Centrally Prime Rings which are Commutative

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Abstract

In this paper the definition of centrally prime rings is introduced , our main purpose is to classify those centrally prime rings which are commutative and so that several conditions are given each of which makes a centrally prime ring commutative.

The Fundamentals

Let R be a ring .A non-empty subset S of R is said to be a multiplicative closed set in R if $a,b \in S$ implies $ab \in S$, and a multiplicative closed set S is called a multiplicative system if $0 \notin S$,(Larsen &McCarthy ,1971).Let S be a multiplicative system in R such that $[S,R]=\{0\}$,where $[S,R]=\{[s,r]: s \in S, r \in R\}$ and $[s,r]$ is the commutator defined by $sr-rs$. Define a relation (~) on $R \times S$ as follows :
If
$(a,s),(b,t) \in R \times S$ *then* $(a,s) \sim (b,t)$ *iff* $\exists x \in S$ *such that* $x(at-bs)=0$
Since $[S,R]=\{0\}$,it can be shown that (~) is an equivalence relation on $R \times S$.Now denote the equivalence class of (a,s) in $R \times S$ by a_S, that is $a_S = \{(b,t) \in R \times S : (a,s) \sim (b,t)\}$ (this equivalence class is also denoted by $\frac{a}{s}$ (Larsen & McCarthy,1971) or by $s^{-1}a$, and then denote the set of all equivalence classes determined under this equivalence relation by R_S, that is let $R_S = \{a_s : (a,s) \in R \times S\}$.Note that R_S is also denoted by $S^{-1}R$ (Larsen & McCarthy,1971 ;Ranicki,2006).
On $R \times S$ we define addition $(+)$ and multiplication $(.)$ as follows:
$a_S + b_t = (at+bs)_{st}$.
$a_S . b_t = (ab)_{st}, \forall a_S, b_t \in R_S$.

It can be shown that these two operations are well-defined and that $(R_S,+,.)$ forms a ring which is known as the localization of R at S ,(Fahr,2002).
Now we mention to some basic definitions:
Let R be a ring.Then:
R is called a prime ring if $aRb=\{0\}$ for $a,b \in R$ then $a=0$ or $b=0$ (Herstein,1969,Tsai,2004 and Ashraf,2005), where $aRb=\{arb : r \in R\}$.
An additive mapping $D : R \to R$ is called a derivation on R if $D(ab)=D(a)b+aD(b), \forall a,b \in R$ (Martindale &Miers,1983; Vukman,1999 and Jung & Park,2006), in other words a mapping $D : R \to R$ is called a derivation on R if:
1-$D(a+b)=D(a)+D(b)$, and
2-$D(ab)=aD(b)+D(a)b, \forall a,b \in R$.
An element $r \in R$ is called a zero divisor if $rx=0$ or $xr=0$ for some nonzero x in R, and a zero divisor is called a proper zero divisor if it is nonzero.By the center of R we mean the set $Z(R)=\{x \in R : xr=rx, \forall r \in R\}$,it can be shown that $Z(R)$ is a subring of R.

Some Remarks

If R is a ring and S is a multiplicative system in R such that $[S,R]=\{0\}$, then:
i) R_S has the identity element though R does not have , in fact if $s \in S$ then s_s is the identity element of R_S, since if a_t is any element of R_S, then $ast=ats$ which gives $ast-ats=0$, then $s(ast-ats)=0$ and hence $s((as)t-a(ts))=0$,thus $(as,ts) \sim (a,t)$ which means $(as)_{ts}=a_t$ or $a_t s_s = a_t$ and using the same technique we can show that $s_s a_t = a_t$ which means $\forall s \in S, s_s$ is the identity element of R_S .Note that this identity does not depend on the choice of elements of S that is $s_s = t_t, \forall s,t \in S$. To prove this, since we have $[S,R]=\{0\}$, so $[s,t]=\{0\}, \forall s \in S, t \in R$ and hence we get $st-ts=0, \forall s \in S, t \in R$, or $st-ts=0, \forall s,t \in S$,(since $S \subseteq R$) then we get $s(st-ts)=0, \forall s,t \in S$

which means $(s,s)\sim(t,t),\forall s,t\in S$, and thus $s_s = t_t, \forall s,t\in S$.

