Parameterization Techniques for Quadratic Continuous Optimal Control Problems

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Abstract

The main purpose of this work is to propose the parameterization techniques for solving quadratic optimal control (OC) problem with aid of both Chebyshev and Hermite Polynomials as a basis function to find the approximate solution for OC problem. Some examples are given as applications of the proposed.

Keywords: Optimal control, Parameterization technique, Chebyshev polynomials, Hermite polynomials.

اساليب المعلمة لمسائل السيطرة المثلى المستمرة التربيعية

الخلاصة

الغرض الاساسي في هذا العمل هو اقتراح اساليب المعلمات لحل مسألة السيطرة المثلى التربيعية باستخدامكل من متعددات حدود شيبشف وهرمت كدوال اساسية لايجاد الحل التقريبي لمسألة السيطرة المثلى .بعض الامثلة اعطيت كتطبيق للطريقة المقترحة.

1-Introduction:[7]

Optimal control theory, an extension of the calculus of variations, is a mathematical optimization method for deriving control policies. The method is largely due to the work of Lev Pontryagin and his collaborators in the Soviet Union and Richard Bellman in the United States.

Optimal control deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved. A control problem includes a cost functional that is a function of state and control variables. An optimal control is a set of differential equations describing the paths of the control variables that minimize the cost functional. The optimal control can be derived using Pontryagin's maximum principle (a necessary condition), or by solving the Hamilton-Jacobi-Bellman equation (a sufficient condition).[10]

The linear quadratic control is a special case of the general nonlinear optimal control problem. The LQ problem is stated as follows.

Minimize the quadratic continuous-time cost functional

$$J = \int_{t_0}^{\infty} \left(x^T Qx + u^T Ru\right)dt \qquad \text{…. (1)}$$

Subject to the linear first-order dynamic constraints

$$\dot{x} = Ax(t) + Bu(t) \qquad \text{…. (2)}$$

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and the initial condition

$$x(t_0) = x_0 \qquad \text{…..(3)}$$

A particular form of the LQ problem that arises in many control system problems is that of the linear quadratic regulator (LQR) where all of the matrices (i.e., $A, B, Q, and R$) are constant, the initial time is arbitrarily set to zero, and the terminal time is taken in the limit (∞) (this last assumption is what is known as infinite horizon).

In the finite-horizon case the matrices are restricted in that Q and R are positive semi-definite and positive definite, respectively. In the infinite-horizon case, however, the matrices Q and R are not only positive-semi definite and positive-definite, respectively, but are also constant.

2. Chebyshev polynomial

Chebyshev polynomials are important in approximation theory because the roots of the Chebyshev polynomials of different kind are used as nodes in polynomial interpolation. The resulting interpolation polynomial minimizes the problem of Runge's phenomenon and provides an approximation that is close to the polynomial of best approximation to a continuous function under the maximum norm. This approximation leads directly to the method of Clenshaw â€“ Curtis quadrature.[5]

The Chebyshev polynomials have many beautiful properties and countless applications, arising in a variety of continuous settings. They are a sequence of orthogonal polynomials appearing in approximation theory, numerical integration, and differential equations.

In this paper we deal with the second , third and fourth Chebyshev polynomials.

2.1The Second Chebyshev Polynomials

A modified set of Chebyshev polynomials defined by a slightly different generating function. They arise in the development of four-dimensional spherical harmonics in angular momentum theory. They are a special case of the Gegenbauer polynomial with $\alpha = 1$. They are also intimately connected with trigonometric multiple-angle formulas. The Chebyshev polynomials of the second kind are denoted by $U_n(t)$, for $t \in [-1,1]$.[6]

The defining generating function of the Chebyshev polynomials of the second kind is

$$s(x,t) = \frac{1}{1-2xt+x^2} = \sum_{n=0}^{\infty} U(t)x^n$$

for $|t| < 1$ and $|x| < 1$.

