The Stability Analysis of the Shimizu–Morioka System with Hopf Bifurcation

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Abstract

In this work, we study new system with a rich structure (the Shimizu-Morioka system), which is exhibiting the Lorenz-like dynamics.

$$\dot{x} = y$$
$$\dot{y} = x(1-z) - \beta y$$
$$\dot{z} = -\alpha z + x^2$$
$$\alpha, \beta > 0 \qquad \ldots(1)$$

Where the dot denotes $\frac{d}{dt}$. the system obtained a Hopf bifurcation (Supercritical and subcritical) for some values of β. For the analysis we use the center manifold and normal form theorem. A computer algebra system using Maple (version 9) was used to derive all the formulas and verifying the results presented in this work [Char, David].

Introduction

The Shimizu-Morioka mode was considered in which complex behavior of trajectories has been discovered [Shimizu] by means of computer simulation. This equations were put forward in [Shimizu] as a model for studying the dynamics of the Lorenz system for large Rayleigh number. A detailed exposition of the plethora of bifurcational phenomena in that system can be found in (Shilinikov 1989, 1991).It was shown in (Sil'nikov 1993, 1991) that there are two types of Lorenz-like attractors in this model. The first is an orientable Lorenz-like attractor and the second is non orientable containing a countable set of saddle periodic orbits with negative multipliers.

Basically there are two ways of investigating periodic solutions of more than two coupled ODE. One is to use the fixed-point theorem to establish the existence, but not the stability of periodic solutions in the large. The other method is to investigate the bifurcation of an isolated equilibrium point, as

some parameter changes, into an equilibrium point surrounded by a small periodic orbit. The obvious parameter here is α. This method can demonstrate the stability, as well as the existence, of a periodic solution in the small. This Hopf bifurcation can go in two ways. A stabile equilibrium point can go to an unstable equilibrium point surrounded by a small stable periodic orbit. Or an unstable equilibrium point can go to a stable equilibrium point surrounded by a small unstable periodic orbit. In this work one use Hopf bifurcation theorem for locating the limit cycles of Shimizu-Morioka system. We have shown that system (1) possesses a stable limit cycle for some value of β and unstable limit cycle for others value of β. Of course it is important to give an analytical proof for this result. A detailed analysis of the Hopf bifurcation, using the methods of local bifurcation theory, especially the center manifold and normal form theorem.

This work is organized as follows: In section 2, we introduce some concepts of background. In the succeeding section, the stability of the equilibrium points of this model was analyzed. The Hopf bifurcation for the Shimizu-Morioka system was studied in section 4. In section 5 we study the spatial cases for occurring the limit cycle. The conclusions are finally made.

Some Concepts of background

The term bifurcation was originally used by poincare to describe the "splitting"equilibrium (equilibrium) solutions in a family of differential equations.If

$$\dot{x} = f_{\mu}(x); \qquad x \in R^{n}, \mu \in R^{k} \qquad \ldots (2)$$

Is a system of differential equations depending on the k-dimensional parameter μ then the equilibrium solutions of (2) are given by the solutions of the equation $f_{\mu}(x) = 0$. As μ varies, the implicit function theorem implies that these equilibria are described by smooth functions of μ away from those points at which the Jacobian derivative of $f_{\mu}(x)$ with respect to x, $D_x f_x$, has a zero eigenvalue, the grave of each of these functions is a branch of equilibria of (2). At an equilibrium (x_0, μ_0) where $D_x f_x$, has a zero eigenvalue, several branches of equilibria may com together, and one says that (x_0, μ_0) is a point of bifurcation.

We have the following important theorems

Theorem (1): (Ronald, 1998)

The equilibrium point E is asymptotically stable if all the eigenvalues of $Df(E)$ have negative real parts.

Theorem (2): (Ronald, 1998)

The equilibrium point E is unstable if at least one of the eigenvalues of $Df(E)$ has positive real part.

Theorem (3) :(Rama Mohana, 1980)

A necessary and sufficient condition for the negativity of the real parts of all the roots of the polynomial

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ...+ a_{n-1}\lambda + a_n$$

With real coefficients is the positivity of all the principal diagonals of the minors of the Hurwitz matrix

$$H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & 0 & ..... & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & 0 & 0 & ..... & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & 0 & ..... & 0 \\ . & . & . & . & . & . & . & & . \\ . & . & . & . & . & . & . & & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & ..... & a_n \end{bmatrix}.$$

It should be noted that the principal diagonal of the Hurwitz matrix H_n exhibits the coefficients of the polynomial $p(\lambda)$ in the order of their numbers from a_1 *to* a_n, denote the principal diagonal minors of the Hurwitz matrix by

$$D_1 = |a_1|, D_2 = \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix}, \ ....., D_n = \det(H_n).$$

If n =3, then the Hurwitz conditions reduce to

$$a_1 > 0, a_2 > 0, a_3 > 0 \text{ , and } \quad a_1a_2 - a_3 > 0$$

The center manifold theorem reduces the original system to a center manifold which may have smaller dimensions than the original system.

