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Positive Solution For Eigen value Problems

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ABSTRACT

Studying the boundary value problem :-

$$-y'' = \lambda g(t) f(y) \quad , \quad a < t < b$$
$$y(a) = y(b) = 0$$

Values of the parameter (λ) are determined for which this problem has a positive solution. The methods used here extend recent works by a simple application of a Fixed Point Theorem in cones . I show the existence of at least one positive solution of this boundary value problem.

Introduction:

In this paper we consider the second - order boundary value problem (BVP)

$$\left.\begin{aligned} -y'' &= \lambda g(t) f(y) \quad , \quad a<t<b \\ y(a) &= y(b) = 0 \end{aligned}\right\} \quad ....(1.1)$$

The following conditions will be assumed throughout:-

A- $f : [0 , \infty) \to [0 , \infty)$ is continuous ,

B- $g : [0 , 1] \to [0 , \infty)$ is continuous and does not vanish identically on any subinterval ,

C- $f_0 = \lim_{x\to 0} \frac{f(x)}{x}$ and $f_\infty = \lim_{x\to\infty} \frac{f(x)}{x}$ exist ,

D- $a \geq 0 , b \leq 1$

The BVP (1.1) is called a Sturm - Liouville Problem and it is describe many phenomena of applied mathematics and physics ; see[1-3,5,8] for some references along this line.

For the case when α = 1 , β = 0 , γ =1 , δ = 0 and 0 < t < 1 , Johnny Henderson and Haiyan Wang [6] obtained solutions that are positive for an open interval of eigenvalues.

In this paper we obtaining solutions of (1.1) for certain λ involve concavity properties of solutions, which are employed in defining a cone on which a positive integral operator is defined . A Krasnosel'skii fixed point theorem [7] is applied to yield positive solutions of (1.1) , for λ beloninging to an open interval.

In Section 2 , we present some properties of Green's functions that are used in defining a positive operator . we also state the Krasnosel'skii fixed point theorem .

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In Section 3 , we give an appropriate Banach space and construct a cone to which we apply the fixed point theorem yielding solutions of (1.1) , for an open interval of eigenvalues .

2- SOME PRELIMINARIES

In this section , we state the above mentioned Krasnosel'skii fixed point theorem.We shall apply this fixed point theorem to completely continuous integral operator , whose kernal , G (t , s) , is the Green's function for

$- y'' = 0$

$y(a) = y(b) = 0$

In particular

$$G(t,s) = \begin{cases} t(1-s) & a \leq t \leq s \leq b \\ s(1-t) & a \leq s \leq t \leq b \end{cases} \quad ....(2.1)$$

from which

$G(t , s) > 0$ on $(0 , 1) \times (0 , 1)$, ...(2.2)

$G(t , s) \leq G(s , s) = s(1-s)$, $a \leq t \leq b , a \leq s \leq b$, ...(2.3)

and it is shown in [4] that

$G(t , s) \geq \frac{1}{4} G(s , s) = \frac{1}{4} s(1-s)$ $\frac{2a+1}{4} \leq t \leq \frac{2b+1}{4}$, $a \leq s \leq b$, ...(2.4)

I shall apply the following fixed point theorem to obtain solutions of 1.1,for certain λ □

THEOREM 1. [7] :Let B a Banach space , and let P be a cone in B . Assume N , K open subset of B with are be a completely continuous $0 \in N \subset \overline{N} \subset K$, and let $T : P \cap (\overline{K} \setminus N) \to P$ operator such that , either

1- $\| Tu \| \leq \| u \|$, $u \in P \cap \partial N$, and $\| Tu \| \geq \| u \|$, $u \in P \cap \partial K$, or

2- $\| Tu \| \geq \| u \|$, $u \in P \cap \partial N$, and $\| Tu \| \leq \| u \|$, $u \in P \cap \partial K$

Then T has a fixed point in $P \cap (\overline{K} \setminus N)$.