The Rodrigues representation for U_n is

$$U_n(t) = \frac{(-1)^n (n+1)\sqrt{p}}{2^{n+1}\left(n+\frac{1}{2}\right)!(1-t^2)^{1/2}} \frac{d^n}{dt^n}\left[(1-t^2)^{n+1/2}\right]$$

The polynomials can also be defined in terms of the sums

$$U_n(t) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n-i}{i} (2t)^{n-2i}$$
$$= \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2m+1} t^{n-2m} (t^2-1)^m$$

where $\lfloor x \rfloor$ is the floor function and $\lceil x \rceil$ is the ceiling function, or in terms of the product

$$U_n(t) = 2^n \prod_{k=1}^{n}\left[t - \cos\left(\frac{kp}{n+1}\right)\right]$$

Chebyshev polynomials of the second kind $U_n(t)$ also obey the interesting determinant identity

$$U = \begin{vmatrix} 2t & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2t & 1 & 0 & \ddots & 0 & 0 \\ 0 & 1 & 2t & 1 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 2t & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2t \end{vmatrix}$$

The Chebyshev polynomials of the second kind are a special case of the Jacobi polynomials $P_n^{(a,b)}$ with $\alpha = \beta = 1/2$,

$$U_n(t) = (n+1)\frac{P_n^{(1/2,1/2)}(t)}{P_n^{(1/2,1/2)}(1)}$$

$$= (n+1)\,{}_2F_1(-n, n+2; 3/2; 1/2(1-t))$$

where ${}_2F_1(a,b;c;x)$ is a hypergeometric function .

2.2The third- and fourth -kind Chebyshev polynomials:

The polynomials $V_n(t)$ *and* $W_n(t)$ are, in fact rescaling of two particular Jacobi[2]polynomials $P_n^{(a,b)}$ with $a = -1/2, b = 1/2$, and vice versa. Explicitly [5]

$$\binom{2n}{n} V_n(t) = 2^{2n} P_n^{(-1/2,1/2)}(t) \quad ,$$

$$\binom{2n}{n} W_n(t) = 2^{2n} P_n^{(1/2,-1/2)}(t)$$

These polynomials to may be efficiently generated by use of a recurrence relation. Since

$$V_n(t) = 2tV_{n-1}(t) - V_{n-2}(t)$$
$$V_n(t) = U_n(t) - U_{n-1}(t)$$
$$W_n(t) = 2tW_{n-1}(t) - W_{n-2}(t)$$
$$W_n(t) = U_n(t) + U_{n-1}(t)$$
$$W_n(t) = V_n(-t) \qquad (\text{n even})$$
$$W_n(t) = -V_n(-t) \qquad (\text{n odd})$$

3. Shifted Polynomials U_n^*, V_n^*, W_n^*:

since the range [0,1],is quite often more convenient to use than the range[-1,1],we sometimes map the independent variable t in [0,1] to the variable s in [-1,1] by the transformation

$$s = 2t - 1 \qquad t = \frac{1}{2}(1+s)$$

And this leads to a shifted polynomials U_n^*, V_n^*, W_n^* of the second, third and fourth kinds may be defined in precisely analogous ways:

$$U_n^*(t) = U_n(2t-1), \quad V_n^*(t) = V_n(2t-1), \quad W_n^*(t) = W_n(2t-1)$$

4. Hermite polynomials $H_n(t)$

The Hermite polynomials $H_n(t)$ are set of orthogonal polynomials over the domain $(-\infty, \infty)$ with weighting function e^{-t^2}, illustrated above for n=1, 2, 3, and 4. Hermite polynomials are implemented in Mathematica as Hermite H[*n*, *t*]. [4] The *nth* Hermite polynomials are defined as follows

$$H_n(t) = (-1)^n \frac{d^n}{dt^n}\left(e^{-t^2}\right)$$

Also $H_n(t)$ can be expressed as

$$H_n(t) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \frac{n!}{k!(n-2k)!}(2t)^{n-2k}$$

and $\lfloor s \rfloor$ denotes the largest integer $\leq$ s.

The values $H_n(0)$ may be called Hermite numbers.