Theorem (4) :(Center Manifold Theorem for Flows) (Guckenheimer & Holmes, 2002)

Let f be a C^r vector field on R^n vanishing at the origin ($f(0) = 0$) and let $A = Df(0)$. Divide the spectrum of A in to three parts, $\sigma_s, \sigma_c, \sigma_u$ with

$$\operatorname{Re}\lambda \begin{cases} <0 & if\ \lambda \in \sigma_S, \\ =0 & if\ \lambda \in \sigma_C, \\ >0 & if\ \lambda \in \sigma_u . \end{cases}$$

Let the (generalized) eignspaces of $\sigma_s, \sigma_c, and\ \sigma_u$ be $E^s, E^c, and\ E^u$, respectively. Then there exist C^r stable and unstable invariant manifolds $W^s\ and\ W^u$ tangent to $E^s and\ E^u$ at 0 and a C^{r-1} center manifold W^c tangent to E^c at 0. The manifolds $W^s, W^u, and\ W^c$ are invariant for the flow of f .The stable and unstable manifolds are unique, but W^c need not be.

The center manifold theorem implies that the bifurcating system is locally topologically equivalent to

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x})$$
$$\dot{\tilde{y}} = -\tilde{y} \quad ; (\tilde{x}, \tilde{y}, \tilde{z}) \in (w^c, w^s, w^u). \qquad ...(3)$$
$$\dot{\tilde{z}} = \tilde{z}$$

At the bifurcating point. We now tackle the problem of computing the "reduced" vector field $\tilde{f}$. For simplicity, and because it is the most interesting case physically, we assume that the unstable manifold is empty and that the linear part of the bifurcating system is in block diagonal form:

$$\dot{x} = Bx + f(x, y)$$
$$\dot{y} = Cy + g(x, y)\ ; \qquad (x, y) \in R^n \times R^m \qquad ...(4)$$

Where B and C are $n \times n$ and $m \times m$ matrices whose eigenvalues have, respectively, zero real pars and negative real parts, and *f* and g vanish, along with there first partial derivatives, at the origin. Since the center manifold is tangent to E^c (the $y = 0$ space) we can represent it as a (local) graph

$$w^c = \{(x, y) / y = h(x)\};\ h(0) = Dh(0) = 0 \qquad ... (5)$$

Where $h : U \to R^m$ is defined on some neighborhood $U \subset R^n$ of the origin, we now consider the projection of the vector field on $y = h(x)\ onto\ E^c$:

$$\dot{x} = Bx + f(x, h(x)). \qquad ...(6)$$

Since h(x) is tangent to $y = 0$, the solutions of equation (6) provided a good approximation of the flow of $\dot{\tilde{x}} = \tilde{f}(\tilde{x})$ restricted to w^c. In fact we have

Theorem (5): (Carr, 1981)

If the origin $x = 0$ of equation (6) is locally asymptotically stable (respectively unstable) then the origin of equation (4) is also locally asymptotically stable (respectively unstable).

Now, one can show how h(x) can be calculated or at least approximated. Substituting $y = h(x)$ in the second component of equation (4) and using the chain rule, the followings are obtained:

$$\dot{y} = Dh(x)\dot{x} = Dh(x)[Bx + f(x,h(x))] = Ch(x) + g(x,h(x))$$

Or

$$N(h(x)) = Dh(x)[Bx + f(x,h(x))] - Ch(x) - g(x,h(x)) = 0. \quad \ldots(7)$$

With boundary conditions $h(0) = Dh(0) = 0$. This (partial)differential equation for $h(x)$ can not of course, be solved exactly in most cases, but its solution can be approximated arbitrarily close as a Taylor series at x = 0, in most cases of interest, an approximation of degree two or three suffices.

Theorem (6): (Guckenheimer & Holmes, 2002)

If a function $\phi(x)$ with $\phi(0) = D\phi(0) = 0$ can be found such that $N(\phi(x)) = O(|x|^{\rho})$ for some $\rho > 1$ as $|x| \to 0$ then it follows that $h(x) = \phi(x) + O(|x|^{\rho})$ *as* $|x| \to 0$.

Thus according to the above theorem one can approximate the center manifold to any degree of approximation by solving the equation (7) with the same degree of approximation.

One of the basic tools in the study of dynamic behavior of system governed by non linear differential equations near a bifurcation point is the theory of normal forms. The fundamental idea of the method of normal forms is to employ successive coordinate transformations to systematically construct a form of the original differential equations to be as a simple as possible. The normal form theory is usually applied together with the center manifold theorem [Yu].

Guckenheimer and Holmes have explicitly shown that on the basis of the normal form theorem, one finds a non linear coordinate transformation which transforms every system with the structure

$$\dot{x} = -\omega y + O(|x|,|y|)$$
$$\dot{y} = \omega x + O(|x|,|y|) \quad \ldots (8)$$

In to the system

$$\dot{u} = -\omega v + (au - bv)(u^2 + v^2) + O(4)$$
$$\dot{v} = \omega u + (av + bu)(u^2 + v^2) + O(4) \quad \ldots (9)$$

This is expressed in polar coordinates as:

$$\dot{r} = ar^3$$
$$\dot{\theta} = \omega + br^2 \quad \ldots (10)$$

It can be seen that the sign of "a" determine the stability of the equilibrium point at the Hopf bifurcation point.

Guckenheimer and Holmes carried out the procedure for calculating the stability coefficient "a" and gave the formula:

$$a = \frac{1}{16}[f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} + (\frac{1}{\omega})(f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy})] \quad \ldots (11)$$

Where f_{xy} denotes $(\frac{\partial^2 f}{\partial x \partial y})(0,0)$, etc. and f, g are the functions containing the non linear terms of equation (8).

