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Solving the Boundary Value Problems of Ordinary Differential Equation 4^{th} order using RK4 and RK-Butcher Techniques

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Abstract

The two-point boundary value problems for the 4th order ordinary differential equations with a positive coefficient multiplying at least one of derivative terms are solved with two numerical methods. These numerical methods are the (Rung- Kutta of 4th Order) and (Rung–Kutta Butcher of 6th Order). The 4th order ordinary differential Equations problem had been transformed to pair of second Order differential equations, which were solved together by the suggested methods. An initial value of the dependent variable had been predicted and corrected to some error. The two studied methods were tested on a physical model problem from the literature for comparing results. Solutions were presented in Tables and figures. good agreements were appeared in applying the studied methods.

Keywords: Boundary Value Problems, Ordinary Differential Equation, RK4, RK-Butcher

حل مسائل القيمة الحدية ذوات المعادلة التفاضلية من المرتبة الرابعة باستخدام طريقتي RK4 و RK-Butcher

الخلاصة

تم حل مسائل القيمة الحدية (BVPs) التي تحتوي على معادلات تفاضلية من المرتبة الرابعة والتي تكون على الأقل إحدى مشتقاتها مضروبة بأمثال موجبة عند شروطها المحددة . إذ طبقنا كلا من الطرائق التالية (طريقة رانج – كوتا Rung-Kutta) من المرتبة الرابعة وطريقة (Rung - Kutta Butcher) من المرتبة السادسة) لحل مثل هذه المسائل في شروط معينة . تم تحويل المعادلة التفاضلية من المرتبة الرابعة إلى معادلتين تفاضليتين من المرتبة الثانية ليتم حلهما معا باستخدام كل من الطريقتين كلا على حده . يتم التنبؤ بالقيم الأولية للمتغيرات المعتمدة ويتم تعديلها وتصحيحها وفق قيمة خطأ معين . تم تطبيق الطريقتين المدروستين لحل بعض المسائل ثم قارنا النتائج الحاصلة مع طرائق أخرى . دونت النتائج في جداول ومثلت بيانيا التي أظهرت نتائج جيدة.

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1. Introduction

Several models of mathematical physics and applied mathematics contain Boundary Value Problems BVPs in the 4th order ordinary differential equations (ODEs) [1].

Let consider the 4th order (ODEs) is of the following equation:

$$a\frac{d^4y}{dx^4}+b\frac{d^2y}{dx^2}+gy=f(x,y) \qquad (1)$$

$$y(0)=0 \qquad (2)$$

$$y''(0)=x \qquad (3)$$

$$y(1)=y \qquad (4)$$

$$y''(1)=z \qquad (5)$$

where a,b,g and x,y,z are constants and $f(x,y)$ continuous and $\frac{\partial f}{\partial y}$ exists and continuous and $\frac{\partial f}{\partial y}\geq 0$.

Different analytical and numerical methods are used to solve the 4th order (ODEs) this can be concerned by Kapur [1]. Some of the numerical methods applied by Ortner [2] which gave approximate solutions. Okey [3] used the GEM (Green Element Method) to solve these problems.

In this paper we proposed to apply Rung-Kutta (RK4)[4] and (RK-Butcher)[5] to solve BVPs with there conditions as solving Eqs.(1) under the conditions (2)-(5). Special program is designed to apply the proposal methods. Some physical problems were studied before from [1],[2],[3], and [6] are solved by RK4 and RK-Butcher methods. Results are presented by tables and figures to compare the error with another solution which show good agreements.

2. The Proposal Solution:

To solve Eqs.(1) under the condition in (2)-(5) , it will be transform to two pair of 2nd (ODEs) by assuming:

$$\frac{d^2y}{dx^2}=M \qquad (6)$$

Then Eqs.(1) becomes:

$$\frac{d^2M}{dx^2}+\frac{b}{a}.M=(f(x,y)-g.y)/a \qquad (7)$$

Where:

$$y(0)=0 \qquad (8)$$

$$M(0)=x \qquad (9)$$

$$y(1)=y \qquad (10)$$

$$M(1)=z \qquad (11)$$

The solution $y(x)$ for this problem from $x_0=0$ to $x_n=1$ after assuming h and n where:

$$h=\frac{x_n-x_0}{n} \qquad (12)$$

The proposal methods are:

2.1 (Rung – Kutta of 4th Order):

The BVPs of 4th order (ODEs) is solved by Rung – Kutta of 4th Order (RK4) to solve Eqs.(6) and (7) after transforming them to two pair of 1st order (ODEs) as the following:

$$\frac{dy}{dx}=p_1=fnf_1(x,y,M,p_1,p_2) \qquad (13)$$

$$\frac{dp_1}{dx}=M=fnp_1(x,y,M,p_1,p_2) \qquad (14)$$

$$\frac{dM}{dx}=p_2=fnf_2(x,y,M,p_1,p_2) \qquad (15)$$

$$\frac{dp_2}{dx} = fnp_2(x, y, M, p_1, p_2)$$
$$= -\frac{b}{a}.M + (f(x, y) - g.y)/a \quad (16)$$

The set of 1st order (ODEs) (13) to (16) are solver together from the following:

$$y^{n+1} = y^{n+1} + \frac{1}{6}(k_{11} + 2k_{21} + 2k_{31} + k_{41}) \quad (17)$$

$$p_1^{n+1} = p_1^n + \frac{1}{6}(L_{11} + 2L_{21} + 2L_{31} + L_{41}) \quad (18)$$

$$M^{n+1} = M^{n+1} + \frac{1}{6}(k_{12} + 2k_{22} + 2k_{32} + k_{42}) \quad (19)$$

$$p_2^{n+1} = p_2^n + \frac{1}{6}(L_{12} + 2L_{22} + 2L_{32} + L_{42}) \quad (20)$$

Constants $k_{11}, k_{21}, k_{31}, k_{41}, L_{11}, L_{21}, L_{31}, L_{41}, k_{12}, k_{22}, k_{32}, k_{42}, L_{12}, L_{22}, L_{32}, L_{42}$ in Eqs.(17) to (20) are calculated from the following :

$$k_{11} = h.fnf_1(x, y, M, p_1, p_2) \quad (21)$$

$$L_{11} = h.fnp_1(x, y, M, p_1, p_2) \quad (22)$$

$$k_{12} = h.fnf_2(x, y, M, p_1, p_2) \quad (23)$$

$$L_{12} = h.fnp_2(x, y, M, p_1, p_2) \quad (24)$$

$$k_{21} = h.fnf_1(x + h/2, y + k_{11}/2, M + k_{12}/2, p_1 + L_{11}/2, p_2 + L_{12}/2) \quad (25)$$

$$L_{21} = h.fnp_1(x + h/2, y + k_{11}/2, M + k_{12}/2, p_1 + L_{11}/2, p_2 + L_{12}/2) \quad (26)$$

$$k_{22} = h.fnf_2(x + h/2, y + k_{11}/2, M + k_{12}/2, p_1 + L_{11}/2, p_2 + L_{12}/2) \quad (27)$$

$$L_{22} = h.fnp_2(x + h/2, y + k_{11}/2, M + k_{12}/2, p_1 + L_{11}/2, p_2 + L_{12}/2) \quad (28)$$

$$k_{31} = h.fnf_1(x + h/2, y + k_{21}/2, M + k_{22}/2, p_1 + L_{21}/2, p_2 + L_{22}/2) \quad (29)$$

$$L_{31} = h.fnp_1(x + h/2, y + k_{21}/2, M + k_{22}/2, p_1 + L_{21}/2, p_2 + L_{22}/2) \quad (30)$$

$$k_{32} = h.fnf_2(x + h/2, y + k_{21}/2, M + k_{22}/2, p_1 + L_{21}/2, p_2 + L_{22}/2) \quad (31)$$

$$L_{32} = h.fnp_2(x + h/2, y + k_{21}/2, M + k_{22}/2, p_1 + L_{21}/2, p_2 + L_{22}/2) \quad (32)$$

$$k_{41} = h.fnf_1(x + h, y + k_{31}, M + k_{32}, p_1 + L_{31}, p_2 + L_{32}) \quad (33)$$

$$L_{41} = h.fnp_1(x + h, y + k_{31}, M + k_{32}, p_1 + L_{31}, p_2 + L_{32}) \quad (34)$$

$$k_{42} = h.fnf_2(x + h, y + k_{31}, M + k_{32}, p_1 + L_{31}, p_2 + L_{32}) \quad (35)$$

$$L_{42} = h.fnp_2(x + h, y + k_{31}, M + k_{32}, p_1 + L_{31}, p_2 + L_{32}) \quad (36)$$