The Hermite polynomials satisfy the symmetry condition

$$H_n(-1) = (-1)^n H_n(t)$$

They also obey the recurrence relations

$$H_n(t) = \frac{2t}{2(n+1)} H_{n+1}(t) - \frac{1}{2(n+1)} H_{n+2}(t)$$

$$H_n(t) = 2tH_{n-1}(t) - 2(n-1)H_{n-2}(t)$$

$$2tH_n(t) = 2nH_{n-1}(t) + H_{n+1}(t)$$

The Product and Differentiation properties can be expressed as

$$H_n(t)H_m(t) = \sum_{k=0}^{\min(n,m)} \binom{m}{k}\binom{n}{k} 2^k k! H_{-2k+m+n}(t)$$

$$\frac{dH_n(t)}{dt} = 2nH_{n-1}(t)$$

$$\frac{d^m H_n(t)}{dt^m} = \frac{2^m n!}{(n-m)!} H_{n-m}(t) \qquad m \in N^+$$

5. Parameterization Techniques for Optimal Control Problems:

The work in this paper is based on using parameterization technique to convert the optimal control problem into a mathematical programming problem.

The parameterization technique can be applied in one of the following three forms

-Control-State Parameterization:

The control- state parameterization approach is based on approximating both the state variables and the control variables by a sequence of known functions with unknown parameters.

-State Parameterization:

The state parameterization is based on approximating the state variables by sequence of known functions with unknown parameters and the control variables are obtained from the state equations.

The parameterization technique can be employed different basis functions. In this work the Hermite polynomials and Second, Three and Fourth Chebyshev polynomials will be used to parameterize the system state and control variables.[1]

The parameterization technique for state and control or state variables are proposed to solve OCP.

5.1: Using State and Control Parameterization technique

The control-state parameterization is based on approximating both the state variable and control variables by a sequence of Hermite polynomials ($H_n(t)$) with unknown parameters as follows:

$$x(t) \approx \sum_{i=0}^{n} a_i H_i(t) \qquad n=1,2,3,\ldots$$

$0 \leq t \leq 1$

$$u(t) \approx \sum_{i=0}^{n} b_i H_i(t)$$

n=1,2,3,… $0 \leq t \leq 1$

where a_i, b_i are unknown parameters

First, when n=1, yields

$$x(t) = \sum_{i=0}^{1} a_i H_i(t) \qquad 0 \leq t \leq 1 \qquad \text{………(4)}$$

$$u(t) = \sum_{i=0}^{1} b_i H_i(t) \qquad \text{…….(5)}$$

Substitute $H_0(t), H_1(t)$ into eqn.(4)-(5) t0 get

$$x(t) = \sum_{i=0}^{1} a_i \sum_{i=0}^{\left\lfloor \frac{1}{2} \right\rfloor} (-1)^i \frac{1!}{i!(1-2i)!} (2t)^{1-2i} \qquad \text{….(6)}$$

$$u(t) = \sum_{i=0}^{1} b_i \sum_{i=0}^{\left\lfloor \frac{1}{2} \right\rfloor} (-1)^i \frac{1!}{i!(1-2i)!} (2t)^{1-2i} \qquad \text{……..(7)}$$

The control points $a_i, b_i \quad i = 0,1$ can be evaluated as follows:

Put t=0 and t=1 into (6)-(7) to get a_0, a_1, b_0, b_1 respectively,

i.e

$$x(0) = a_0 \quad , x(0) = x_0 \qquad \text{………(8)}$$

$$x(1) = a_0 + 2a_1 \qquad \text{……..(9)}$$

$$u(0) = b_0 \qquad \text{……..(10)}$$

$$u(1) = b_0 + 2b_1 \qquad \text{…….(11)}$$

And differentiation (6) with respect to t and put t=0 i.e

$$\frac{dx(t)}{dt}\bigg|_{t=0} = 2a_1 \qquad \text{…….(12)}$$

Substitute eqn.(12)and(7) into eqn.(2)

$$2a_1 = a_0 + b_0 \qquad \text{……(13)}$$

Rewrite eqn.(11),(13),(9)and (10) in the matrix form as

$$D = \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & -1 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} a_1 \\ b_0 \\ b_1 \\ x(1) \\ u(0) \\ u(1) \end{bmatrix} \text{ and } F = \begin{bmatrix} 0 \\ x(0) \\ x(0) \\ 0 \end{bmatrix} \qquad \text{…(14)}$$