Thomas [1996] determines the formula for calculating the coefficient "b":

$$b = \frac{1}{16}[g_{xxx} + g_{xyy} - f_{xxy} - f_{yyy} + \frac{1}{3\omega}[5(f_{xx}g_{xy} + f_{xy}g_{yy} - f_{xx}f_{yy} - f_{yy}^2 - g_{xx}g_{yy} - g_{xx}^2) - 2(f_{xx}^2 + f_{xy}^2 + g_{xy}^2 + g_{yy}^2) + f_{yy}g_{xy} + f_{xy}g_{xx}]] \quad \ldots (12)$$

Stability analysis of the equilibrium points

The Shimizu-Morioka system will be investigated by the following three non linear differential equations:

$$\dot{x} = y$$
$$\dot{y} = x(1-z) - \beta y$$
$$\dot{z} = -\alpha z + x^2$$

Where α, β are the positive real parameters of the system.

Proposition 1:

If $\alpha \geq 0$ the system (1) has three isolated equilibrium points O(0,0,0), $A_1(\sqrt{\alpha},0,1)$, and $A_2(-\sqrt{\alpha},0,1)$ and fore $\alpha < 0$ it has only one isolated equilibrium point O(0,0,0).

Proof:

Solving the system

$$y = 0$$
$$x(1-z) - \beta y = 0 \quad \ldots (13)$$
$$-\alpha z + x^2 = 0$$

The system (13) leads to

$$x(1-z) = 0$$
$$-\alpha z + x^2 = 0$$

Which yields $x = 0, y = 0, z = 0$ x=0, where $\alpha < 0$ and fore $\alpha \geq 0$, $x = \mp\sqrt{\alpha}$, $y = 0$, $z = 1$.

Therefore, the system (1) has only one equilibrium point $O(0,0,0)$ for $\alpha < 0$ but for $\alpha \geq 0$ has three isolated equilibrium points $O(0,0,0),\ A_1(\sqrt{\alpha},0,1), A_2(-\sqrt{\alpha},0,1)$.

Proposition 2:

The equilibrium point $O(0,0,0)$ is unstable for all $\alpha \in R$.

Proof:

The Jacobian matrix of system (1) at $O(0,0,0)$ is:

$$J_0 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -\beta & 0 \\ 0 & 0 & -\alpha \end{bmatrix}$$

The characteristic polynomial of J_0 is $\lambda^3 + (\alpha+\beta)\lambda^2 + (\alpha\beta - 1)\lambda - \alpha = 0$.

Then the eigenvalues of J_0 are: $\lambda_1 = -\alpha,\ \lambda_{2,3} = \dfrac{-\beta \mp \sqrt{\beta^2+4}}{2}$. It is clear that $\lambda_2 = \dfrac{-\beta + \sqrt{\beta^2+4}}{2}$ is positive for every $\alpha \in R$ then $O(0,0,0)$ is unstable by theorem (2) (see the figure).

Next, consider the stability of system (1) at $A_1(\sqrt{\alpha},0,1), A_2(-\sqrt{\alpha},0,1)$. for $\alpha \geq 0$.

Because the system is invariant under the transformation $(x, y, z) \to (-x,-y, z)$, one only needs to consider the stability of system (1) at $A_1(\sqrt{\alpha},0,1)$.

Under the linear transformations $(x, y, z) \to (x_1, y_1, z_1)$

$$x = x_1 + \sqrt{\alpha}$$
$$y = y_1$$
$$z = z_1 + 1$$

The system (1) becomes:

$$\dot{x}_1 = y_1$$
$$\dot{y}_1 = -(x_1 + \sqrt{\alpha})z_1 - \beta y_1 \qquad \ldots (14)$$
$$\dot{z}_1 = -\alpha z_1 + 2\sqrt{\alpha}x_1 + x^2$$

Hence, one has to consider the stability of system (14) at $O(0,0,0)$.

The Jacobian matrix for system (14) at the point $O(0,0,0)$ is:

$$J(A_1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\beta & -\sqrt{\alpha} \\ 2\sqrt{\alpha} & 0 & -\alpha \end{bmatrix}$$

With characteristic equation:

$$\lambda^3 + (\alpha + \beta)\lambda^2 + \alpha\beta\lambda + 2\alpha = 0 \qquad \ldots(15)$$

Then, for Routh-Hurwitz condition [Rama], this equation has all roots with negative real parts if and only if

$$\alpha > 0\ ,\ \beta > 0$$

$$\beta^2 + \alpha\beta - 2 > 0 \qquad \ldots(16)$$

Proposition 3:

The equilibrium point $A_1(\sqrt{\alpha},0,1)$ is asymptotically stable if and only if $\alpha > \alpha_o = \dfrac{2-\beta^2}{\beta}$ *where* $\beta \in (0,\sqrt{2})$.

Proof:

Suppose $A_1(\sqrt{\alpha},0,1)$ is asymptotically stable, then equation (15) has no roots with positive real parts. Since $\det(J(A_1)) = -2\alpha$ *for* $\alpha > 0$, then equation (15) also has no zero roots for $\alpha > 0$.