ii)if $a,b\in R$ and $s\in S$, then $a_s + b_s = (a+b)_s$, to show this we have $(as+bs)s = (a+b)ss$ or $(as+bs)s-(a+b)ss = 0$, which means $s[(as+bs)s-(a+b)ss]=0$, and hence $(as+bs,ss)\sim(a+b,s)$,
that means $(as+bs)_{ss} = (a+b)_s$ and then we get
$a_s + b_s = (as+bs)_{ss} = (a+b)_s$.

iii) $\forall s\in S, 0_s$ is the zero of the ring R_S , since if $a_t \in R_S$ is any element, where
$a\in R, t\in S$, then $a_t + 0_s = (as+0t)_{ts} = (as)_{ts} = a_t s_s = a_t$.
Similarly it can be shown that $0_s + a_t = a_t$.(we will denote 0_s simply by 0).
It is necessary to mention that $0_s = 0_t, \forall s,t\in S$.That is the zero of R_S also does not depend on the choice of the elements of S ,to prove this it is known that $0t = 0 = 0s, \forall s,t\in S$, hence we get $s(0t-0s)=0, \forall s,t\in S$ and thus $(0,s)\sim(0,t),\forall s,t\in S$ which means $0_s = 0_t, \forall s,t\in S$.

iv)if $a_s\in R_S$, where $a\in R$ and $s\in S$ then $(-a)_s$ is the additive identity of a_s in R_S ,that is $-a_s = (-a)_s$.Note that
$a\in R \Rightarrow -a\in R \; and \; s\in S \Rightarrow (-a)_s \in R_S$ and then
$a_s + (-a)_s = (as+(-a)s)_{ss} = 0_{ss} = 0_s = 0$.

v)if $a_s = 0$ in R_S,where $a\in R, s\in S$, then $\exists t\in S$ such that $ta = 0$.To show this,we have $a_s = 0 = 0_s$, hence $(a,s)\sim(0,s)$ which means $\exists x\in S$ such that $x(as-0s)=0$ or $xas = 0$.If we let $t = xs$ then $x,s\in S$ implies $t = xs\in S$ and then $ta = xsa = xas = 0$.

vi)If $D: R\to R$ is a mapping then by D^2 we mean $D\circ D$.In general D^n will mean $D\circ D\circ\ldots\circ D$ (n times) and finally if $x\in R$ then by xD we mean the mapping $xD: R\to R$ which is defined by $(xD)(r) = x(D(r)), \forall r\in R$.

The Main Results

Our main purppose in this paper is to transfere some results on prime rings to centrally prime rings , in fact these results known as the commutativity conditions for prime rings and here we will prove the alternative results for centrally prime rings , that is we will determine the conditions which make centrally prime rings commutative but first it is necessary to indicate to the following theorems:

Theorem A:(Chung et al. ,1979)

Let R be a prime ring and $D: R \to R$ is a derivation on R.
If $D(x) \in Z(R), \forall x \in R$ then either $D = 0$ or R is commutative.

Theorem B:(Felzenszwalb & Giambruno,1982)

Let R be a prime ring and U is a nonzero ideal of R.If $D: R \to R$ is a nonzero derivation on R such that $D(u)u = uD(u), \forall u \in U$ then R is commutative.

Theorem C:(Lee & Lee,1986)

Let R be a prime ring and I a nonzero ideal of R.Suppose that $D: R \to R$ is a derivation on R and n is a possitive integer such that $D^n(I) \subseteq Z(R)$ then either $D^n = 0$ or R is commutative.

Theorem D:(Daif & Bell,1992)

Let R be a prime ring and $D: R \to R$ is a derivation on R.If there exists a nonzero ideal K of R such that either
$xy + D(xy) = yx + D(yx), \forall x, y \in K$ or
$xy - D(xy) = yx - D(yx), \forall x, y \in K$, then R is commutative.

Theorem E:(Bresar,1993)

Let R be a prime ring and U is a nonzero left ideal of R, and suppose that $D, G: R \to R$ are derivations on R satisfying
$D(u)u - uG(u) \in Z(R), \forall u \in U$.
If $D \neq 0$ then R is commutative.

Theorem F:(Filipps,1999)

Let R be a prime ring and I is a nonzero ideal of R.If $D: R \to R$ is a nonzero derivation on R such that $D([x, y]) = [x, y], \forall x, y \in I$ then R is commutative.

Now we introduce the following definitions.

Definitions

Let R be a ring.Then we say:

1-R is a centrally prime ring if R_S is a prime ring for each multiplicative system S in R with $[S,R]=\{0\}$.