Or DC=F

D is singular matrix, we will use theorem (Existence of the Moor-Penrose Inverse). To find the inverse matrix of D

Theorem: Let $D = BC$ be a full rank factorization of a nonzero matrix D. then $D^+ = C^T (CC^T)^{-1} (B^T B)^{-1} B^T$

Proof:[3]

From which we get the following system

$$C = D^+ F \qquad \text{….(15)}$$

Finally, Gauss elimination procedure is used to solve the above system to find

$c_0, c_1, c_2, c_3, c_4, c_5$.

when n=2 or 3 or ….. the same step follow

5.2: Using State Parameterization technique:

The idea of the state parameterization, using the polynomials Chebychev ($U_n^*(t), V_n^*(t), W_n^*(t)$) as a basis function, is to approximate the state variables as follows:

$$x(t) \approx \sum_{i=0}^{n} a_i U_i^*(t) \qquad \text{….(16)}$$

where, a_i are the unknown parameters. The control variables $u(t)$ are determined from the system state equation as function of the unknown parameters of the applied parameterization technique. Two case are considered, if the numbers of states and control variables are equal to n=m, that each state variable will be approximated by finite length polynomials series and control vector is obtained as function of the state variables. If the number of the state variables is greater than the number of control variables n>m, in this case, a set of the state variables is approximated which will enable us to find the remaining state variables and control variables as functions of this set.[1]

First, when n=1 , we have

$$x(t) \approx \sum_{i=0}^{1} a_i U_i^*(t) \qquad 0 \le t \le 1 \qquad \text{….(17)}$$

Substitute $U_0^*(t), U_1^*(t)$ into eqn.(17) to get

$$x(t) = \sum_{i=0}^{1} a_i \sum_{i=0}^{\lfloor 1/2 \rfloor} (-1)^i \binom{1-i}{i} (2(2t-1))^{1-2i} \qquad \text{…….(18)}$$

The control points a_i i=0,1 can be evaluated as follows:

Put t=0 and t=1 into (18) to get a_0, a_1 respectively, i.e

$$x(0) = a_0 - 2a_1 \qquad , x(0) = x_0 \qquad \text{….(19)}$$

$$x(1) = a_0 + 2a_1 \qquad \text{…..(20)}$$

And differentiation (18) with respect to t and put t=0 i.e

$$\frac{dx(t)}{dt}\Big|_{t=0} = 4a_1 \qquad \text{…..(21)}$$

Substitute eqn.(21) into eqn.(2)

$$4a_1 = x(0) + u(0) \qquad \text{……(22)}$$

Rewrite eqn.(19)-(20)and (22) in the matrix form as

$$D=\begin{bmatrix} 1 & -2 & 0 & 0 \\ 1 & 2 & 0 & -1 \\ 0 & 4 & -1 & 0 \end{bmatrix}, \quad C=\begin{bmatrix} a_0 \\ a_1 \\ u(0) \\ x(1) \end{bmatrix} \quad \text{and} \quad F=\begin{bmatrix} x(0) \\ 0 \\ x(0) \end{bmatrix} \qquad \text{….(23)}$$

or DC=F

from which we get the following system

or DC=F …… (24)

Finally, Gauss elimination procedure is used to solve the above system equation (24) to find c_0, c_1, c_2.

when n=2 or 3 or ….. the same step follow.

6. Examples of OC Problems:

In this section the performance of the proposed methods discussed in the previous section will be compared using several text examples.