Comparing the coefficients of equation $\lambda^3 - T\lambda^2 - K\lambda - D = 0$ and equation (15) to obtain

$$T = -(\alpha + \beta)$$

$$k = -\alpha\beta$$

$$D = -2\alpha$$

and $$TK + D = \alpha(\beta^2 + \alpha\beta - 2) \qquad \ldots(17)$$

Since $TK + D = 0$ if and only if $\alpha = 0$ *or* $\alpha = \dfrac{2-\beta^2}{\beta}, \beta \neq 0.$ then equation (15) has no pure imaginary roots for $\alpha \neq \alpha_0 = \dfrac{2-\beta^2}{\beta}, \beta \neq 0.$. From above we can obtain that, equation (15) has no roots with zero real parts when $\alpha \neq \alpha_0 = \dfrac{2-\beta^2}{\beta}, \beta \neq 0.$. Then equation (15) has all roots with negative real parts for $\alpha \neq \alpha_0 = \dfrac{2-\beta^2}{\beta}, \beta \neq 0.$, therefore the condition (16) holds, it follows that $\alpha > \dfrac{2-\beta^2}{\beta}$ *where* $\beta \in (0,\sqrt{2})$.

Conversely, suppose $\alpha > \alpha_o = \dfrac{2-\beta^2}{\beta}$ *where* $\beta \in (0,\sqrt{2})$, it is easy to see that the condition (16) holds, this means that the equation (15) has all roots with

negative real parts, and the theorem (1) guarantees that the equilibrium point $A_1(\sqrt{\alpha},0,1)$ is a asymptotically stable (see the figure).

Proposition 4:

The equilibrium point $A_1(\sqrt{\alpha},0,1)$ is unstable if and only if $\alpha < \alpha_o = \frac{2-\beta^2}{\beta}$ where $\beta \in (0,\sqrt{2})$.

Proof:

Let the equilibrium point $A_1(\sqrt{\alpha},0,1)$ be unstable, then not all roots of equation (15) have negative real parts, this means that the equation (15) dose not satisfy the condition (16).

Now if $\beta^2+\alpha\beta-2<0 \Rightarrow \alpha < \frac{2-\beta^2}{\beta}$, $\beta \in (0,\sqrt{2})$. Then its shown that if $A_1(\sqrt{\alpha},0,1)$ be unstable then $\alpha < \alpha_o = \frac{2-\beta^2}{\beta}$ where $\beta \in (0,\sqrt{2})$.

Conversely, suppose $\alpha < \alpha_o = \frac{2-\beta^2}{\beta}$ where $\beta \in (0,\sqrt{2})$, we get $\beta^2+\alpha\beta-2<0$, this is equivalent to the equation (15) which dose not satisfy the condition (16), and then there exists at least eigenvalue with non negative real part.

Since

$$\det(J(A_1)) = -2\alpha \neq 0 \text{ and } TK+D = \alpha(\beta^2+\alpha\beta-2) \neq 0 \text{ for } \alpha < \alpha_o = \frac{2-\beta^2}{\beta}, \beta \in (0,\sqrt{2}),$$

then the equation (15) has no eigenvalues with zero real parts.

From above one can obtain, equation (15) has at least one eigenvalue with positive real part, theorem (2) guarantees that the equilibrium point $A_1(\sqrt{\alpha},0,1)$ is unstable (see the figure).

Hopf bifurcation analysis

In the following we will prove that the system (1) displays a Hopf bifurcation at the equilibrium point $A_1(\sqrt{\alpha},0,1)$.

Proposition 5:

The equation (15) has purely imaginary roots if and only if $\alpha = \alpha_0 = \frac{2-\beta^2}{\beta}$, $\beta \in (0,\sqrt{2})$. In this case the solutions of equation (15) are

$$\lambda_1 = \frac{-2}{\beta},\ \lambda_{2,3} = \mp\omega i \text{ where } \omega = \sqrt{2-\beta^2}. \qquad \text{... (18)}$$

Proof:

Since equation (15) has pure imaginary roots if and only if $TK + D = 0$, Thomas [1995]. From equation (17) yields:

$TK + D = 0$ *if and only if* $\alpha = 0$ *or* $\alpha = \alpha_0 = \frac{2-\beta^2}{\beta}$, $\beta \in (0, \sqrt{2})$ (But if $\alpha = 0$ then equation (15) becomes $\lambda^3 + \beta\lambda^2 = 0$ and it has no any complex eigenvalues).

To find eigenvalues at $\alpha = \alpha_0 = \frac{2-\beta^2}{\beta}$, $\beta \in (0, \sqrt{2})$, let $\lambda_{2,3} = \mp i\omega$ be complex solutions and λ_1 the real solution of equation (15) then,

$\lambda_1 + \lambda_2 + \lambda_3 = Tr(J(A_1)) = \frac{-2}{\beta} \Rightarrow \lambda_1 = \frac{-2}{\beta}$

Since $\lambda_1\lambda_2\lambda_3 = Det(J(A_1)) \Rightarrow \frac{-2}{\beta}\omega^2 = -2\,(\frac{2-\beta^2}{\beta}) \quad \Rightarrow \omega = \sqrt{2-\beta^2}$.

So the first condition for a Hopf bifurcation is fulfilled. Nevertheless, for applying the hopf bifurcation theorem a second condition must be fulfilled, i.e.