2.2 (Rung – Kutta Butcher of 6^{th} Order):

The BVPs of 4^{th} order (ODEs) is solved by Rung – Kutta Butcher of 6^{th} Order (RK-Butcher) to solve Eqs.(6) and (7) after transforming them to two pair of 1^{st} order (ODEs) as the following:

$$y^{n+1} = y^n + \frac{h}{90}(7k_{11} + 32k_{31} + 12k_{41} + 32k_{51} + 7k_{61}) \quad (37)$$

$$p_1^{n+1} = p_1^n + \frac{h}{90}(7L_{11} + 32L_{31} + 12L_{41} + 32L_{51} + 7L_{61}) \quad (38)$$

$$M^{n+1} = M^n + \frac{h}{90}(7k_{12} + 32k_{32} + 12k_{42} + 32k_{52} + 7k_{62}) \quad (39)$$

$$p_2^{n+1} = p_1^n + \frac{h}{90}(7L_{12} + 32L_{32} + 12L_{42} + 32L_{52} + 7L_{62}) \quad (40)$$

Constants in Eqs.(37) to (40) are calculated

$k_{11}, k_{21}, k_{31}, k_{41}, L_{11}, L_{21}, L_{31}, L_{41}, k_{12}, k_{22}, k_{32}, k_{42}, L_{12}, L_{22}, L_{32}, L_{42}$

from the following :

$$k_{11} = fnf_1(x_n, y_n, M_n, p_{1n}, p_{2n}) \quad (41)$$

$$L_{11} = fnp_1(x_n, y_n, M_n, p_{1n}, p_{2n}) \quad (42)$$

$$k_{12} = fnf_2(x_n, y_n, M_n, p_{1n}, p_{2n}) \quad (43)$$

$$L_{12} = fnp_2(x_n, y_n, M_n, p_{1n}, p_{2n}) \quad (44)$$

$$k_{21} = fnf_1(x_n, y_n, M_n, p_{1n}, p_{2n}) + \frac{h}{4}L_{11} \quad (45)$$

$$L_{21} = fnp_1(x_n + \frac{h}{4}, y_n + \frac{hk_{11}}{4}, M_n + \frac{hk_{12}}{4}, p_{1n} + \frac{hL_{11}}{4}, p_{2n} + \frac{hL_{12}}{4}) \quad (46)$$

$$k_{22} = fnf_2(x_n, y_n, M_n, p_{1n}, p_{2n}) + \frac{h}{4}L_{12}(\quad (47)$$

$$L_{22} = fnp_2(x_n + \frac{h}{4}, y_n + \frac{hk_{11}}{4}, M_n + \frac{hk_{12}}{4}, p_{1n} + \frac{hL_{11}}{4}, p_{2n} + \frac{hL_{12}}{4}) \quad (48)$$

$$k_{31} = fnf_1(x_n, y_n, M_n, p_{1n}, p_{2n}) + \frac{h}{8}L_{11} + \frac{h}{8}L_{21} \quad (49)$$

$$L_{31} = fnp_1(x_n + \frac{h}{4}, y_n + \frac{hk_{11}}{8} + \frac{hk_{21}}{8}, M_n + \frac{hk_{12}}{8} + \frac{hk_{22}}{8}, p_{1n} + \frac{hL_{11}}{8} + \frac{hL_{21}}{8}, p_{2n} + \frac{hL_{12}}{8} + \frac{hL_{22}}{8}) \quad (50)$$

$$k_{32} = fnf_2(x_n, y_n, M_n, p_{1n}, p_{2n}) + \frac{h}{8}L_{12} + \frac{h}{8}L_{22} \quad (51)$$