3. SOLUTIONS IN THE CONE

In this section , we shall apply Theorem 1 to the eigenvalue problem (1.1) .We note that y(t) is a solution of (1.1) if , and only if ,

$$y(t) = \lambda \int_a^b G(t,s)\, g(s)\, f(y(s))ds \quad , \quad a \leq t \leq b$$

For our construction , let B = C[a, b] , with norm ,

$$\|x\| = \sup_{a \leq t \leq b} |x(t)|$$

Define a cone , P, by

$$P = \left\{ x \in B : x(t) \geq 0 \text{ on } [a,b] \ , \min_{\frac{2a+1}{4} \leq t \leq \frac{2b+1}{4}} x(t) \geq \|x\| \right\}$$

Also , let the number $h \in [a,b]$ be defined by

$$\int_{\frac{2a+1}{4}}^{\frac{2b+1}{4}} G(h,s)\, g(s)\, ds = \max \int_{\frac{2a+1}{4}}^{\frac{2b+1}{4}} G(t,s)\, g(s)\, ds \qquad ....(3.1)$$

THEOREM 2 .

Assume that condition (A),(B),(C) and (D) are satisfied .Then , for each λ satisfying ……... (3.2)

$$\frac{4}{\left(\int_{(2a+1)/4}^{(2b+1)/4} G(h,s)\, g(s)\, ds\right) f_\infty} < \lambda < \frac{1}{\left(\int_a^b G(s,s)\, g(s)\, ds\right) f_0}$$

there exists at least one solution of (1.1) in P .

Proof.: Let λ be given as in (3.2) . Now , let $\varepsilon > 0$ be chosen such that

$$\frac{4}{\left(\int_{(2a+1)/4}^{(2b+1)/4} G(h,s)\, g(s)\, ds\right)(f_\infty - \varepsilon)} < \lambda < \frac{1}{\left(\int_a^b G(s,s)\, g(s)\, ds\right)(f_0 + \varepsilon)} \qquad ....(3.3)$$

Define an integral operator $T : P \to B$ by

$$Ty(t) = \lambda \int_a^b G(t,s)\, g(s)\, f(y(s))ds \ , \quad y \in P \qquad ....(3.4)$$

We seek a fixed point of T in the cone P.

Notice from (2.2) that , for $y \in P$, $Ty(t) \geq 0$ on [a,b] .

Also , for $y \in P$, we have from (2.3) that

$$Ty(t) = \lambda \int_a^b G(t,s)\, g(s)\, f(y(s))ds$$
$$\leq \lambda \int_a^b G(s,s)\, g(s)\, f(y(s))ds$$

so that

$$\|Ty\| \leq \lambda \int_a^b G(s,s)\, g(s)\, f(y(s))ds \qquad ....(3.5)$$

And next , if $y \in P$, we have by (2.4) and (3.5) , $\to p$. In addition , standard arguments show that T is As a consequence , T : p completely continuous.

Now, turning to f_0 , there exist an $H_1 > 0$ such that $f(x) \leq (f_0 + \varepsilon)\, x$, for $0 < x \leq H_1$.

So , choosing $y \in P$ with $\| y \| = H_1$, we have from (2.3) and (3.3)

$$Ty(t) \leq \lambda \int_a^b G(s,s)\, g(s)\, f(y(s))ds$$
$$\leq \lambda \int_a^b G(s,s)\, g(s)\, (f_0 + \varepsilon)\, y(s)ds$$
$$\leq \lambda \int_a^b G(s,s)\, g(s)\, ds\, (f_0 + \varepsilon)\, y(s)\|y\|$$
$$\leq \|y\|$$

Consequently , $\|Ty\| \leq \|y\|$. So , if we set $\Omega_1 = \{x \in B \mid \|x\| < H_1\}$

then

$\|Ty\| \leq \|y\|$, for $y \in P \cap \partial\Omega_1$. ……….(3.6)

Next , considering f_∞ , there exist an $H_2 > 0$ such that $f(x) \geq (f_\infty - \varepsilon)\, x$,for all $x > H_2$.