$\frac{d}{d\alpha}(\mathrm{Re}(\lambda_{2,3}(\alpha)))\Big|_{\alpha=\alpha_0} = d \neq 0,$

Where $\mathrm{Re}(\lambda_{2,3}(\alpha))$ denotes the real part of λ which is a smooth function of α.

Proposition 6:

The derivative of the real part of complex solution of equation (15) with respect to α at $\alpha = \alpha_0 = \frac{2-\beta^2}{\beta}$, $\beta \in (0, \sqrt{2})$ is non zero and equal to $\frac{2-\beta^2}{2(\beta^2 - 4\beta - 2)}$. This means that $\frac{d}{d\alpha}(\mathrm{Re}(\lambda_{2,3}(\alpha)))\Big|_{\alpha=\alpha_0} = d = \frac{2-\beta^2}{2(\beta^2 - 4\beta - 2)} \neq 0.$

Proof:

We calculate d without solving (15) explicitly. Let $\lambda_2 = u_1 + iv_1$ $\lambda_3 = \overline{\lambda_2} = u_1 - iv_1$ and λ_1 be eigenvalues. As $J(A_1)$ has two non-zero pure imaginary eigenvalues when $\alpha = \alpha_0$, it follows that for α near α_0 two of the eigenvalues will be complex conjugates.

$\lambda_2, \overline{\lambda_2}$ *and* λ_1 Satisfy

$x^3 - (2u_1 + \lambda_1)x^2 + (|\lambda_2|^2 + 2u_1\lambda_1)x - |\lambda_2|^2\lambda_1 = 0$ [Stephen].

Equating coefficients with equation (15) result is

$$-(\alpha+\beta)=2u_1+\lambda_1$$

$$-2\alpha=|\lambda_2|^2\lambda_1$$

$$\textit{and}\qquad \alpha\beta=|\lambda_2|^2+2u_1\lambda_1$$

$$\textit{then}\ \frac{2\alpha}{\alpha+\beta+2u_1}-2u_1(\alpha+\beta+2u_1)=\alpha\beta$$

Implicitly differentiating $u_1=u_1(\alpha)$ the following is obtained:

$$\frac{2(\alpha+\beta+2u_1)-2\alpha(1+2\dot{u}_1)}{(\alpha+\beta+2u_1)^2}-2(\alpha+\beta+4u_1)\dot{u}_1=\beta$$

$$\textit{then}\ \dot{u}_1=\frac{(\alpha+\beta+2u_1)(\beta(\alpha+\beta+2u_1)-2)+2\alpha)}{-(4\alpha+2(\alpha+\beta+4u_1)(\alpha+\beta+2u_1)^2)}$$

At $\alpha=\alpha_0$, where $\mathrm{Re}(\lambda_2)=u_1=0$, after some calculations we obtain:

$$\dot{u}_1(\alpha_0)=d=\frac{2-\beta^2}{2(\beta^2-4\beta-2)}. \qquad ...(19)$$

Since $\beta\in(0,\sqrt{2})$, we have

$$\dot{u}_1(\alpha_0)=d<0,\text{then}\ \frac{d}{d\alpha}(\mathrm{Re}(\lambda_{2,3}(\alpha)))\Big|_{\alpha=\alpha_0}=d=\frac{2-\beta^2}{2(\beta^2-4\beta-2)}<0.$$

Thus, also the second condition for a Hopf bifurcation is fulfilled and system (14) displays a Hopf bifurcation at the point A_1.

We now analyses the Hopf bifurcation of system (1) in detail. At first we give an expression for the flow in the center manifold W^c at the bifurcation point which is two-dimensional (W^c has some dimension as the eignspase of the conjugate complex eigenvalue with zero real part).

Using the eigenvectors as the basis for a new coordinate system equation, with

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=M\begin{bmatrix}u\\v\\w\end{bmatrix}\qquad \text{Where}\quad M=\begin{bmatrix}\beta & \frac{i}{\omega} & -\frac{i}{\omega}\\ -2 & -1 & -1\\ -2\sqrt{\alpha_0} & \frac{\beta+i\omega}{\alpha_0} & \frac{\beta-i\omega}{\alpha_0}\end{bmatrix}$$

System (14) is transformed into the diagonals system:

$$\begin{bmatrix}\dot{u}\\ \dot{v}\\ \dot{w}\end{bmatrix}=\begin{bmatrix}-2/\beta & 0 & 0\\ 0 & -i\omega & 0\\ 0 & 0 & i\omega\end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix}+\begin{bmatrix}h_1u^2+h_2v^2+\bar{h}_2w^2+h_3uv+\bar{h}_3uw+h_4vw\\ k_1u^2+k_2v^2+k_3w^2+k_4uv+k_5uw+k_6vw\\ \bar{k}_1u^2+\bar{k}_3v^2+\bar{k}_2w^2+\bar{k}_5uv+\bar{k}_4uw+\bar{k}_6vw\end{bmatrix} \quad \text{... (20)}$$