$$L_{32} = fnp_2(x_n + \frac{h}{4}, y_n + \frac{hk_{11}}{8} + \frac{hk_{21}}{8}, M_n + \frac{hk_{12}}{8} + \frac{hk_{22}}{8}, p_{1n} + \frac{hL_{11}}{8} + \frac{hL_{21}}{8}, p_{2n} + \frac{hL_{12}}{8} + \frac{hL_{22}}{8}) \quad (52)$$

$$k_{41} = fnf_1(x_n, y_n, M_n, p_{1n}, p_{2n}) - \frac{h}{2}L_{21} + hL_{31} \quad (53)$$

$$L_{41} = fnp_1(x_n + \frac{h}{2}, y_n - \frac{hk_{21}}{2} + hk_{31}, M_n - \frac{hk_{22}}{2} + hk_{32}, p_{1n} - \frac{hL_{21}}{2} + hL_{31}, p_{2n} - \frac{hL_{22}}{2} + hL_{32}) \quad (54)$$

$$k_{42} = fnf_2(x_n, y_n, M_n, p_{1n}, p_{2n}) - \frac{h}{2}L_{22} + \frac{h}{8}L_{32} \quad (55)$$

$$L_{42} = fnp_2(x_n + \frac{h}{2}, y_n - \frac{hk_{21}}{2} + hk_{31}, M_n - \frac{hk_{22}}{2} + hk_{32}, p_{1n} - \frac{hL_{21}}{2} + hL_{31}, p_{2n} - \frac{hL_{22}}{2} + hL_{32}) \quad (56)$$

$$k_{51} = fnf_1(x_n, y_n, M_n, p_{1n}, p_{2n}) + \frac{3h}{16}L_{11} + \frac{9h}{16}L_{41} \quad (57)$$

$$L_{51} = fnp_1(x_n + \frac{3h}{4}, y_n + \frac{3hk_{11}}{16} + \frac{9hk_{41}}{16}, M_n + \frac{3hk_{12}}{16} + \frac{9hk_{42}}{16}, p_{1n} + \frac{3hL_{11}}{16} + \frac{9hL_{41}}{16}, p_{2n} + \frac{3hL_{12}}{16} + \frac{9hL_{42}}{16}) \quad (58)$$

$$k_{52} = fnf_2(x_n, y_n, M_n, p_{1n}, p_{2n}) + \frac{3h}{16}L_{12} + \frac{9h}{16}L_{42} \quad (59)$$

$$L_{52} = fnp_2(x_n + \frac{3h}{4}, y_n + \frac{3hk_{11}}{16} + \frac{9hk_{41}}{16}, M_n + \frac{3hk_{12}}{16} + \frac{9hk_{42}}{16}, p_{1n} + \frac{3hL_{11}}{16} + \frac{9hL_{41}}{16}, p_{2n} + \frac{3hL_{12}}{16} + \frac{9hL_{42}}{16}) \quad (60)$$

$$k_{61} = \begin{cases} fnf_1(x_n, y_n, M_n, p_{1n}, p_{2n}) - \\ \frac{h}{7}(3L_{11} - 2L_{21} - 12L_{31} + 12L_{41} - 8L_{51}) \end{cases} \quad (61)$$

$$L_{61} = \begin{cases} fnp_1[x_n + h, y_n - \frac{h}{7}(3k_{11} - 2k_{21} - \\ 12k_{31} + 12k_{41} - 8k_{51}), \\ M_n - \frac{h}{7}(3k_{12} - 2k_{22} - 12k_{32} + \\ 12k_{42} - 8k_{52}), \\ p_{1n} - \frac{h}{7}(3L_{11} - 2L_{21} - 12L_{31} + \\ 12L_{41} - 8L_{51}), \\ p_{2n} - \frac{h}{7}(3L_{12} - 2L_{22} - 12L_{32} \\ + 12L_{42} - 8L_{52})] \end{cases} \quad (62)$$

$$k_{62} = \begin{cases} fnf_2(x_n, y_n, M_n, p_{1n}, p_{2n}) \\ -\frac{h}{7}(3L_{12} - 2L_{22} \\ -12L_{32} + 12L_{42} - 8L_{52}) \end{cases}$$