Example (6.1)[7]

The first example contains one state variable and one control variable. The performance index to be minimized is

$$J = \frac{1}{2}\int_0^1 (2x^2 + u^2)dt \qquad \text{…..(25)}$$

Subject

$$\dot{x} = \frac{x}{2} + u \qquad x(0) = 1 \qquad \text{…..(26)}$$

This problem is solved by expanding $x(t)$ *and* $u(t)$ into three order Hermite series.
n=3 the state and control variables can be written as

$$x(t) = \sum_{i=0}^{3} a_i H_i(t) \qquad 0 \le t \le 1 \quad \text{…..(27)}$$

$$u(t) = \sum_{i=0}^{3} b_i H_i(t) \quad \text{….(28)}$$

Substitute $H_0(t), H_1(t), H_2(t), H_3(t)$ into eqn.(27)-(28) to get

$$x(t) = \sum_{i=0}^{3} a_i \sum_{i=0}^{\left\lfloor \frac{3}{2} \right\rfloor} (-1)^i \frac{3!}{i!(3-2i)!}(2t)^{3-2i} \quad \text{….(29)}$$

$$u(t) = \sum_{i=0}^{3} b_i \sum_{i=0}^{\left\lfloor \frac{3}{2} \right\rfloor} (-1)^i \frac{3!}{i!(3-2i)!}(2t)^{3-2i} \quad \text{…..(30)}$$

The control points $a_i, b_i \quad i = 0,1,2,3$ can be evaluated as follows:
Put t=0 and t=1 into (29)-(30) to get $a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3$ respectively,
i.e

$$x(0) = a_0 - 2a_2 \quad , x(0) = x_0 \quad \text{….(31)}$$

$$x(1) = a_0 + 2a_1 + 2a_2 - 4a_3 \quad \text{….(32)}$$

$$u(0) = b_0 - 2b_2 \quad \text{….(33)}$$

$$u(1) = b_0 + 2b_1 + 2b_2 - 4b_3 \quad \text{….(34)}$$

And differentiation (29) with respect to t and put t=0 i.e

$$\frac{dx(t)}{dt}\Big|_{t=0} = 2a_1 - 12a_3 \quad \text{…..(35)}$$

Substitute eqn.(35)and(28) into eqn.(26)

$$2a_1 - 12a_3 = \frac{1}{2} + b_0 - 2b_2 \quad \text{…..(36)}$$

To evaluate b_3, we needed two equations

$$\ddot{x}(0) = \frac{\dot{x}(0)}{2} + \dot{u}(0)$$

$$8a_2 = 2b_1 - 12b_3 \quad \text{…..(37)}$$

$$\dddot{x}(0) = \frac{\ddot{x}(0)}{2} + \ddot{u}(0)$$

$$48a_3 = 8b_2 \quad \text{…..(38)}$$

Rewrite eqn.(31)-(32)-(33)-(34)-(36)-(37)and(38)in the matrix form as

$$D = \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & -4 & 0 & -1 \\ 0 & 2 & 0 & -12 & 0 & -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 & -2 & 0 & 12 & 0 & 0 \\ 0 & 0 & 0 & 48 & 0 & 0 & 0 & -8 & 0 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} a0 \\ a1 \\ a2 \\ a3 \\ x(1) \\ b0 \\ b1 \\ b2 \\ b3 \\ u(0) \\ u(1) \end{bmatrix} \quad F = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}$$

Or DC=F ……(39)

Finally, Gauss elimination procedure is used to solve the above system equation (39) to find $c_i \quad i = 0,1,\ldots,11$.
The result using the state –control parameters are shown in tables (1).
The optimal values of J^* for each case are listed in table (2)
From table (2) we can conclude that, the results which are obtain from using state parameterization technique more accurate results than other.

Example (6.2)

Consider the same finite time linear quadratic problem which is Minimize

$$J = \int_{-1}^{1.58} \left(4x_1^2 + u^2\right) dt$$

subject to

$$\dot{x}_1 = x_2 \qquad x_1(0) = 2$$

$$\dot{x}_2 = u \qquad x_2(0) = 1$$

The first step in solving this problem by the proposed method is to transform the time interval to $t \in [0,1]$.this will lead to the following problem

$$J = 2.58\int_0^1 \left(4x_1^2 + u^2\right) dt \qquad \text{….(40)}$$

subject to

$$\dot{x}_1 = 2.58x_2 \qquad x_1(1/2.58) = 2 \qquad \text{…..(41)}$$

$$\dot{x}_2 = 2.58u \qquad x_2(1/2.58) = 1 \qquad \text{…..(42)}$$

This example contain two state variables $x_1(t)$ and $x_2(t)$ and one control variable $u(t)$, i.e, n=2 and m=1.

