# Generalized Sum of Fuzzy Subgroup and α-cut Subgroup

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**Abstract:**

*In this paper we study some results of the generalized sum of a fuzzy subgroup and α-cut subgroup ,we define a α-cut subset and α-cut subgroup, and then .We study some of their properties.*

**الخلاصة:-**

في هذا البحث ندرس بعض النتائج في تعميم الجمع للزمر الجزئية الضبابية ومستوي القطع -$\alpha$ للزمر الجزئية الضبابية ، عرفنا مستوي القطع -$\alpha$ للمجموعات الجزئية و مستوي القطع -$\alpha$ للزمر الجزئية ،وبعد ذلك ندرس البعض من خصائصهم.

## 1-Introduction:

*In 1965 Zadeh [5] mathematically formulated the fuzzy subset concept. He defined fuzzy subset of a non-empty set as a collection of objects with grade of membership in a continuum, with each object being assigned a value between 0 and 1 by a membership function. Fuzzy set theory was guided by the assumption that classical sets were not natural, appropriate or useful notions in describing the real life problems, because every object encountered in this real physical world carries some degree of fuzziness. Further the concept of grade of membership is not a probabilistic concept. We introduce the concept of fuzzy set ,fuzzy subgroup and T-norme.*
*In this work, we first generalize the results of the sum of two fuzzy subsets and fuzzy subgroup .We also α-cut subsets and α-cut subgroup , and then we study some of their properties.*

## 2-Preliminaries :

*We record here same basic concepts and clarify notions used in the sequel.*

***Definition (2-1):* [5,4,1]**

*A fuzzy subset μ of a group (G, +) is said to be a fuzzy sub group of G if for all x,y in G*

1-$\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$

2-$\mu(-x) = \mu(x)$

*Where the addition of x and y is denoted by x + y and the inverse of x by -x*

***Definition(2 -2):* [4,3]**

*A triangular norm ( briefly a t-norm ) is a function $T:[0,1]\times[0,1] \rightarrow [0,1]$ satisfying for each (a , b , r , s) ϵ[0,1]*

1-$T(a,1)=a$

2-$T(a,b) \leq T(r,s)$ if $a \leq r$ and $b \leq s$

3-$T(a,b) = T(b,a)$

4-$T(a, T(a,r)) = T(T(a,b),r)$

***Definition (2-3):* [1]**

*Let G be a groupoid and T a t-norm .*

*A function B:→ [0.1] is a subgroupoid of G iff for every x,y in G , $B(x,y) \geq T(B(x),B(y))$.*

*If G is a group , a t- fuzzy subgroupoid B is at -fuzzy subgroup of G iff for each x ϵ G , B(-x)= B(x)*

***Definition (2-4):* [5,1]**

***Let** μ be fuzzy subset of a set S and let α ϵ[0,1].the set*

$\mu_\alpha=\{x \in S : \mu(x) \geq \alpha\}$ *is called α-cut subset of μ*

***Definition (2-5):[2]***

*For each $i = 1,2,\dots,n$, let $\mu_i$ be a fuzzy sets of set $X_i$, $\lambda_1, \lambda_2, \dots \lambda_n \in F$ then*

*1-$\lambda_1 \mu_1 + \lambda_2 \mu_2 + \dots + \lambda_n \mu_n \subset \mu$*

*2-for all $x_1, \dots, x_n$ $X$, we have*

$$\mu(\lambda_1 x_1, \lambda_2 x_2, \dots, \lambda_n x_n) \geq \{\min\{\mu_1(x_1), \mu_2(x_2), \dots, \mu_n(x_n)\}$$

***Definition (2-6): [5]***

*Let $f:X\rightarrow Y$ be a function for a fuzzy set $\mu$ in $Y$, we define*
*$(f^{-1}(\mu))(x)= \mu(f(x))$ for every $x \in X$*
*For a fuzzy set $\lambda$ in $X$, $f(\lambda)$ is defined by*

$$(f(\lambda))(y)= \begin{cases} \sup \lambda(x) & \text{if } f(z) = y, z \in X \\ 0 & \text{otherwise} \end{cases}$$

*where $y \in Y$*

***Definition (2-7): [2]***

*Let $\mu_1, \mu_2, \dots, \mu_n$ be a fuzzy sets of set $X$, define $f(\mu)= \mu_1 + \mu_2 + \dots + \mu_n$, where $\mu = \mu_1 \times \mu_2 \times \dots \times \mu_n$, and $f: X^n \rightarrow X$, where $f(x_1, x_2, \dots, x_n) = x_1 + x_2 + \dots + x_n$, $X^n = X_1 \times X_2 \times \dots \times X_n$*

