$

and

$$\sigma_{xx} = \sigma_{xx_0} + \varepsilon \sigma_{xx_1}(x, y) + \varepsilon^2 \sigma_{xx_2}(x, y) + \dots \quad \dots (19)$$

here  $\varepsilon$  is related to  $h'$ . The functions  $u_0(x)$  and  $\sigma_{xx_0}$  are unknown at this point and will be derived later in the analysis. Further below we show that if  $h' \ll 1$ , then  $\varepsilon \ll 1$ , and the function  $u$  and  $\sigma_{xx}$  are weakly dependent on  $y$ , when two variables have decomposition slices that contain statements in common, the variable are said to be weakly dependent.

With the decomposition equations (16) and (17), the continuity equation (2) can be integrated over the film thickness to give

$$\int_0^h \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dy = 0$$

$$\int_0^h \frac{\partial}{\partial x} u_0(x) dy + \varepsilon \int_0^h \frac{\partial}{\partial x} u_1(x, y) dy + \varepsilon^2 \int_0^h \frac{\partial}{\partial x} u_2(x, y) dy + \int_0^h \frac{\partial v}{\partial y} dy = 0$$

For the first approximation  $\varepsilon, \varepsilon^2, \varepsilon^3, \dots$  tends to zero .We have

$$u_0'(x)h(x) + v(x, h(x)) - v(x, 0) + O(\varepsilon) = 0 \quad \dots (20)$$

Since the liquid film is symmetric as (Rutayna, 2005), then

$v(x,0) = 0$  and then equation (20) gives

$$h \frac{du_0}{dx} + v(x, h(x)) + O(\varepsilon) = 0 \quad \dots (21)$$

substituting the boundary condition eq.(9) in the equation (21), we get

$$h \frac{du_0}{dx} + u \frac{dh}{dx} + O(\varepsilon) = 0$$

Now, using the decomposition eq.(18) of  $u$ , to obtain

$$h \frac{du_0}{dx} + u_0 \frac{dh}{dx} = 0 \quad \dots(22)$$

$$\rightarrow d(u_0 \cdot h) = 0 \quad \dots(23)$$

Integrate eq.(23) with respect to  $x$ , yields the global form of mass conservation

$$Q = u_0 h \quad \dots(24)$$

where  $Q$  is a constant representing the volumetric mass flow and  $u_0$  is, therefore, the average velocity defined by

$$u_0 \equiv \frac{1}{h} \int_0^h u dy \quad \dots(25)$$

Consistent with the integral form of the continuity equation (2) and without loss of generality, it can be assumed that  $Q=1$  and therefore equation (24) gives,

$$u_0 h = 1 \quad \dots(26)$$

which gives the x-component of the velocity in terms of the film thickness.

Now we integrate the momentum equation over the film thickness in the same manners. The integrals of the non-linear inertia terms in the x-component of the momentum equation eq. (3) are

$$\int_0^h u \frac{\partial u}{\partial x} dy = \int_0^h (u_0 u'_0 + \varepsilon u_1 u'_0 + \varepsilon^2 u_2 u'_0 + \varepsilon u_0 u'_1 + \varepsilon^2 u_1 u'_1 + \varepsilon^3 u_2 u'_1 + \varepsilon^2 u_0 u'_2 + \varepsilon^3 u_1 u'_2 + \varepsilon^4 u_2 u'_2 + \dots) dy$$

$$= u_0 u'_0 h + O(\varepsilon)$$

Restricts to the first order, we get  $\int_0^h u \frac{\partial u}{\partial x} dy = u_0 u'_0 h \quad \dots(27)$

Similarly the integral over the film thickness of the right hand side of equation (3) gives,

$$\begin{aligned} \int_0^h \frac{\partial \sigma_{xx}}{\partial x} dy &= \int_0^h \frac{\partial}{\partial x} (\sigma_{xx_0} + \varepsilon \sigma_{xx_1} + \varepsilon^2 \sigma_{xx_2} + \dots) dy \\ &= \int_0^h \frac{\partial \sigma_{xx_0}}{\partial x} dy + \varepsilon \int_0^h \frac{\partial \sigma_{xx_1}}{\partial x} dy + \varepsilon^2 \int_0^h \frac{\partial \sigma_{xx_2}}{\partial x} dy + \dots \\ &= \sigma'_{xx_0} h(x) + O(\varepsilon) \end{aligned}$$