(63)

$$L_{62}=\begin{cases} fnp_2[x_n+h, y_n-\frac{h}{7}(3k_{11}-2k_{21}- \\ 12k_{31}+12k_{41}-8k_{51}), \\ M_n-\frac{h}{7}(3k_{12}-2k_{22}-12k_{32}+ \\ 12k_{42}-8k_{52}), \\ p_{1n}-\frac{h}{7}(3L_{11}-2L_{21}-12L_{31}+ \\ 12L_{41}-8L_{51}), \\ p_{2n}-\frac{h}{7}(3L_{12}-2L_{22}-12L_{32}+ \\ 12L_{42}-8L_{52})] \end{cases} \quad (64)$$

$p_{1(0)}$ and $p_{2(0)}$ are predicted from:

$$p^r_{1(0)}=[y_{(1)}-y_{(0)}]/h \quad (65)$$

$$p^r_{2(0)}=[M_{(1)}-M_{(0)}]/h \quad (66)$$

Then $p_{1(0)}$ and $p_{2(0)}$ are modified by :

$$p^{r+1}_{1(0)}=p^r_{1(0)}+(y^r_{(1)}-y_{(1)})/h \quad (67)$$

$$p^{r+1}_{2(0)}=p^r_{2(0)}+(M^r_{(1)}-M_{(1)})/h \quad (68)$$

Error is accepted as:

$\left|y^r_{(1)}-y_{(1)}\right|\le 10^{-5}$ and $\left|M^r_{(1)}-M_{(1)}\right|\le 10^{-5}$ (or any tolerance you need) (69)

3. Numerical applications:

Some of the physical problems are solved to assign the effectiveness and accuracy of the proposal methods. Results are presented in tables and figures.

Example 1. To solve the nonlinear 4th order (ODEs) represented by:

$$\frac{d^4y}{dx^4}=6.e^{-4y}-\frac{12}{(1+x)^4} \quad (70)$$

where:

$$y(0)=0 \quad (71)$$

$$y''(0)=-1.0 \quad (72)$$

$$y(1)=\ln 2 \quad (73)$$

$$y''(1)=-0.25 \quad (74)$$

This problem was studied by Okey [3] by applying the (Green Element Method). The Exact solution for this problem is $y(x)=Ln(1+x)$. Results are presented in Table(1). Figure(1) represent the comparison between errors for using RK4 and RK-Butcher for solving the problem. Figure (2) present the solution of problem in example (1) by different methods.

Example2. Consider the following set of 1st order nonlinear (ODEs) [6]:

$$\frac{dy_1}{dx}=2e^{4x}y_4^2 \quad (75)$$

$$\frac{dy_2}{dx}=y_1-y_3+\cos(x)-e^{2x} \quad (76)$$

$$\frac{dy_3}{dx}=y_2-y_4+e^{-x}-\sin(x) \quad (77)$$

$$\frac{dy_4}{dx}=-e^{-5x}y_1^2 \quad (78)$$

$y_1(0)=1$, $y_2(0)=1$, $y_3(0)=0$, $y_4(0)=1$

This problem was studied by Biazar [6] by applying the (Variational Iteration Method). The Exact Solution was given by:

$y_1(x) = e^{2x}$, $y_2(x) = \sin x + \cos x$,

$y_3(x) = \sin x$, $y_4(x) = e^{-x}$

Results of calculating $y_1(x)$, $y_2(x)$, $y_3(x)$, $y_4(x)$ by using RK4 and RK-Butcher were presented in Tables (2) to (5). Figures (3), (4), (5), and (6) show errors between different methods.

4. Results & Discussion

The suggested methods RK4 and RK-Butcher for solving 4th order (ODEs) gave good agreements comparing results of Example (1) with the Exact solution as shown from Table(1) and Figure(1).

Results of solving the problem in Example (2) by the three methods VIM, RK4, and RK-Butcher were presented in Tables (2) to (5) and Figures (3) to (6). RK4 was very accurate in the three methods for its least error. The Vim method was better than RK-Butcher due to its error. Finally the two proposal method RK4 and RK-Butcher gave good agreements in solving BVPs of 4th order (ODEs) in the required conditions.