2-R satisfies central commutation property (CCP) if R_S is commutative for each multiplicative system S in R with $[S,R]=\{0\}$.

3-A derivation $D:R\rightarrow R$ is centrally-zero derivation on R if $D(S)=\{0\}$ for each multiplicative system S in R with $[S,R]=\{0\}$.

Before giving the main results of this paper we will prove the following two lemmas which play the basic role in the proof of results of the present paper.In fact the first lemma proves that if R is a ring and S is a multiplicative system in R with $[S,R]=\{0\}$ then each centrally-zero derivation on R induces some derivation on R_S.

Lemma 1:

Let R be a ring and S a multiplicative system in R such that $[S,R]=\{0\}$.

If $D:R\rightarrow R$ is a centrally-zero derivation on R, then $D_*:R_S\rightarrow R_S$ defined by $D_*(r_s)=(D(r))_s, \forall r_s\in R_S$, is a derivation on R_S.

Proof:

First to show D_* is a mapping.It is clear that $\forall r\in R$ we have $D(r)\in R$ and thus $D_*(r_s)=(D(r))_s\in R_S, \forall r_s\in R_S$.Now let $a_t=b_s$ for $a_t,b_s\in R_S$, then $(a,t)\sim(b,s)$.

Hence there exists $x\in S$ *such that* $x(as-bt)=0$.So we have $xas=xbt$ or $xsa=xtb$ (since $[a,s]=0$ *and* $[b,t]=0$) and thus $D(xsa)=D(xtb)$ or $(xs)D(a)+D(xs)a=(xt)D(b)+D(xt)b$, then we get

$(xs)D(a)=(xt)D(b)$ (since $D(xs)=0=D(xt)$).Hence

$x(sD(a)-tD(b))=0$ which yields $x(D(a)s-D(b)t)=0$(since $[S,R]=\{0\}$) and so $(D(a),t)\sim(D(b),s)$.

Hence $(D(a))_t=(D(b))_s$ and thus $D_*(a_t)=D_*(b_s)$ which means D_* is a mapping.Now we will show that:

1- $D_*(a_s + b_t) = D_*(a_s) + D_*(b_t)$ and

2- $D_*(a_s b_t) = a_s D_*(b_t) + D_*(a_s) b_t, \forall a_s, b_t \in R_S$

For the proof of the first we have

$D_*(a_s + b_t) = D_*((at+bs)_{st}) = (D(at+bs))_{st}$

$= (D(at) + D(bs))_{st} = (D(at))_{st} + (D(bs))_{st} = (D(a)t)_{st} + (D(b)s)_{ts}$

$(D(a))_s t_t + (D(b))_t s_s = (D(a))_s + (D(b))_t = D_*(a_s) + D_*(b_t)$.

Also we have

$D_*(a_s b_t) = D_*((ab)_{st}) = (D(ab))_{st} = (aD(b) + D(a)b)_{st} =$

$(aD(b))_{st} + (D(a)b)_{st} = a_s (D(b))_t + (D(a))_s b_t =$

$a_s D_*(b_t) + D_*(a_s) b_t$

Hence D_* is a derivation on R_S .♦

Remark

We call the derivation $D_* : R_S \to R_S$ as constructed above from the centrally-zero derivation $D : R \to R$, the induced derivation on R_S .

Next we give the second lemma, the importance of which for getting the results is not less than importance of the first one.

Lemma 2:

Let R be a ring for which $Z(R)$ contains no proper zero divisors of R then:

1-If $t \in Z(R) - \{0\}$ and $r \in R$ such that $tr = 0$ then $r = 0$.

2- $Z(R) - \{0\}$ is a multiplicative system in R.

3-If R satisfies (CCP) then it is commutative.

4- $(Z(R))_S = Z(R_S)$, for all multiplicative systems S in R with $[S, R] = \{0\}$.

Proof:

1: Since $0 \neq t \in Z(R)$ so if $r \neq 0$ then t is a proper zero divisor of R which is
a contradiction and hence $r = 0$.