To the first order approximation, we get  $\int_0^h \frac{\partial \sigma_{xx}}{\partial x} dy = \sigma'_{xx_0} h(x)$  ... (28)

and

$$\int_0^h \frac{\partial \sigma_{xy}}{\partial y} dy = \sigma_{xy}(x, y) \Big|_0^h = \sigma_{xy}(x, h) - \sigma_{xy}(x, 0) \quad \dots (29)$$

But  $\sigma_{xy}(x, 0) = 0$  due to the symmetry of the flow in liquid film with respect to the center line and thus eq.(29) reduces to give

$$\int_0^h \frac{\partial \sigma_{xy}}{\partial y} dy = \sigma_{xy}(x, h) \quad \dots (30)$$

Now from equation (14),  $\sigma_{xy}$  is given by

$$\sigma_{xy} = (\gamma K - \sigma_{xx_0}) \frac{n_x}{n_y}$$

Using eq.(11), the following can be obtained

$$\sigma_{xy} = (\gamma K - \sigma_{xx_0})(-h'(x)) \quad \dots (31)$$

on the free surface  $y = h(x)$ , eq.(31) gives

$$\sigma_{xy}(x, h) = (\sigma_{xx_0}(x, h) - \gamma K)h'(x). \quad \dots (32)$$

Therefore the integral of the x-component of the momentum equation over the film thickness can now be written as:

$$u_0 u'_0 h = \frac{1}{\rho} [h(x) \sigma'_{xx} + \sigma_{xy}(x, h)] \quad \dots (33)$$

Substituting  $\sigma_{xy}(x, h)$  from equation (32) into equation (33), we get

$$u_0 u'_0 = \frac{1}{\rho} \sigma'_{xx_0} + \frac{1}{\rho h} [\sigma_{xx_0}(x, h) - K\gamma]h'(x). \quad \dots (34)$$

Using equation (5), we can express the y-component of equation (15) by

$$\frac{\partial v(x, h)}{\partial x} + K\gamma h'(x) - 2\mu \left( \frac{\partial v}{\partial y} \right)_{y=h(x)} h'(x) = 0 \quad \dots (35)$$

Now if the magnitude of  $x$  and  $u$  are assumed to be  $O(1)$ , then for a thin film  $y$  is  $O(h)$  and  $h'(x)$  is also of order  $h(x)$ , where  $h'(x) \ll 1$ . Also from

eq.(10),  $v$  is of order of  $h$ , and from equation (4) we can find that  $\frac{\partial \sigma_{yy}}{\partial y}$

is of order  $h$ , and hence we can assume that

$$\sigma_{yy} = -P_0 \quad \dots (36)$$

where  $P_0$  is a constant representing the ambient pressure .

Now from eq.(5), we have

$$\sigma_{yy} = -P(x) + 2\mu \frac{\partial v}{\partial y} \quad \dots (37)$$

Using equation (36) and equation (37), thus give

$$P(x) = P_0 + 2\mu \frac{\partial v}{\partial y} \quad \dots (38)$$

From the continuity equation (2), equation (38) becomes

$$P(x) = P_0 - 2\mu \frac{\partial u}{\partial x} \quad \dots (39)$$

Furthermore, the x-component of the normal stress can now be written as

$$\sigma_{xx} = -P(x) + 2\mu \frac{\partial u}{\partial x} \quad (40)$$

Substituting the expression for  $P(x)$  from equation (39) in the equation (40), we get

$$\sigma_{xx} = -P_0 + 4\mu \frac{\partial u}{\partial x} \quad \dots(41)$$

Now by substituting equation (41) for the x-component of the normal stress tensor in equation (34), we get

$$u_0 u_0' = \frac{4\mu}{\rho} u'' - \frac{P_0}{\rho} \frac{h'}{h} + \frac{4\mu}{\rho} \frac{u'h'}{h} - \frac{K\gamma}{\rho} \frac{h'}{h} \quad \dots(42)$$

Using equation (12) into equation (42), the following can be obtained:

$$u_0 u_0' - \frac{4\mu}{\rho} u'' + \frac{P_0}{\rho} \frac{h'}{h} - \frac{4\mu}{\rho} \frac{u'h'}{h} - \frac{\gamma}{\rho} \frac{h'}{h} h'' = 0 \quad \dots (43)$$

where the term  $h'^2(x)$  is very small to give  $K = h''(x)$ .