Let $H_3 = \max\{2H_1 , 4H_2\}$ and let $\Omega_2 = \{ x \in B \mid \|x\| < H_3\}$

if $y \in P$ with $\|y\| = H_3$, then $\min_{\frac{2a+1}{4} \leq t \leq \frac{2b+1}{4}} y(t) \geq \frac{1}{4}\|y\| \geq H_2$,

and we have from (3.1) and (3.3) that

$$Ty(h) = \lambda \int_a^b G(h,s)\, g(s)\, f(y(s))ds$$
$$\geq \lambda \int_{(2a+1)/4}^{(2b+1)/4} G(h,s)\, g(s)\, f(y(s))ds$$
$$\geq \lambda \int_{(2a+1)/4}^{(2b+1)/4} G(h,s)\, g(s)\, (f_\infty - \varepsilon)\, y(s)ds$$
$$\geq \frac{\lambda}{4} \int_{(2a+1)/4}^{(2b+1)/4} G(h,s)\, g(s)\, ds\, (f_\infty - \varepsilon)\, \|y\|$$
$$\geq \|y\|$$

Thus , $\|Ty\| \geq \|y\|$. Hence ,

$\|Ty\| \geq \|y\|$, for $y \in P \cap \partial\Omega_2$ ……….(3.7)

Applying (1) of theorem 1 to (3.6) and (3.7) yields that T has a fixed point y(t) $\in P \cap (\overline{\Omega_2} \setminus \Omega_1)$. As such , y(t) is a desired solution of 1.1 for the given λ . Further , since G (t , s) > 0 , it follows that y(t) > 0 for a < t < b . This completes the proof of the theorem .

THEOREM 3 .

Assume that condition (A),(B),(C) and (D) are satisfied . Then ,for each λ satisfying

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$$\frac{4}{\left(\int_{(2a+1)/4}^{(2b+1)/4} G(h,s)\, g(s)\, ds\right) f_0} < \lambda < \frac{1}{\left(\int_a^b G(s,s)\, g(s)\, ds\right) f_\infty} \qquad \text{……….(3.8)}$$

there exists at least one solution of 1.1 in P .

Proof.: Let λ be given as in (3.8) . Now , let $\varepsilon > 0$ be chosen such that

$$\frac{1}{\left(\int_{(2a+1)/4}^{(2b+1)/4} G(h,s)\, g(s)\, ds\right)(f_0-\varepsilon)} < \lambda < \frac{1}{\left(\int_a^b G(s,s)\, g(s)\, ds\right)(f_\infty+\varepsilon)} \qquad \text{.........(3.9)}$$

Let T be the cone preserving , completely continuous operator that was define by (3.4) . Beginning with f_0 , there exist an $H_4 > 0$ such that $f(x) \geq (f_0 - \varepsilon)\, x$, for $0 < x \leq H_4$.

$y \in P$ with $\| y \| = H_4$, we have from (3.1) and (3.9) so So , for , for

Thus , $\|Ty\| \geq \|y\|$. So , if we let

$\Omega_3 = \{x \in B \mid \|x\| < H_4\}$

$$\begin{aligned} Ty(h) &= \lambda \int_a^b G(h,s)\, g(s)\, f(y(s))ds \\ &\geq \lambda \int_{(2a+1)/4}^{(2b+1)/4} G(h,s)\, g(s)\, f(y(s))ds \qquad \text{then} \\ &\geq \lambda \int_{(2a+1)/4}^{(2b+1)/4} G(h,s)\, g(s)\, (f_0-\varepsilon)\, y(s)ds \\ &\geq \frac{\lambda}{4} \int_{(2a+1)/4}^{(2b+1)/4} G(h,s)\, g(s)\, ds\, (f_0-\varepsilon)\, \|y\| \\ &\geq \|y\| \end{aligned}$$

$\|Ty\| \geq \|y\|$ for $y \in P \cap \partial\Omega_3$ ……. (3.10)

It remains to consider f_∞ , there exist an $H_5 > 0$ such that $f(x) \leq (f_\infty + \varepsilon)\, x$, for all $x > H_5$. There are the two cases , (a) f is bounded , and (b) f is unbounded .