2: Clearly $0 \notin Z(R) - \{0\}$ and if $a, b \in Z(R) - \{0\}$ then $a, b \in Z(R)$, and $a \neq 0$, $b \neq 0$ but since $Z(R)$ is a subring of R, so $ab \in Z(R)$.Since

$Z(R)$ has no proper zero divisors we get $ab \neq 0$.
Hence $ab \in Z(R)-\{0\}$.Thus $Z(R)-\{0\}$ is a multiplicative system in R.
3:By **(2)** we have $Z(R)-\{0\}$ is a multiplicative system in R.Let $S = Z(R)-\{0\}$.Then clearly $[S,R]=\{0\}$ and hence by the (CCP) we get that R_S is commutative.Now let $a,b \in R$ be any elements and since $S \neq \phi$ so there exists an $s \in S$.Then $a_S, b_S \in R_S$ and hence $a_S b_S = b_S a_S$ or $(ab)_{SS} = (ba)_{SS}$ and then $(ab, ss) \sim (ba, ss)$ so that $\exists t \in S$ such that $t(abss - bass) = 0$ which implies $tss(ab - ba) = 0$.But then $t, s \in S$ implies $tss \in S$ so that $tss \neq 0$ and hence $tss \in Z(R)-\{0\}$.By applying (**1**) we get $ab - ba = 0$.That is $ab = ba$, which means that R is commutative.
4:It is clear that $Z(R)$ is a ring (in fact it is a subring of R) and since $[S,R]=\{0\}$ so $[S,Z(R)]=\{0\}$ and also $S \subseteq Z(R)$ which means that S is a multiplicative system in the ring $Z(R)$ with $[S,Z(R)]=\{0\}$ which means talking about localization of $Z(R)$ at S meaningfull.
Next we will show $(Z(R))_S \subseteq Z(R_S)$.
Let $a_S \in (Z(R))_S$, $where\ a \in Z(R), s \in S$ then let $b_t \in R_S$ be any element ,
where $b \in R, t \in S$ .Since $a \in Z(R)\ and\ b \in R$ so $ab = ba$ and then $a_S b_t = (ab)_{st} = (ba)_{ts} = b_t a_S$ which means $[a_S, b_t] = 0$ for all $b_t \in R_S$.
Hence $[a_S, R_S] = \{0\}$ thus $a_S \in Z(R_S)$ and so that $(Z(R))_S \subseteq Z(R_S)$.
It remains to show $Z(R_S) \subseteq (Z(R))_S$, so let $r_S \in Z(R_S)$ where $r \in R, s \in S$. Now $\forall x \in R$ we have $x_S \in R_S$ and hence $r_S x_S = x_S r_S$ or $(rx - xr)_{SS} = 0$ and so that $\exists u \in S$ such that $u(rx - xr) = 0$,where $u \in S \subseteq Z(R)-\{0\}$ and hence by (**1**) we get $rx - xr = 0$ or $rx = xr$, thus $r \in Z(R)$ and so that $r_S \in (Z(R))_S$ which gives $Z(R_S) \subseteq (Z(R))_S$.Hence $(Z(R))_S = Z(R_S)$. ♦
Now it is the time for giving our main results.

Theorem 1:
Let R be a centrally prime ring in which $Z(R)$ contains no proper zero divisors and $D: R \to R$ is a centrally-zero derivation on R with $D \neq 0$. If $D(x) \in Z(R), \forall x \in R$ then R is commutative.

Proof:
To show R satisfies (CCP).So let S be any multiplicative system in R such that
$[S, R] = \{0\}$.Consider the induced derivation $D_* : R_S \to R_S$ on R_S, where $D_*(r_S) = (D(r))_S, \forall r_S \in R_S$.
Let $D_* = 0$.Now fix $s \in S$ (since $S \neq \phi$) and let $r \in R$ then $r_S \in R_S$.Hence $(D(r))_S = D_*(r_S) = 0$.Then there exists $t \in S$ such that $tD(r) = 0$ but $0 \neq t \in S \subseteq Z(R)$.Hence $t \in Z(R) - \{0\}$.So by lemma **(2),** $D(r) = 0$.Hence this result is true for all $r \in R$ and thus $D = 0$ which is a contradiction.So $D_* \neq 0$. Now if $x_S \in R_S$ where $x \in R$ *and* $s \in S$ then we have $D_*(x_S) = (D(x))_S \in (Z(R))_S = Z(R_S)$.
Hence R_S is a prime ring and $D_* : R_S \to R_S$ is a nonzero derivation on R_S such that $D_*(x_S) \in Z(R_S), \forall x_S \in R_S$.Hence by theorem (**A**), R_S is commutative.
So R satisfies (CCP).Since $Z(R)$ has no proper zero divisors we get R is commutative.♦

Theorem 2:
Let R be a centrally prime ring with $Z(R)$ has no proper zero divisors and $D: R \to R$ be a centrally-zero derivation on R with $D \neq 0$.If U is a nonzero ideal of R such that $D(u)u = uD(u), \forall u \in U$ then R is commutative.