With

$$h_1=-\frac{3\beta^2\sqrt{\alpha_0}}{A},\ h_2=\frac{\alpha-\omega^2\beta-i\omega\beta^2}{A\omega^2\sqrt{\alpha_0}},\ h_3=\frac{\beta^3\omega+i(\beta^2\omega^2-4\beta\alpha_0)}{A\omega\sqrt{\alpha_0}},\ h_4=\frac{2(\omega^2\beta-\alpha_0)}{A\omega^2\sqrt{\alpha_0}},$$

$$k_1=\frac{2\beta\sqrt{\alpha_0}d_2+\beta^2d_3}{A},\ k_2=\frac{-\sqrt{\alpha_0}d_3+\omega^2d_2+i\omega\beta d_2}{A\omega^2\sqrt{\alpha_0}},\ k_3=\frac{-\sqrt{\alpha_0}d_3+\omega^2d_2-i\omega\beta d_2}{A\omega^2\sqrt{\alpha_0}},$$

$$k_4=\frac{-\beta^2\omega d_2-i(2\alpha_0d_2-\beta\omega^2d_2+2\beta\sqrt{\alpha_0}d_3)}{A\omega\sqrt{\alpha_0}},\ k_5=\frac{-\beta^2\omega d_2+i(2\alpha_0d_2-\beta\omega^2d_2+2\beta\sqrt{\alpha_0}d_3)}{A\omega\sqrt{\alpha_0}},$$

$$k_6=\frac{2(\sqrt{\alpha_o}d_3-\omega^2d_2)}{A\omega^2\sqrt{\alpha_o}},\ A=\beta\omega^2+2\beta+2\alpha,$$

$$d_1=-\omega^2+i\omega(\beta+\alpha_o),\ d_2=-\frac{1}{2}(\beta\omega^2+2\alpha_o)+\frac{1}{2}i\beta^2\omega,\ d_3=\sqrt{\alpha_o}+\frac{1}{2}i\beta\omega\sqrt{\alpha_o}.$$

Where an over bar denotes complex conjugation.

According to the center manifold theorem the center manifold W^c is tangent to eigenspace E^c = span {v, w}. Therefore W^c an be approximated for the two variables v, w by equation

$$u=h(v,w)=a_1v^2+a_2vw+a_3w^2+O(3) \quad \text{...(21)}$$

Where O (3) denotes terms of order v^3, v^2w, vw^2 *and* w^3. With

$$\dot{u}=\frac{\partial h}{\partial v}\dot{v}+\frac{\partial h}{\partial w}\dot{w} \quad \text{... (22)}$$

It follows together with system (21) and equation (22), after comparison of the coefficients for v^2, vw and w^2 obtains:

$$a_1=\frac{h_2}{2/\beta-2i\omega},\ a_2=\frac{h_4}{2/\beta},\quad \textit{and}\quad a_3=\frac{\bar{h}_2}{2/\beta+2i\omega}.$$

Substitute the value of a_1, a_2 and a_3 in equation (21) to get:

$$u=h(v,w)=\frac{h_2}{2/\beta-2i\omega}v^2+\frac{h_4}{2/\beta}vw+\frac{\bar{h}_2}{2/\beta+2i\omega}w^2+O(3) \quad \text{...(23)}$$

After inserting $u=h(v,w)$ into the equation for v, w in equation (20), one obtains an approximated expression for the flow in the center manifold:

$$\begin{bmatrix}\dot{v}\\ \dot{w}\end{bmatrix}=\begin{bmatrix}-i\omega v\\ i\omega w\end{bmatrix}+\begin{bmatrix}k_2v^2+k_3w^2+k_6vw+(k_4v+k_5w)u\\ \bar{k}_3v^2+\bar{k}_2w^2+\bar{k}_6vw+(\bar{k}_5v+\bar{k}_4w)u\end{bmatrix}. \qquad ...(24)$$

In a second step we simplify the expression for the flow in the center manifold by removing all of the redundant non linear terms. The simplest expression is the normal form which still contains all information about the qualitative behavior of the system at the bifurcation point. With a further linear coordinate transformation system (23) can be rewritten into a form which only contains real numbers giving the so-called standard form.

With $$\begin{bmatrix}v\\ w\end{bmatrix}=T\begin{bmatrix}\zeta\\ \chi\end{bmatrix};\ T=\begin{bmatrix}1 & -i\\ 1 & i\end{bmatrix}$$

It follows

$$\dot{\zeta}=-\omega\chi+\frac{1}{2}(-k_3-\bar{k}_3-k_2-\bar{k}_2+k_6+\bar{k}_6)\,\chi^2+\frac{1}{2}(k_3+\bar{k}_3+k_2+\bar{k}_2+k_6+\bar{k}_6)\zeta^2$$
$$+i(k_3-\bar{k}_3-k_2+\bar{k}_2)\,\zeta\chi$$
$$+\frac{1}{2}[(k_4+\bar{k}_5)(\frac{h_2}{2/\beta-2i\omega}+\frac{h_4}{2/\beta}+\frac{\bar{h}_2}{2/\beta+2i\omega})+(k_5+\bar{k}_4)(\frac{h_2}{2/\beta-2i\omega}+\frac{h_4}{2/\beta}+\frac{\bar{h}_2}{2/\beta+2i\omega})]\,\zeta^3$$
$$+\frac{i}{2}[(k_4+\bar{k}_5)(\frac{-3h_2}{2/\beta-2i\omega}-\frac{h_4}{2/\beta}+\frac{\bar{h}_2}{2/\beta+2i\omega})+(k_5+\bar{k}_4)(-\frac{h_2}{2/\beta-2i\omega}+\frac{h_4}{2/\beta}+\frac{3\bar{h}_2}{2/\beta+2i\omega})]\,\zeta^2\chi$$
$$+\frac{1}{2}[(k_4+\bar{k}_5)(\frac{-3h_2}{2/\beta-2i\omega}+\frac{h_4}{2/\beta}+\frac{\bar{h}_2}{2/\beta+2i\omega})+(k_5+\bar{k}_4)(\frac{h_2}{2/\beta-2i\omega}+\frac{h_4}{2/\beta}-\frac{3\bar{h}_2}{2/\beta+2i\omega})]\,\zeta\chi^2$$
$$+\frac{i}{2}[(k_4+\bar{k}_5)(\frac{h_2}{2/\beta-2i\omega}-\frac{h_4}{2/\beta}+\frac{\bar{h}_2}{2/\beta+2i\omega})+(k_5+\bar{k}_4)(-\frac{h_2}{2/\beta-2i\omega}+\frac{h_4}{2/\beta}-\frac{\bar{h}_2}{2/\beta+2i\omega})]\,\chi^3$$