For case (a) , suppose $H_6 > 0$ is such that $f(x) \leq H_6$, for all $0 < x < \infty$.

Let $H_7 = \max \{2H_4 , H_6$ $\lambda\int_a^b G(s,s)\, g(s)\, f(y(s))ds\}$. Then , for $y \in P$ with $\|y\| = H_7$, we have from (2.3) and (3.2)

$$\begin{aligned} Ty(t) &= \lambda \int_a^b G(t,s)\, g(s)\, f(y(s))ds\} \\ &\leq \lambda H_6 \int_a^b G(s,s)\, g(s)\, ds \\ &\leq \|y\| \end{aligned}$$

so that $\|Ty\| \leq \|y\|$. So if

$\Omega_4 = \{x \in B \mid \|x\| < H_7\}$

then

$\|Ty\| \leq \|y\|$, for $y \in P \cap \partial\Omega_4$ ……….(3.11)

For case (b) , let $H_8 > \max \{2H_4 , H_5\}$ be such that $f(x) \leq f(H_8)$, for $0 < x \leq H_8$.

Choosing $y \in P$ with $\|y\| = H_8$ and we have from (2.3),(3.2) and (3.9)

$$\begin{aligned} Ty(t) &= \lambda \int_a^b G(t,s)\, g(s)\, f(y(s))ds\} \\ &\leq \lambda \int_a^b G(s,s)\, g(s)\, f(y(s))ds \\ &\leq \lambda \int_a^b G(s,s)\, g(s)\, f(H_8)\, ds \\ &\leq \lambda \int_a^b G(s,s)\, g(s)\, ds\, (f_\infty+\varepsilon) H_8 \end{aligned}$$

But

$$\lambda \int_a^b G(s,s)\, g(s)\, ds\, (f_\infty+\varepsilon) H_8 = \lambda \int_a^b G(s,s)\, g(s)\, ds\, (f_\infty+\varepsilon)\|y\|$$

Therefore

$$Ty(t) \leq \lambda \int_a^b G(s,s)\, g(s)\, ds\, (f_\infty+\varepsilon)\|y\|$$

and so $\|Ty\| \leq \|y\|$. For this case , if we let

$\Omega_4 = \{x \in B \mid \|x\| < H_8\}$

then $\|Ty\| \leq \|y\|$, for $y \in P \cap \partial\Omega_4$ ……….(3.12)

Thus , in either of the case , an applying of part (2) of theorem 1 to (3.10),(3.11) and (3.12) yields that T has a fixed point $y(t) \in P \cap (\overline{\Omega_4} \setminus \Omega_3)$. As such , y(t) is a desired solution of 1.1 for the given λ . Further , since G (t , s) > 0 , it follows that y(t) > 0 for a < t <b . This completes the proof of the theorem .

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الحلول الموجبة لمسائل القيم الذاتية

صالح محمد حسين

الخلاصة

يدرس هذا البحث المسألة الحدودية التالية :-

$$-y'' = \lambda\, g(t)\, f(y) \quad , \quad a < t < b$$
$$y(a) = y(b) = 0$$

ومعرفة قيم المعلمة (λ) التي تمتلك عندها المسألة الحدودية حل موجب . استخدمت في بحثي هذا تطبيق نظرية النقطة الثابتة في المخروط وتوصلت إلى أن هذه المسألة الحدودية تمتلك على الأقل حل واحد موجب .