Proof:
To show that R satisfies (CCP).So let S be any multiplicative system in R such that $[S, R] = \{0\}$.To show the ideal U_S of R_S is nonzero.If $U_S = 0$, then let u be any element in U and fix $s \in S$ (this is possible because $S \neq \phi$) so that $u_S \in U_S$ and so $u_S = 0$, hence $\exists t \in S$ such that

$tu = 0$,where $0 \neq t \in S \subseteq Z(R)$, or $t \in Z(R) - \{0\}$.Hence $u = 0$ by lemma (**2**) and this, in consequence implies that $U = 0$ which is a contradiction and thus $U_S \neq 0$.

Next let $D_* : R_S \to R_S$ be the induced derivation on R_S, where $D_*(r_S) = (D(r))_S, \forall r_S \in R_S$.To show $D_* \neq 0$. If $D_* = 0$ then fix $s \in S$, so $\forall r \in R$ we have $r_S \in R_S$.Hence $(D(r))_S = D_*(r_S) = 0$.Thus $\exists t \in S$ such that $tD(r) = 0$, where $0 \neq t \in S \subseteq Z(R)$ which means that $t \in Z(R) - \{0\}$.Hence by lemma (**2**) we get $D(r) = 0$ and this is true $\forall r \in R$. Hence $D = 0$ which is a contradiction,thus $D_* \neq 0$.Next $\forall u_S \in U_S$ we have $D_*(u_S)u_S = (D(u))_S u_S = (D(u)u)_{SS} = (uD(u))_{SS} = u_S(D(u))_S = u_S D_*(u_S)$.

Thus we have R_S is a prime ring (since R is centrally prime), U_S is a nonzero ideal of R_S and D_* is a nonzero derivation on R_S such that $D_*(u_S)u_S = u_S D_*(u_S), \forall u_S \in U_S$.Hence by theorem (**B**) we get that R_S is commutative.Hence R satisfies (CCP) and $Z(R)$ being without proper zero divisors R is commutative.♦

Theorem 3:

Let R be a centrally prime ring in which $Z(R)$ has no proper zero divisors and I a non zero ideal of R.Suppose that $D : R \to R$ is a centrally-zero derivation on R and n is a positive integer such that $D^n(I) \subseteq Z(R)$ and $D^n \neq 0$ then R is commutative.

Proof:

If R does not satisfy (CCP) then there exists a multiplicative system S with $[S, R] = \{0\}$ for which R_S is not ommutative.Note that $[S, R] = \{0\}$ means that $S \subseteq Z(R)$.Now suppose that $I_S = 0$.

Since $I \neq 0$ *and* $S \neq \phi$ so $\exists 0 \neq x \in I$ *and* $s \in S$ .Then $x_S \in I_S$, and hence $x_S = 0$ then $\exists t \in S$ such that $tx = 0$, where $0 \neq t \in S \subseteq Z(R)$.

Since $x \neq 0$ so t is a proper zero divisor of R.Thus $Z(R)$ contains a proper zero divisor of R which is a contradiction.Hence we get that $I_S \neq 0$.

Next let $D_* : R_S \to R_S$ be the induced derivation on R_S (of lemma **1**), where $D_*(r_S) = ((D(r))_S, \forall r_S \in R_S$.

To show $D_*^n(I_S) \subseteq Z(R_S)$.Let $\lambda \in D_*^n(I_S)$.Then there exists $r_S \in I_S$,where $r \in I, s \in S$, such that $\lambda = D_*^n(r_S)$.But $r \in I$ implies $D^n(r) \in D^n(I) \subseteq Z(R)$.Thus $\lambda = D_*^n(r_S) = (D^n(r))_S \in (Z(R))_S = Z(R_S)$.

Hence we get $D_*^n(I_S) \subseteq Z(R_S)$.

Now R_S is a prime ring , I_S is a non zero ideal of R_S and D_* is a derivation on R_S such that $D_*^n(I_S) \subseteq Z(R_S)$ so by theorem (**C**) we get either $D_*^n = 0$ or R_S is commutative and R_S being noncommutative so we get $D_*^n = 0$.