$$\dot{\chi}=\omega\zeta-\frac{i}{2}(k_3-\bar{k}_3+k_2-\bar{k}_2-k_6+\bar{k}_6)\chi^2-\frac{i}{2}(-k_3+\bar{k}_3-k_2+\bar{k}_2-k_6+\bar{k}_6)\zeta^2$$
$$+(-k_3-\bar{k}_3+k_2+\bar{k}_2)\,\zeta\chi$$
$$+\frac{i}{2}[(k_4-\bar{k}_5)(\frac{h_2}{2/\beta-2i\omega}+\frac{h_4}{2/\beta}+\frac{\bar{h}_2}{2/\beta+2i\omega})+(k_5-\bar{k}_4)(\frac{h_2}{2/\beta-2i\omega}+\frac{h_4}{2/\beta}+\frac{\bar{h}_2}{2/\beta+2i\omega})]\,\zeta^3$$
$$-\frac{1}{2}[(k_4-\bar{k}_5)(-\frac{3h_2}{2/\beta-2i\omega}-\frac{h_4}{2/\beta}+\frac{\bar{h}_2}{2/\beta+2i\omega})+(k_5-\bar{k}_4)(-\frac{h_2}{2/\beta-2i\omega}+\frac{h_4}{2/\beta}+\frac{3\bar{h}_2}{2/\beta+2i\omega})]\,\zeta^2\chi$$
$$+\frac{i}{2}[(k_4-\bar{k}_5)(-\frac{3h_2}{2/\beta-2i\omega}+\frac{h_4}{2/\beta}+\frac{\bar{h}_2}{2/\beta+2i\omega})+(k_5-\bar{k}_4)(\frac{h_2}{2/\beta-2i\omega}+\frac{h_4}{2/\beta}-\frac{3\bar{h}_2}{2/\beta+2i\omega})]\,\zeta\chi^2$$
$$-\frac{1}{2}[(k_4-\bar{k}_5)(\frac{h_2}{2/\beta-2i\omega}-\frac{h_4}{2/\beta}+\frac{\bar{h}_2}{2/\beta+2i\omega})+(k_5-\bar{k}_4)(-\frac{h_2}{2/\beta-2i\omega}+\frac{h_4}{2/\beta}-\frac{\bar{h}_2}{2/\beta+2i\omega})]\chi^3 \qquad ...(25)$$

Applying the equations (11), (12) to expression (25) which has the same as structure equation (8), one obtains:

$$a=\frac{\beta^2(3\beta^4-5\beta^2-1)}{(\beta^4-2\beta^2-1)(\beta^4-2\beta^2-4)},\quad b=\frac{\beta(3\beta^8+8\beta^6-76\beta^4+82\beta^2+36)}{3(2-\beta^2)^{3/2}.(\beta^4-2\beta^2-1).(\beta^4-2\beta^2-4)} \qquad ...(26)$$

Remark:

Maple program tell us $b<0\ \forall\beta\in(0,\sqrt{2})$. Since $\omega>0$ *and* $b<0$ it can be seen from (10) that there is an equilibrium value $r=r^*$ where the direction of rotation changes.

Special cases

In this section we consider the different values for the constant β. By changing instantaneously the dissociation constant β the response of the system is different.

For $\beta=1$, we obtain: $\alpha_0=1$, the coefficients which describe the limit cycle:

$a=-0.3<0$,

$b=-1.766666667<0, d=-0.1<0, \omega=1>0.$

Since $a<0$, then the equilibrium point A_1 is stable and system (1) has stable limit cycle with periodic $P=2\pi$.

Since $d<0$ *and* $-\frac{a}{d}<0$, then the Hopf bifurcation is supercritical and the bifurcating periodic solutions exist for $\alpha<1$.