***Definition (2-8):[2]***

*Let $\mu_1, \mu_2, \dots, \mu_n$ be a fuzzy sets of set $X_1, X_2, \dots, X_n$ respectively. Define $\mu = \mu_1 \times \mu_2 \times \dots \times \mu_n$, by $\mu(x_1, x_2, \dots, x_n) = \min\{\mu_1(x_1), \mu_2(x_2), \dots, \mu_n(x_n)\}$*

## ***3- Generalized Sum of Fuzzy Subgroup***

***Theorem(3-2) :-[2]***

*Let $\mu_1, \mu_2, \dots, \mu_n$ be a fuzzy subsets of the sets $G_1, G_2, \dots, G_n$ respectively. Then*

$$(\mu_1 + \mu_2 + \dots + \mu_n)(Z) = Sup_{x_1+x_2+\dots+x_n=Z}\{\min\{\mu_1(x_1), \mu_2(x_2), \dots, \mu_n(x_n)\}$$

***Proof :-***

*Since $f(\mu) = \mu_1 + \mu_2 + \dots + \mu_n$, using Definition(2-7)and Definition(2-6)*

$$(\mu_1+\mu_2 + \dots + \mu_n)(Z) = sup_{f(x)=z}\, \mu(x),\ x = (x_1, x_2, \dots, x_n) \in G^n$$

*$f(x) = Z$*

*Since $f(x) = f(x_1, x_2, \dots, x_n) = x_1 + x_2 + \dots + x_n$, using Definition(2-8)*

$$\mu(x) = \mu(x_1, x_2, \dots, x_n) = \min\{\mu_1(x_1), \mu_2(x_2), \dots, \mu_n(x_n)\}$$

*Then* $(\mu_1+\mu_2+\ldots+\mu_n)(Z) =$
$Sup_{x_1+x_2+\cdots+x_n=z}\{min\{\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n)\}$

**Theorem(3-2) :-**

*Let* $\mu_1, \mu_2, \ldots, \mu_n$ *be a fuzzy subgroups of the groups* $G_1, G_2, \ldots, G_n$ *respectively then* $(\mu_1+\mu_2+\ldots+\mu_n)$ *is fuzzy subgroup of* $G_1, G_2, \ldots, G_n$

**Proof :-**

*We must show that* $(\mu_1+\mu_2+\ldots+\mu_n)$ *is fuzzy subgroup of* $G_1, G_2, \ldots, G_n$ *for all elements* $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in G_1, G_2, \ldots, G_n$

*We get*

$(\mu_1+\mu_2+\ldots+\mu_n)(x_1, x_2, \ldots, x_n)+(y_1, y_2, \ldots, y_n) = (\mu_1+\mu_2+\ldots+\mu_n)(x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$

*Let* $(x_1+y_1, x_2+y_2, \ldots, x_n+y_n) = Z$

$= \underset{(x_1+y_1, x_2+y_2, \ldots, x_n+y_n)=z}{Sup} min\{\mu_1(x_1+y_1), \mu_2(x_2+y_2), \ldots, \mu_n(x_n+y_n)\}$

$\geq \underset{(x_1+y_1, x_2+y_2, \ldots, x_n+y_n)=z}{Sup} min\{min(\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n)), min(\mu_1(y_1), \mu_2(y_2), \ldots, \mu_n(y_n))\}$

$= min\{(\mu_1+\mu_2+\ldots+\mu_n)(x_1, x_2, \ldots, x_n), (\mu_1+\mu_2+\ldots+\mu_n)(y_1, y_2, \ldots, y_n)\}$

*Also*

$(\mu_1+\mu_2+\ldots+\mu_n)(Z) = \underset{x_1+x_2+\cdots+x_n=Z}{Sup} min\{\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n)\}\}$

*Since* $(\mu_1+\mu_2+\ldots+\mu_n)$ *is fuzzy subgroups of* $G_i$

$(\mu_1+\mu_2+\ldots+\mu_n)(Z) = \underset{x_1+x_2+\cdots+x_n=Z}{Sup} min\{\mu_1(-x_1), \mu_2(-x_2), \ldots, \mu_n(-x_n)\}$