If $r \in R$ is any element then fix $s \in S$ (since $S \neq \phi$) and thus $r_S \in R_S$ then $(D^n(r))_S = D_*^n(r_S) = 0$ which means $\exists t \in S$ such that $t(D^n(r)) = 0$,where $0 \neq t \in S \subseteq Z(R)$, and thus $t \in Z(R) - \{0\}$,hence by lemma(2), $D^n(r) = 0$ and this result is true $\forall r \in R$ so that $D^n = 0$ which is a contradiction and hence R must satisfy (CCP) and since $Z(R)$ contains no proper zero divisor we get that R is commutative (Lemma **2**).♦

As a corollary to this theorem we give:

Corollary 1:

Let R be a centrally prime ring with $Z(R)$ contains no proper zero divisors and $D : R \to R$ is a centrally-zero derivation on R.If n is a positive integer such that $D^n(R) \subseteq Z(R)$ and $D^n \neq 0$ then R is commutative.

Proof:

Putting $I = R$ in theorem (**3**) the result will follows.♦

Theorem 4:

Let R be a centrally prime ring with $Z(R)$ contains no proper zero divisors and $D : R \to R$ is a centrally-zero derivation on R.If there exists a nonzero ideal J of R such that either

$xy + D(xy) = yx + D(yx), \forall x, y \in J$ or

$xy - D(xy) = yx - D(yx), \forall x, y \in J$, then R is commutative.

Proof:

If possible suppose that R does not satisfy (CCP), so there exists a multiplicative system S in R with $[S,R]=\{0\}$ but R_S is not commutative. Then R_S is a prime ring (since R is centrally prime).Now let $D_*:R_S \to R_S$ be the induced derivation on R_S (of Lemma **1**)that is $D_*(r_S)=(D(r))_S, \forall r_S \in R_S$. To show that the ideal J_S is nonzero.Let $J_S=0$.Now if $x \in J$ then since $S \neq \phi$ so $\exists s \in S$. Hence $x_S \in J_S$ which implies that $x_S=0$ so $\exists t \in S$ such that $tx=0$,where $0 \neq t \in S \subseteq Z(R)$, which means $t \in Z(R)-\{0\}$.Hence $x=0$ (by lemma **2**) which implies $J=0$ and this is a contradiction and thus $J_S \neq 0$.

We take the first case which is $xy+D(xy)=yx+D(yx), \forall x,y \in J$.

Let $x_S, y_t \in J_S$ *for* $x,y \in J$ *and* $s,t \in S$.

Then $x_S y_t + D_*(x_S y_t)=(xy)_{st}+D_*((xy)_{st})=(xy)_{st}+(D(xy))_{st}=$ $(xy+D(xy))_{st}=(yx+D(yx))_{ts}=(yx)_{ts}+(D(yx))_{ts}=y_t x_S + D_*((yx)_{ts})$ $=y_t x_S + D_*(y_t x_S)$.That means J_S is a nonzero ideal of the prime ring R_S and $D_*:R_S \to R_S$ is a derivation on R_S such that $x_S y_t + D_*(x_S y_t)=y_t x_S + D_*(y_t x_S), \forall x_S, y_t \in J_S$.Hence from theorem (**D**) we get that R_S is commutative which is a contradiction.Hence R must satisfy (CCP),and since $Z(R)$ has no proper zero divisors so by lemma(2), we get R is commutative.And if we take the second case , that is $x_S y_t - D_*(x_S y_t)=y_t x_S - D_*(y_t x_S), \forall x_S, y_t \in J_S$ then by the same technique we get that R is again commutative.♦

As a corollary to this theorem we give:

Corollary 2:

Let R be a centrally prime ring in which $Z(R)$ has no proper zero divisors and $D:R \to R$ is a centrally-zero derivation on R.

If $xy+D(xy)=yx+D(yx), \forall x,y \in R$ or

$xy-D(xy)=yx-D(yx), \forall x,y \in R$ then R is commutative.

Proof: Taking $J=R$ in the theorem (**4)** we get the result.♦

Theorem 5:
Let R be a centrally prime ring in which $Z(R)$ has no proper zero divisors and U a nonzero left ideal of R.Suppose that $D:R\to R$ and $G:R\to R$ are two centrally-zero derivations on R with $D\neq 0$ and such that $D(u)u-uG(u)\in Z(R),\forall u\in U$, then R is commutative.