References

[1]Kapur K.K. , Vibrations of a Timoshenko Beam. Using a finite element approach , J Acoust. Soc Am , 40(1996) 1058-63 .

[2]Ortner N. Wagner P. , Solution of the initial-boundary value problem for simply supported semi-infinite Timoshenko Beam, JElast , 42, (1996)217-241

[3]Okey O. , A Green element method for fourth order ordinary differential equations, Advances in Engineering Software 35 (2004)517-525.

[4]Stroud K. A. , Advanced Engineering Mathematics, Palgrave MacMillan ,2003 .

[5]Sekar S., Murugesh V. ,and Murugesan K. , Numerical Strategies for the system of Second order IVPs Using the RK-Butcher Algorithms, Strategies for the system of Second order IVPs Using the RK-Butcher Algorithms, International journal of Computer Science & Application Vol 1. No. 11(2004) 96-117 .

[6]Biazar J. ,Ghazvini H. , He's Variational Iteration Method for solving linear and non-linear systems of ordinary different ional equations, Applied Mathematics and Computation 191 (2007)287-297.

Table(1) Results of solving the problem in Example (1).

X	*Exact Solution*	*Rung- kutta*	*Rung- kutta Error*	*RK-Butcher*	*RK-Butcher Error*
0.0	0	0	0	0	0
0.1	.0861786	.0861776	-9.0E-07	.0861776	-9.1E-07
0.2	.1823224	.1823215	-8.4E-07	.1823215	-8.6E-07
0.3	.262365	.2623643	-7.7E-07	.2623642	-8.0E-07
0.4	.3364729	.3364722	-6.5E-07	.3364722	-6.8E-07
0.5	.4054657	.4054652	-4.7E-07	.4054652	-5.0E-07
0.6	.4700041	.4700038	-2.9E-07	.4700038	-3.2E-07
0.7	.5306286	.5306286	-5.9E-08	.5306285	-1.1E-07
0.8	.587787	.5877872	2.3E-07	.5877872	1.7E-07
0.9	.6418541	.6418547	5.9E-07	.6418546	5.3E-07
1.0	.6931474	.6931484	1.0E-06	.6931483	9.5E-07

Table(2) Solutions of problem in Example(2) of $y_1(x)$:

X	Exact Solution	Rung-Kutta Solution	Rung-Kutta Error	RK-Butcher solution	RK-Butcher Error	VIM solution	VIM Error
0.0	1	1	0	1	0	1	0
0.1	1.221405	1.221404	1.3E-06	1.21966	1.7E-03	1.221403	2.1E-6
0.2	1.491828	1.491827	5.9E-07	1.488053	3.7E-03	1.491824	4.3E-6
0.3	1.822122	1.822123	7.1E-07	1.816108	6.0E-03	1.822113	9.5E-6
0.4	2.225545	2.225547	2.3E-06	2.217288	8.2E-03	2.225473	6.7E-5
0.5	2.718286	2.718291	4.7E-06	2.708086	1.0E-02	2.717802	4.9E-4
0.6	3.320122	3.320129	6.9E-06	3.308678	1.1E-02	3.317646	2.5E-3
0.7	4.055205	4.055214	8.5E-06	4.043724	1.1E-02	4.045049	1.0E-2
0.8	4.953038	4.953048	1.0E-05	4.943336	9.7E-03	4.917836	3.5E-2
0.9	6.049653	6.049664	1.0E-05	6.044241	5.4E-03	5.943003	1.1E-1
1.0	7.389061	7.389072	1.0E-05	7.391204	2.1E-03	7.101202	2.8E-1

Table(3) Solutions of problem in Example(2) of $y_2(x)$:

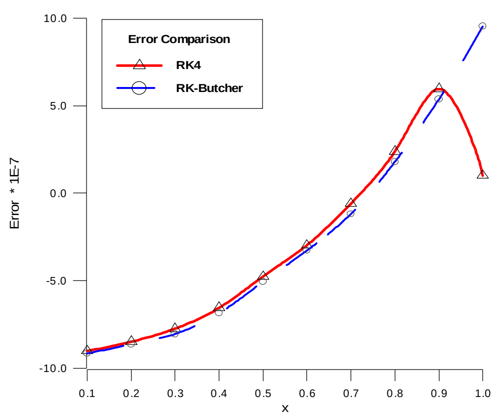
X	Exact Solution	Rung-Kutta Solution	Rung-Kutta Error	RK-Butcher solution	RK-Butcher Error	*VIM* solution	*VIM Error*
0.0	1	1	0	1	0	1	0
0.1	1.094839	1.094837	1.1E-06	1.084316	1.0E-02	1.094837	2.0E-6
0.2	1.178737	1.178736	9.5E-07	1.167627	1.1E-02	1.178736	1.1E-6
0.3	1.250857	1.250857	7.1E-07	1.239046	1.1E-02	1.250859	2.9E-6
0.4	1.31048	1.310479	3.5E-07	1.297827	1.2E-02	1.310511	3.2E-5
0.5	1.357008	1.357009	1.1E-07	1.343378	1.3E-02	1.357220	2.1E-4
0.6	1.389978	1.389979	1.0E-06	1.375277	1.4E-02	1.390995	1.0E-3
0.7	1.40906	1.409062	2.0E-06	1.393286	1.5E-02	1.412974	3.9E-3
0.8	1.414063	1.414066	3.2E-06	1.397362	1.6E-02	1.426865	1.3E-2
0.9	1.404937	1.404941	4.4E-06	1.387668	1.7E-02	1.441809	3.6E-2
1.0	1.381773	1.381779	5.3E-06	1.364579	1.7E-02	1.477347	9.5E-2

Table(4) Solutions of problem in Example(2) of $y_3(x)$:

X	Exact Solution	Rung-Kutta Solution	Rung-Kutta Error	RK-Butcher solution	RK-Butcher Error	*VIM* solution	*VIM Error*
0.0	0	0	0	0	0	0	0
0.1	.0998344	.0998331	1.2E-06	1.085171	.9853365	.0998334	1.0 E-6
0.2	.1986703	.1986688	1.4E-06	1.169481	.9708109	.1986693	1.0 E-6
0.3	.2955212	.2955194	1.7E-06	1.241883	.9463617	.2955216	4.0 E-6
0.4	.3894192	.3894174	1.8E-06	1.301617	.9121975	.3894332	1.5 E-5
0.5	.4794263	.4794244	1.8E-06	1.348079	.8686529	.4795191	9.4 E-5
0.6	.5646431	.5646412	1.8E-06	1.380836	.8161924	.5650703	4.3 E-4
0.7	.6442182	.6442165	1.6E-06	1.399635	.7554169	.6457932	1.6 E-3
0.8	.7173564	.7173551	1.3E-06	1.404422	.6870655	.7222996	9.4 E-3
0.9	.7833272	.7833264	7.7E-07	1.395347	.6120194	.7969926	1.3 E-2
1.0	.8414712	.841471	1.7E-07	1.372773	.5313019	.8753804	3.3 E-2

Table(5) Solutions of problem in Example(2) of $y_4(x)$:

x	Exact Solution	Rung-Kutta Solution	Rung-Kutta Error	RK-Butcher solution	RK-Butcher Error	*VIM* solution	*VIM Error*
0.0	1	1	0	1	0	1	0
0.1	.9048365	.9048377	1.1E-06	.9049744	1.3E-04	.9048374	9.0 E-7
0.2	.8187299	.8187312	1.3E-06	.8192065	4.7E-04	.8187306	7.0 E-7
0.3	.7408175	.7408189	1.3E-06	.7417469	9.2E-04	.7408161	1.4 E-6
0.4	.6703194	.6703207	1.2E-06	.6717419	1.4E-03	.6703200	1.8 E-5
0.5	.6065301	.6065311	1.0E-06	.6084277	1.8E-03	.6064379	9.3 E-5
0.6	.5488113	.5488119	5.9E-07	.5511234	2.3E-03	.5486428	3.4 E-4
0.7	.496585	.4965853	3.5E-07	.4992241	2.6E-03	.4955431	1.0 E-3
0.8	.4493287	.4493288	2.9E-08	.4521933	2.8E-03	.4466719	2.7 E-3
0.9	.4065695	.4065693	1.4E-07	.409555	2.9E-03	.4005911	5.9 E-3
1.0	.3678793	.367879	2.9E-07	.370887	3.0E-03	.3557110	1.2 E-2



Figure(1) Comparison of Error of Example(1).