From eq.(8), we have as  $x$  tends to a large positive value ( $x \rightarrow \infty$ ), the film approaches a uniform thickness  $H$ , with velocity  $U$ , and equation (22) gives

$$d(uh) = 0 \quad \text{or} \\ uh = UH = R\mathcal{G} \quad \dots(44)$$

where  $\mathcal{G} = \frac{\mu}{\rho}$  is the Kinematics viscosity and is the Reynolds number which



is determined by the boundary conditions and is taken to be small, that is  $R \ll 1$ , ( Reynolds number is very small).

Now from eq.(44), we have

$$u(x) = \frac{Rg}{h} \quad \dots (45)$$

Thus from continuity equation (2), we get

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \quad \rightarrow \partial v = -\frac{\partial u}{\partial x} \partial y \quad \rightarrow \int \partial v = -\int \frac{\partial u}{\partial x} \partial y$$

$$v(x, y) = \frac{Rgh'}{h^2} y \quad \dots(46)$$

Furthermore equations (39) and (45), gives the pressure  $P(x)$ , where

$$P(x) = P_0 + 2\mu Rg \frac{h'}{h^2} \quad \dots(47)$$

In this section, we obtain the dimensional non-linear equation (43) such governs the thin liquid films on the vertical guide wires.

**2- Dimensional analysis:**

We now introduce non-dimensional variables as follows:

$$h(x) = Hf(\eta) \quad \dots(48)$$

and  $x = \frac{H\eta}{R} \quad \dots(49)$

where  $\eta$  is a non-dimensional variable and  $f(\eta)$  is a non-dimensional function,  $x$  and  $h(x)$  are dimensional variables, function respectively and  $R$  is Reynolds number.

Using equation (45), equation (43), reduces to

$$\left(\frac{Rg}{h}\right)\left(-\frac{Rgh'}{h^2}\right) - \frac{4\mu}{\rho}\left(-Rg\frac{hh'' - 2h'^2}{h^3}\right) + \frac{P_0}{\rho}\frac{h'}{h} - \frac{4\mu}{\rho}\left(-\frac{Rgh'}{h^2}\right)h' - \frac{\gamma}{\rho}\frac{h'}{h}h'' = 0$$

since  $(g = \frac{\mu}{\rho})$

$$-\frac{R^2g^2h'}{h^3} + 4Rg^2\frac{hh'' - 2h'^2}{h^3} + \frac{P_0}{\rho}\frac{h'}{h} + 4\frac{Rg^2h'^2}{h^2} - \frac{\gamma}{\rho}\frac{h'}{h}h'' = 0$$

implies that 
$$-R^2g^2h' + 4Rg^2hh'' - 4Rg^2h'^2 + \frac{P_0}{\rho}h^2h' - \frac{\gamma}{\rho}h^2h'h'' = 0 \quad \dots(50)$$

Now from the transformation eq.(48) and eq.(49), we have

$$h'(x) = Hf'(\eta)\frac{d\eta}{dx}, \quad \frac{d\eta}{dx} = \frac{R}{H}$$

$$\text{hence } \left. \begin{aligned} h' &= Hf'(\eta)\left(\frac{R}{H}\right) = Rf'(\eta) \\ h'' &= \frac{R^2}{H} f''(\eta) \end{aligned} \right\} \dots (51)$$

substitute eq.(50) in the equation (49), we get

$$f' - 4ff'' + 4f'^2 - \alpha f^2 f' - \beta f^2 ff'' = 0 \dots (52)$$

where

$$\alpha = \frac{P_0}{\rho g^2 R^2} H^2 \quad \text{and} \quad \beta = \frac{\gamma}{\rho g^2} H$$

So that, in a given liquid,  $\beta$  is determined by  $H$  alone and the surface tension  $\gamma$  depends on  $\beta$  and  $H$ .