Proof:
To show R satisfies (CCP), let S be any multiplicative system in R with $[S,R]=\{0\}$.Clearly R_S is a prime ring.Consider the induced derivations $D_*,G_*:R_S\to R_S$ on R_S. To show U_S is nonzero left ideal of R_S, let $u_s,v_t\in U_S$, where $u,v\in U,s,t\in S$. Clearly $0_s\in U_S$ so U_S is a non-empty subset of R_S.Now $u_s-v_t=(ut-vs)_{st}\in U_S$(since $ut-vs\in U,st\in S$) , and if $r_x\in R_S$ then $r_xu_s=(ru)_{xs}\in U_S$(since $ru\in U$ *and* $xs\in S$).
Hence U_S is a left ideal of R_S. To show $U_S\neq 0$ and $D_*\neq 0$.If $U_S=0$ then fix $s\in S$ and now if $u\in U$ then $u_s=0$ and thus $\exists t\in S$ such that $tu=0$,where $0\neq t\in S\subseteq Z(R)$, that is $t\in Z(R)-\{0\}$ and hence by lemma (**2**) , $u=0$ which means $U=0$ and this is a contradiction and thus $U_S\neq 0$ and if $D_*=0$ then for any $r\in R$ we have $r_s\in R_S$.Hence $(D(r))_s=D_*(r_s)=0$.So $\exists x\in S$ such that $xD(r)=0$,where $0\neq x\in S\subseteq Z(R)$ or $x\in Z(R)-\{0\}$.Thus by lemma (**2**), $D(r)=0$ and this is true $\forall r\in R$ so $D=0$ which is a contradiction and thus $D_*\neq 0$.Next $\forall u_s\in U_S$ we have $D_*(u_s)u_s-u_sG_*(u_s)=$
$(D(u))_su_s-u_s(G(u))_s=(D(u)u)_{ss}-(uD(u))_{ss}=$
$(D(u)u-uD(u))_{ss}\in Z(R_S)$.Thus R_S is a prime ring , U_S is a nonzero left ideal of R_S and $D_*,G_*:R_S\to R_S$ are derivations on R_S with $D_*\neq 0$ and such that $D_*(u_s)u_s-u_sG_*(u_s)\in Z(R_S),\forall u_s\in U_S$ and hence by theorem (**E**), R_S is commutative so R satisfies (CCP) and since $Z(R)$ is without proper zero divisors so R is commutative.♦

Corollary 3:

Let R be a centrally prime ring in which $Z(R)$ is without proper zero divisors and U is a nonzero left ideal of R.If $D: R \to R$ is a centrally-zero derivation on R with $D \neq 0$ such that $D(u)u + uD(u) \in Z(R), \forall u \in U$, then R is commutative.

Proof:

Define $G: R \to R$ by $G(r) = -D(r), \forall r \in R$.To show G is a derivation on R.First if $a = b \in R$ then $D(a) = D(b)$ or $-D(a) = -D(b)$ and hence $G(a) = G(b)$ So G is a mapping.

Now for $a, b \in R$ we have $G(a+b) = -D(a+b) = -(D(a) + D(b)) = -D(a) + (-D(b)) = G(a) + G(b)$ and also $G(ab) = -D(ab) = -(aD(b) + D(a)b) = a(-D(b)) + (-D(a))b = aG(b) + G(a)b$ and thus G is a derivation on R.Now if S is any multiplicative system in R with $[S, R] = \{0\}$ then $\forall s \in S$ we have $G(s) = -D(s) = 0$.Hence G is also a centrally-zero derivation.Next $\forall u \in U$ we have

$D(u)u - uG(u) = D(u)u - u(-D(u)) = D(u)u + uD(u) \in Z(R)$.

Thus R is a centrally prime ring in which $Z(R)$ has no proper zero divisors , U is a nonzero left ideal of R and $D, G: R \to R$ are centrally-zero derivations with $D \neq 0$ such that $D(u)u - uG(u) \in Z(R), \forall u \in U$.

Hence by the theorem (**5**) we get R is commutative.♦

Theorem 6:

Let R be a centrally prime ring with $Z(R)$ has no proper zero divisors and U is a nonzero left ideal of R.If $D: R \to R$ is a centrally-zero derivation on R with $D \neq 0$ such that $D(U) \subseteq Z(R)$, then R is commutative.