Here $x_1(t)$ is approximate by n=2,3,4,5,6,7 order Chebychev and Hermite series U_n^*, V_n^*, W_n^*, H_n of unknown parameter, then $x_2(t)$ is found from (41) using the differentiation property of the Chebychev and Hermite polynomials U_n^*, V_n^*, W_n^*, H_n that is used. The control variable $u(t)$ is obtained from (42). By substituting $x_1(t)$, $x_2(t)$ and $u(t)$ into (40), an expression of J^* can be found.

The same steps of section (6-2) we are follow to found $x_1(t)$, $x_2(t)$ and $u(t)$. The result using the state parameters are shown in tables (3).

From table (3) we notice that, the algorithms using Chebyshev and Hermite producer are accurate results.

Example (6.3)

Minimize

$$J = \int_0^{\infty} \left(2x_1^2 + x_2^2 + 0.4u^2\right) dt \qquad \text{…..(43)}$$

subject to

$$\dot{x}_1 = x_2 \qquad x_1(0) = -3 \qquad \text{…..(44)}$$

$$\dot{x}_2 = -0.5x_2 + 0.5u \qquad x_2(0) = 0 \qquad \text{….(45)}$$

In this example we will take t=10 and we will use the Hermite polynomials because it's more suitable than Chebyshev polynimoals. Because its limited from 0 to t .This example contain two state variables $x_1(t)$ and $x_2(t)$ and one control variable $u(t)$, i.e, n=2 and m=1. Here $x_1(t)$ is approximate by N=2,3,4,5,6,7,8,9,10 order Hermite series H_n of unknown parameter, then $x_2(t)$ is found from (44) using the differentiation property of the Hermite polynomials H_n that is used. The control variable $u(t)$ is obtained from (45). By substituting $x_1(t)$, $x_2(t)$ and

$u(t)$ into (43), an expression of J^* can be found. The same steps of section (6-2) we are follow to find $x_1(t)$, $x_2(t)$ and $u(t)$.

The values of the cost J for N=2,3,4,5,6,7,8,9,10 for the arbitrary final time t=10 are displayed in table (4).

7. Conclusion:

In the paper, approximated techniques were proposed to solve optimal control problems. These techniques are based on using the parameterization of the system state and state- control using Hermite polynomials and Chebyshev(second, three. fourth kind) polynomials. The State Parameterization technique is better than State-Control Parameterization technique because it requires less time and effort .

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Table (1)

State–Control Parameterization Using Hermite polynomia			
a_0	0.2379	b_0	-0.1066
a_1	0.1967	b_1	0.1193
$a2$	-0.3811	$b2$	0.0942
a_3	0.0157	b_3	0.1755
J^*	0.82527057		

Table (2)

J^*	**Hermite polynomials** State-Control Parameterization	State Parameterization	**Chebyshev polynomials State Parameterization** U_n^*	$,V_n^*$	$,W_n^*$
1	0.89731757	0.96875000	0.96875000	0.96875000	0.96875000
2	0.88024408	0.86472603	0.86472603	0.86472603	0.86472603
3	0.82527057	0.86421807	0.86421807	0.86421807	0.86421807

Table (3)

	Chebyshev Polynomials			**Hermite Polynomials**
n	$U_n^*(t)$	$V_n^*(t)$	$W_n^*(t)$	$H_n(t)$
		J^*		
2	45.66553695	45.66553695	45.66553695	45.66553695
3	45.66534248	45.66534248	45.66534248	45.66534248
4	44.85976925	44.85976925	44.85976925	44.85976925
5	44.85930544	44.85930544	44.85930544	44.85930544
6	44.60621591	44.60621591	44.60621591	44.60621591
7	44.59908693	44.59908693	44.59908693	44.59908693

Table (4)

J^*	**Hermite polynomials**
2	85.32693764
3	53.46918734
4	38.56685909
5	31.57665671
6	29.06042461
7	28.47505531
8	28.40099598
9	28.39702050
10	28.39699903