We have to note here that the transformation eq.(48) and eq.(49) ensures that the domain under consideration is long and thin and that the films are justified in a domain in which  $f(\eta)$  remains bounded, The mathematical origins of the terms in eq.(40) are as follows:

Inertia  $f'$ , Viscosity  $-4ff'' + 4f'^2$ , Surface tension  $-\beta f^2 ff''$  Pressure  $-\alpha f^2 f'$   
 Now as  $\eta \rightarrow \infty$ , the boundary conditions require that

$$f(\eta) = 1 + \varepsilon g(\eta), \quad \varepsilon < 1 \dots (53)$$

The asymptotic condition eq.(8) as  $x \rightarrow -\infty$  with the transformations eq.(48) and eq.(49) become

$$f(\infty) = 1, \quad u(\infty) = 1 \quad \text{and} \quad v(\infty) = 0 \quad \text{since at } x \rightarrow -\infty \text{ also } \eta \rightarrow -\infty \text{ and}$$

$$h(x) = Hf(\eta)$$

$$\rightarrow f(\eta) = \frac{h(x)}{H} \text{ at } \eta \rightarrow -\infty \text{ become } f(\infty) \rightarrow 1$$

where  $\varepsilon$  is an arbitrary constant and  $g(\infty) = 0$ , and the differential equation (52), then gives

$$\begin{aligned} \varepsilon g' - 4\varepsilon g'' - 4\varepsilon^2 g g'' + 4\varepsilon^2 g'^2 - \alpha \varepsilon g' - 2\alpha \varepsilon^2 g g' \\ - \alpha \varepsilon^3 g^2 g' - \beta \varepsilon^2 g' g'' + 2\beta \varepsilon^3 g g' g'' + \beta \varepsilon^4 g^2 g' g'' = 0 \end{aligned} \dots (54)$$

since  $\varepsilon < 1$ , then we have  $\varepsilon^n \rightarrow 0$  for  $n=2, 3, 4, \dots$

Thus equation (54) reduces to

$$4g'' + (\alpha - 1)g' = 0 \dots (55)$$

since  $\varepsilon \neq 0$ , so equation (55) has a fundamental solution of the form

$$g(\eta) = e^{m\eta} \quad \dots (56)$$

Substitute (56) in the equation (55), to get

$$[4m^2 + (\alpha - 1)m]e^{m\eta} = 0 \quad . \text{ Since } e^{m\eta} \neq 0$$

$$\text{implies that } 4m^2 + (\alpha - 1)m = 0 \quad \dots (57)$$

which gives either  $m = 0$  and a trivial solution is obtained or

$$m = \frac{1 - \alpha}{4} \quad \dots (58)$$

The root  $m$  in (58) represents a balance among viscosity, inertia and pressure, the effect of surface tension being negligible. A part from special circumstances, in which the amplitude of the solution representing a viscosity-inertia is zero, the inertia term is almost always not negligible as the film approaches a condition of asymptotic uniform thickness and the importance of inertia, however, small is the Reynolds number

When  $f(\eta)$  departs from its asymptotic value of unity to an appreciable extent, the complete differential equation (52) must be satisfied.

Now the variable  $\eta$  does not appear explicitly in equation (52), so that its differential order can be reduced by one by the following transformations:

$$\left. \begin{array}{l} \xi = f(\eta) \\ g(\xi) = f'(\eta) \end{array} \right\} \quad \dots (59)$$

Thus equation (2.3.5) reduces to

$$1 - 4\xi g' + 4g - \alpha\xi^2 - \beta\xi^2 g g' = 0 \quad \dots(60)$$

since  $g \neq 0$

$$g'(\xi) = \frac{1 + 4g - \alpha\xi^2}{4\xi + \beta\xi^2 g} \quad \dots(61)$$

Furthermore, equation (58) can be written as

$$\frac{1}{\xi^2} - 4\left(\frac{g'}{\xi} - \frac{g}{\xi^2}\right) - \alpha - \beta g g' = 0 \quad \dots(62)$$

which may be integrated with respect to  $\xi$  to give

$$\beta\xi g^2 + 2\alpha\xi^2 + 8g + 2c\xi + 2 = 0 \quad \dots (63)$$

where  $c$  is an arbitrary constant.