Proof:

To show R satisfies (CCP), let S be any multiplicative system in R with $[S, R] = \{0\}$.Clearly R_S is a prime ring (since R is centrally prime).

Now consider the induced derivation $D_*: R_S \to R_S$ on R_S.

As we have done in theorem (**5**) we can show U_S is a left ideal of R_S and $U_S \neq 0, D_* \neq 0$ *and* $D_*(U_S) \subseteq Z(R_S)$.Hence R_S is a prime ring , U_S is a nonzero left ideal of R_S and $D_*: R_S \to R_S$ is a nonzero derivation on R_S such that $D_*(U_S) \subseteq Z(R_S)$.Hence from theorem (**E**) we get R_S is commutative and hence R must satisfy (CCP) and $Z(R)$ being without proper zero divisors, therefore R becomes commutative.♦

Theorem 7:

Let R be a centrally prime ring in which $Z(R)$ has no proper zero divisors and $D: R \to R$ is a centrally-zero derivation on R with $D \neq 0$.If I is a nonzero ideal of R such that $D([x,y]) = [x,y], \forall x,y \in I$ then R is commutative.

Proof:

If R does not satisfy (CCP) then there exists a multiplicative system S in R with $[S,R] = \{0\}$ but R_S is not commutative.Then let $D_*: R_S \to R_S$ be the induced derivation on R_S where $D_*(r_S) = (D(r))_S, \forall r_S \in R_S$.

Since R is centrally prime so R_S is a prime ring.

We will show $I_S \neq 0$ *and* $D_* \neq 0$.If $I_S = 0$, then $S \neq \phi \Rightarrow \exists s \in S$ so for any $x \in I$ we have $x_S = 0$ which implies that $\exists t \in S$ such that $tx = 0$, where $0 \neq t \in S \subseteq Z(R)$.So $t \in Z(R) - \{0\}$.Thus by lemma(**2**) , $x = 0$.

Hence $I = 0$, which is again a contradiction and so $I_S \neq 0$.

Now suppose that $D_* = 0$ then fix $s \in S$ (since $S \neq \phi$), now

if $r \in R$ is any element then we have $r_S \in R_S$.Hence $(D(r))_S = D_*(r_S) = 0$.

Thus $\exists u \in S$ such that $uD(r) = 0$,where $0 \neq u \in S \subseteq Z(R)$, that is $u \in Z(R) - \{0\}$.Hence by lemma (**2**), we get $D(r) = 0$ and this result is true for all $r \in R$ which means that $D = 0$ and this contradicts the fact that D is a nonzero derivation on R.Hence $D_* \neq 0$.

Next let $a_s, b_t \in I_S$, where $a,b \in I, s,t \in S$.Then

$$D_*([a_s,b_t]) = D_*(a_s b_t - b_t a_s) = D_*((ab)_{st} - (ba)_{ts}) =$$
$$D_*((ab-ba)_{st}) = (D(ab-ba))_{st} =$$
$$(D([a,b]))_{st} = ([a,b])_{st} = (ab-ba)_{st}$$
$$= (ab)_{st} - (ba)_{ts} = a_s b_t - b_t a_s = [a_s, b_t].$$

Hence R_S is a prime ring, I_S is a nonzero ideal of R_S and D_* is a nonzero

derivation on R_S such that $D_*([a_s, b_t]) = [a_s, b_t], \forall a_s, b_t \in I_S$.
Hence from theorem (**F**) we get that R_S is commutative which is a contradiction thus R must satisfy (CCP) and $Z(R)$ being without proper zero divisors we get R is commutative.♦

Corollary 4:

Let R be a centrally prime ring in which $Z(R)$ has no proper zero divisors and $D: R \to R$ be a centrally-zero derivation on R with $D \neq 0$ such that $D([x, y]) = [x, y], \forall x, y \in R$ then R is commutative.

Proof:

Putting $I = R$ in the theorem (**7**), the result will follows.♦

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الحلقات الاولية مركزيا والتي تكون تبادلية

عادل قادر جبار
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الخلاصة

في هذا البحث قدمنا تعريف الحلقات الاولية مركزيا حيث ان هدفنا الرئيسي هو تصنيف الحلقات الاولية مركزيا والتي تكون تبادلية وقد تمكننا من اعطاء شروط عديدة والذي يجعل كل واحد منها من الحلقة الاولية مركزيا حلقة تبادلية.