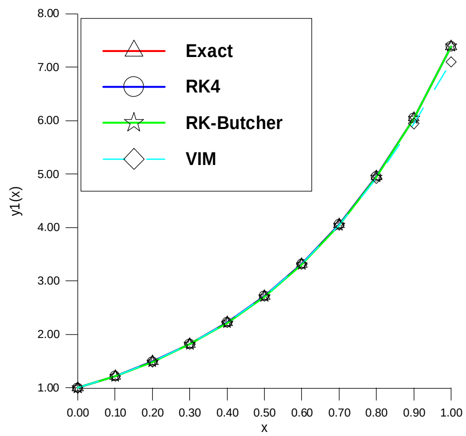


Figure (2) Solving the problem in Example (1) by different methods

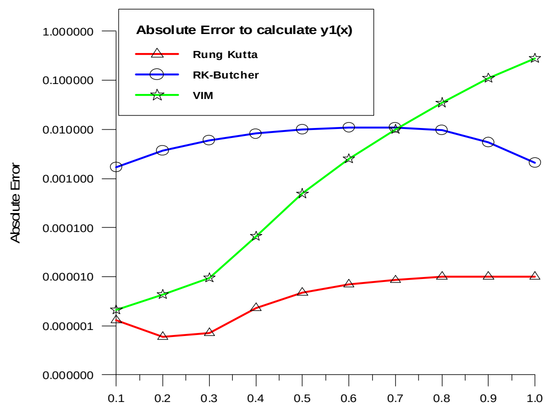


Figure (3) error between methods to calculate y1(x)

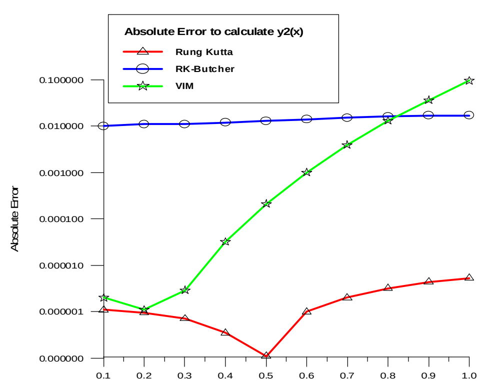


Figure (4) error between methods to calculate y2(x)

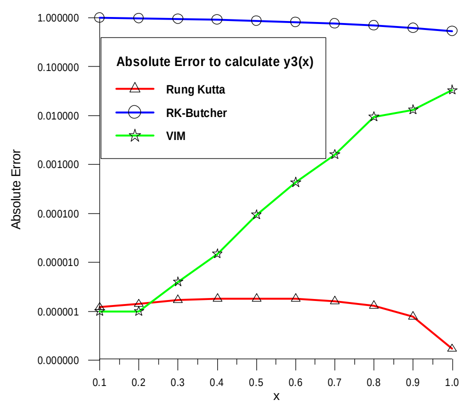


Figure (5) error between methods to calculate y3(x)

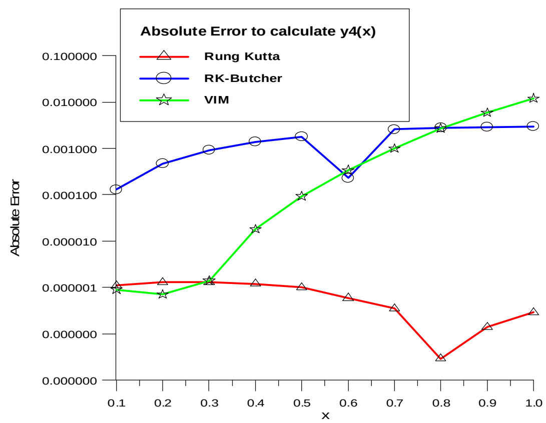


Figure (6) error between methods to calculate y4(x)