Now every solution of the original equation whose asymptotic behavior for large positive value of  $\xi$  is given by eq.(53) must pass through the point  $(1, 0)$  in  $(\xi, g)$  plane since from (59)  $\xi = f(\eta)$  and  $g(\xi) = f'(\eta)$  and from eq.(53) we have  $f(\eta) \rightarrow 1$  as  $\eta \rightarrow -\infty$  this implies that  $\xi \rightarrow 1$  as  $\eta \rightarrow -\infty$ .

Furthermore since  $f'(\eta) \rightarrow 0$ , as  $\eta \rightarrow -\infty$ , then one can have  $g(\xi) \rightarrow 0$  thus equation eq.(61) at (1,0) gives

$$g'(\xi) = \frac{1-\alpha}{4} \quad \dots(64)$$

Integrate eq.(64) with respect to  $\xi$ , and after condition  $g(1) = 0$

$$g(\xi) = \frac{1-\alpha}{4} \xi + c_1 \quad \dots (65)$$

where  $c_1 = \frac{\alpha-1}{4}$  thus equation (65) becomes

$$g(\xi) = \frac{1-\alpha}{4} (\xi - 1) \quad \dots(66)$$

which is the asymptotic behavior in  $(\xi, g)$ -plane.

Now, substitute equation (66) into equation (63) and pass through (1,0), we get

$c = -(\alpha + 1)$  and equation (63) reduces to :

$$\beta \xi g^2 + 8g + 2\alpha \xi^2 - 2\xi(\alpha + 1) + 2 = 0$$

which has the following solution

$$g(\xi) = \frac{-4 \pm \sqrt{16 - \beta \xi [2\alpha \xi^2 - 2(\alpha + 1)\xi + 2]}}{\beta \xi} \quad \dots(67)$$

It is clear that the solution has asymptotic behavior as follows

$$\lim_{\xi \rightarrow 0} g(\xi) = \lim_{\xi \rightarrow 0} \frac{-4 \pm \sqrt{16 - \beta \xi [2\alpha \xi^2 - 2(\alpha + 1)\xi + 2]}}{\beta \xi} = \infty$$

But in the case of  $\alpha = 0$ , there is

$$g_c(\xi) = \lim_{\xi \rightarrow \infty} \left( \frac{-4}{\beta \xi} \pm \sqrt{\frac{16}{\beta^2 \xi^2} + \frac{2}{\beta} - \frac{2}{\beta \xi}} \right) = \left( \frac{2}{\beta} \right)^{1/2},$$

therefore  $g_c(\xi) \rightarrow \left( \frac{2}{\beta} \right)^{1/2}$  as  $\xi \rightarrow \infty$  within the former class of solutions,

there is a critical solution  $g_c(\xi)$  such that  $g_c(\xi) = -\frac{1}{4}$  as  $\xi \rightarrow 0$ .

Note that the locus of points at which  $g'(\xi) = 0$  and from equation (61) for  $\alpha = 0$ , there is

$$g_c(\xi) = -\frac{1}{4} .$$

One has to note that when  $\alpha = 0$  equation (67) has the form

$$g(\xi) = \frac{-4 \pm \sqrt{16 + 2\beta \xi [\xi - 1]}}{\beta \xi} \quad \dots (68)$$

there is the critical solution  $g_c(\xi)$  such that  $g_c(\xi) = -\frac{1}{4}$  as  $\xi \rightarrow 0$

and also  $g_c(\xi) = \left(\frac{2}{\beta}\right)^{1/2}$  as  $\xi \rightarrow \infty$ .

Some of the solution curves of equation (68) in  $(\xi, g)$  plane are presented in figures (2, 3, and 4) for different value of  $\alpha$  and  $\beta$ , we note that  $t = \xi$  in the following figure.