For $\beta=1.37$ we obtain: $\alpha_0=0.092044$, the coefficients which describe the limit cycle:

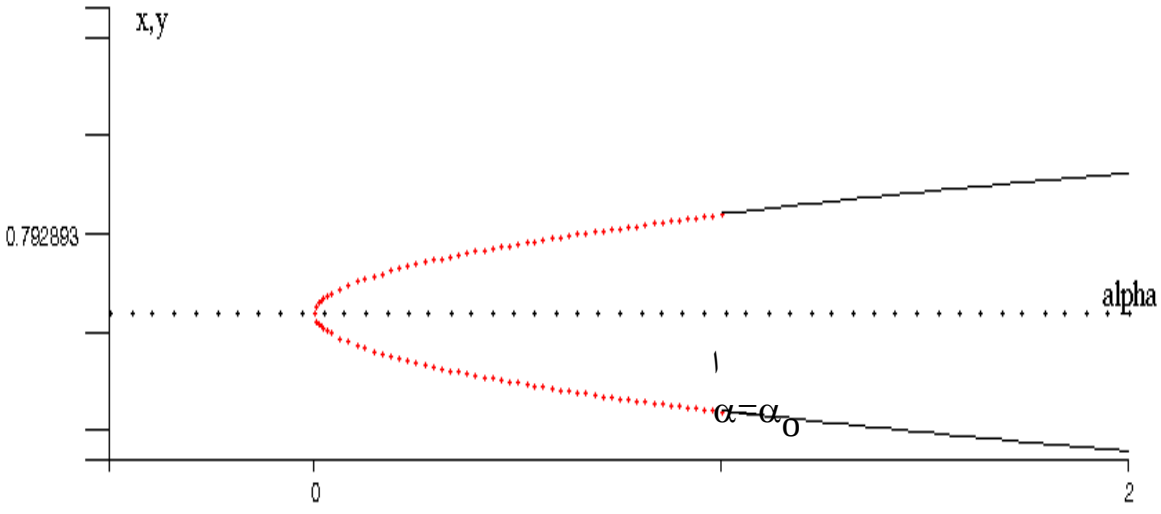
$a=0.06621736154>0, b=-24.97018622<0, d=-0.01098499045<0, \omega=0.3508560959>0.$

Since $\alpha>0$, then the equilibrium point A_1 is unstable and system (1) has unstable limit cycle with periodic $P=5.700342743\pi$.

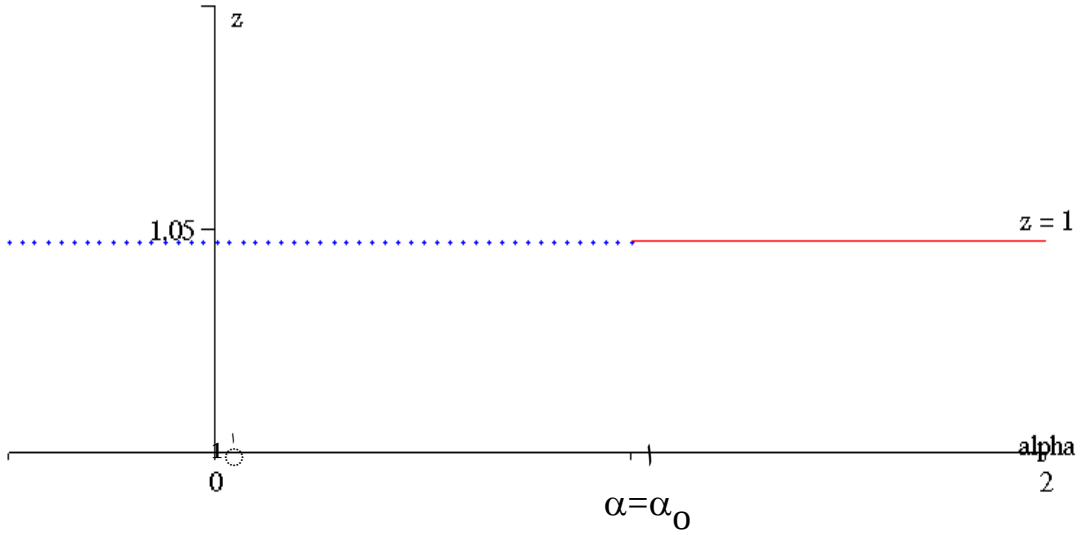
Since $d<0$ *and* $-\frac{a}{d}>0$, then the Hopf bifurcation is subritical and the bifurcating periodic solutions exist for $\alpha>0.092044$.

Conclusions

In this work we analyse the Shimizu-Morioka system. We discuss the local stability and the existence of the Hopf bifurcation; we study the direction and stability of the bifurcating periodic solutions. It was shown that a limit cycle exists and it is characterized by the coefficients from (18), (19) and (25), for different values of the equilibrium constant α which depends on the value of β, in section 4 we obtain stable or unstable limit cycle, via a Hopf bifurcation.



(a)



(b)

Figure: Equilibrium bifurcation diagrams of Shimizu-Morioka system dependence on α of: (a) x, y; (b) z.

The solid curves depict stable behaviour and the doted curves depict unstable behavior.

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تحليل إستقرارية نظام Shimizu-Morioka بتشعيب Hopf

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الخلاصة

في هذا البحث تم دراسة نظام جديد مع تركيب غنى هو نظام (Shimizu-Morioka) الذي يعرض شبه لورنز الدينميكية.

$$\dot{x} = y$$
$$\dot{y} = x(1-z) - \beta y$$
$$\dot{z} = -\alpha z + x^2$$
$$\alpha, \beta > 0 \qquad \ldots(1)$$

من النظام حصلنا على تفرع هوبف (Supercritical and subcritical) لبعض القيم β . استخدمت طريقة نظرية التفرع المحليّة في التحليل و بالأخص نظرية (center manifold) و(normal form). واستخدم نظام الجبري للحاسبة (Maple) لاشتقاق جمع الصيغ ولإثبات النتائج المقدمة في هذه البحث.