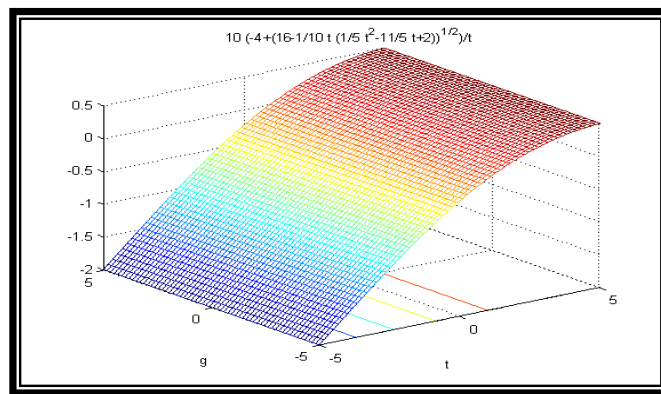


Figure (2): Solution curves in  $(\xi, g)$ -plane for different values of  $\alpha = 0.1, \beta = 0.1$

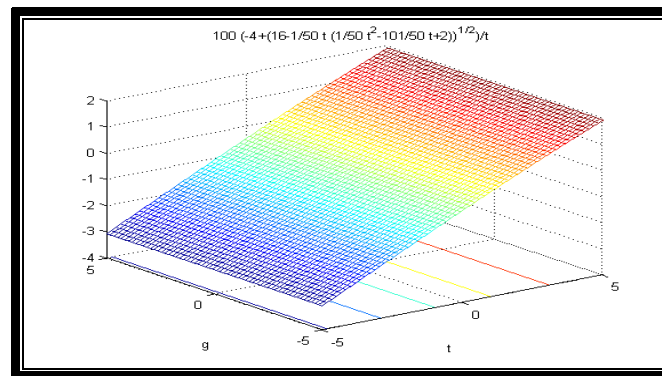


Figure (3): Solution curves in  $(\xi, g)$ -plane for different values of  $\alpha = 0.01, \beta = 0.01$

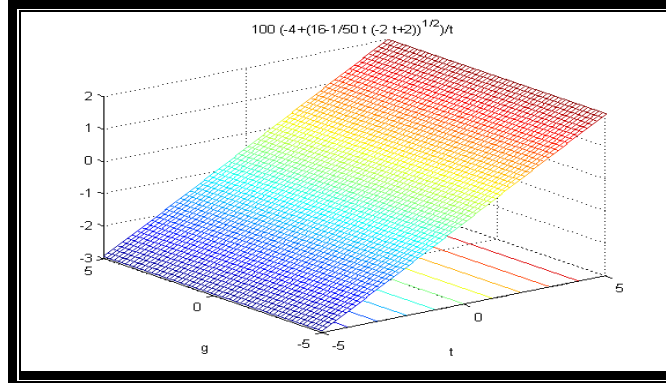


Figure (4): Solution curves in  $(\xi, g)$ -plane for different values of  $\alpha = 0, \beta = 0.01$

This equation will be solved but without the surface tension term namely “ $-\beta f^2 f f'' = 0$ ” and in this case the equation (52) becomes

$$f' - 4ff'' + 4f'^2 - \alpha f^2 f' = 0 \quad \dots (69)$$

the transformation eq.(59), reduces equation (69) to give

$$1 - 4\xi g' + 4g - \alpha \xi^2 = 0 \text{ for } f' \neq 0 \text{ in this case}$$

$$g'(\xi) = \frac{1 + 4g - \alpha \xi^2}{4\xi} \quad \dots (70)$$

equation (70) is integrated, and after some simplifications, the solution of equation (70) has the form

$$g(\xi) = -\frac{1}{4}(\alpha \xi^2 - \alpha \xi - \xi + 1). \quad \dots (71)$$

Suppose that  $p_0 = 0$ , from equation (69) one gets

$$f' - 4ff'' + 4f'^2 = 0 \text{ or}$$

$$f'' = \frac{f' + 4f'^2}{4f} \quad \dots (72)$$

Now equation (43) is solved numerically by using matlab program (William ,2001) with initial conditions given from Brown’s experiment, namely  $f(12) = 4.6$  and  $f'(12) = 0.32$  the solution has the form

$$f(\xi) = \frac{11431}{5} \frac{\exp(-\frac{497}{460}\xi + \frac{1491}{115})}{\left(-32 + 529 \exp(-\frac{497}{460}\xi + \frac{1491}{115})\right)} \quad \dots (73)$$

The solution curve of equation (73) is shown in figure (5) in  $(\xi, f)$  plane; we note that  $t = \xi$  in the figure below.

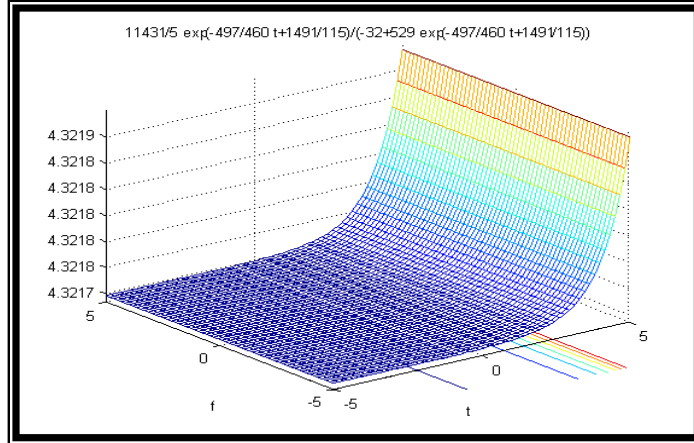


Figure (5): solution curve of equation (45).

Some of the solution curves of equation (70) are shown in figure (6) in  $(\xi, g)$  plane for different value  $\alpha$

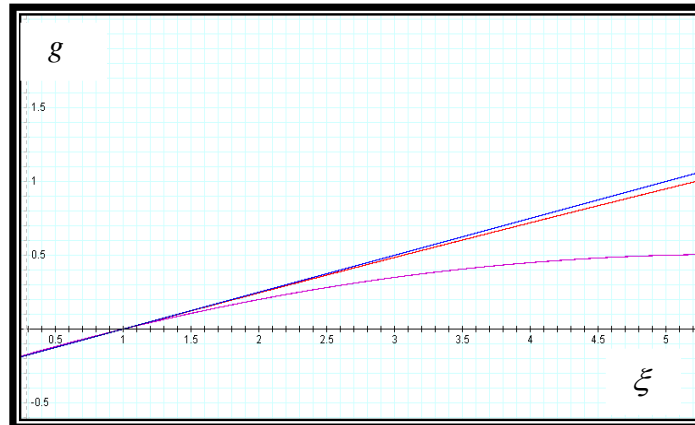


Figure (6): Solution curves in  $(\xi, g)$ -plane for different values of

•  $\alpha = 0.1$ , •  $\alpha = 0.01$ , •  $\alpha = 0$

### **Conclusion**

The dynamics of a free-surface liquid film is very useful in industrial coating and spinning processes. The solution of the thickness of the liquid film is determined for different values of the parameter  $\beta$  and from the solution curves it seems that the thickness or the liquid film increases as the parameter  $\beta$  increases for a thin liquid film flowing steadily between two vertical guide wires where we consider two cases.

In the first case the gravity is taken to be zero and in the second case when gravity is taken into account, the two cases are considered to see what the significant effect of gravity is.

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## ديناميكية الأغشية الرقيقة السائلة

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### الخلاصة

ان ميكانيكية الجريان للطبقة الرقيقة بين دليين عمودين قد درست ولكن بانعدام جهد القص على السطوح الحرة وان الجاذبية ليس لها اي تأثير على غشاء السائل، لقد تم استخدام معادلات (Navier-Stokes) في النظام الثنائي البعد الازمني والغير قابل للانضغاط . لقد تم اشتقاق المعادلات التفاضلية التي تحكم هذا الجريان وقد حصلنا على حل لها عندما يكون فيها المجال واسعاً ورقيقاً. وكذلك تم الحصول على حل المعادلات التي تحكم الجريان تحليليا حيث وجد حل حرج  $g_c(\xi) = (\frac{2}{\beta})^{1/2}$  عندما تكون  $\xi$  كبيرة و المعلمة  $\alpha$  مساوية الى الصفر حيث ان  $\alpha = \frac{P_0}{\rho g^2 R^2} H^2$  تكون مطابقة مع الحالة التي يكون فيها الضغط القياسي  $P_0$  مساوي الى الصفر، و بصورة عامة تم حل المعادلة عندما تكون  $P_0$  غير مساوية الى الصفر وباستخدام قيم مختلفة للمعلمة  $\beta$  تبين بان السمك يزداد بازياد المعلمة  $\beta$ .