$= (\mu_1+\mu_2+\ldots+\mu_n)(-x_1, -x_2, \ldots, -x_n)$

$= (\mu_1+\mu_2+\ldots+\mu_n)(-(x_1, x_2, \ldots, x_n))$

*Thus* $(\mu_1+\mu_2+\ldots+\mu_n)$ *is fuzzy subgroups of* $G_i$

## 4- α-cut subgroup

*In this section, we introduce a definition of α-cut subgroup*

**Definition(4-1) :-**

*Let* $\mu$ *be a fuzzy subset of a set* $G$ *,* $T$ *a t-norm and* $\alpha \in [0,1]$*, then we define* $\alpha$ *–cut subset of a fuzzy subset* $\mu$ *as*

$\mu_\alpha^T = \{x \in G : sup_{x\in G} T(\mu(x), \alpha) \geq \alpha\}$

**Theorem(4-2) :**

*Let G be a group and* $\mu$ *a t- fuzzy subgroup of (G,+) then* ***α –cut*** *subset* $\mu_\alpha^T$ *is fuzzy subgroup of (G,+) where e is the identity of G*

**Proof :**

*Let* $x, y \in \mu_\alpha^T$ *, then* $sup_{x\in G} T(\mu(x), \alpha) \geq \alpha$ *and* $sup_{y\in G} T(\mu(y), \alpha) \geq \alpha$ *,*

*since* $\mu$ *t-fuzzy subgroup of G*

*then $\mu(x+y) \geq T(\mu(x), \mu(y))$ is satisfied .This means*

$$\sup_{x,y\in G} T(\mu(x+y), \alpha) \geq \sup_{x,y\in G} T(T(\mu(x), \mu(y), \alpha))$$
$$= \sup_{x\in G} T(\mu(x), \alpha) \text{ or } \sup_{y\in G} T(\mu(y), \alpha)$$
$$\geq \sup_{x\in G} T(\mu(x), \alpha) \geq \alpha \text{ or } \sup_{y\in G} T(\mu(y), \alpha) \geq \alpha$$

*hence $x+y \in \mu_\alpha^T$*
*A gain $x \in \mu_\alpha^T$ implies $\sup_{x\in G}(\mu(x), \alpha) \geq \alpha$*
*since $\mu$ is a t-fuzzy subgroup $\mu(-x)= \mu(x)$ and hence*
$\sup_{x\in G} T(\mu(-x), \alpha)= \sup_{x\in G} T(\mu(x), \alpha) \geq \alpha$
*this means that $-x \in \mu_\alpha^T$*

***Theorem (4-3)**:*
*Let (G,+) be a group and **$\mu$** a fuzzy subgroup then the α-cut subset $\mu_\alpha^T$*
*For $\alpha \in [0,1]$ is a subgroup of G ,where e is identity of G*

***Proof:***
*Let $x,y \in \mu_\alpha^T$ ,then $\sup_{x\in G} T(\mu(x), \alpha) \geq \alpha$ and $\sup_{y\in G} T(\mu(y), \alpha) \geq \alpha$*
*since $\mu$ is subgroup of G*
*$\mu(x+y) \geq \min(\mu(x), \mu(y))$ is satisfied ,this means*

*$\sup_{x,y\in G} T(\mu(x+y), \alpha) \geq \sup_{x,y\in G} T(\min(\mu(x), \mu(y)), \alpha)$,where there are two cases*
*$\min(\mu(x), \mu(y))= \mu(x)$ or $\min(\mu(x), \mu(y))= \mu(y)$ since $x,y \in \mu_\alpha^T$*
*Also in to case $\sup_{x,y\in G} T(\min(\mu(x), \mu(y)), \alpha) \geq \alpha$*

*therefore $\sup_{x,y\in G} T(\mu(x+y), \alpha) \geq \alpha$,thus we get $x+y \in \mu_\alpha^T$*
*it is easily seen that ,as above $-x \in \mu_\alpha^T$*
*Hence $\mu_\alpha^T$ is a subgroup of G*

***Theorem(4-4)**:*
*Let $\mu$ and $v$ be α-cut subsets of the sets G,H respectively, and $\alpha \in [0,1]$,*
*then $\mu+v$ is also a α-cut subset of G+H*

***Proof:***
*Since any t-norm T is associative ,using definition(4-1)and definition(2-2) we can write ,the following statements.*

$$\sup_{x,y\in G,H} T((\mu+v)(x+y), \alpha)= \sup_{x,y\in G,H} T(\sup \min(\mu(x), v(y)), \alpha)$$
$$= \sup_{x\in G} T(\mu(x), \sup_{y\in G} \min(v(y), \alpha))$$
$$\geq \sup_{x\in G} T(\mu(x), \alpha)$$
$$\geq \alpha$$

***Definition(4-5) :- [1]***

*Let (G,+) be a group and $\mu$ a t-fuzzy subgroup of G.The subgroups $\mu_\alpha^T$,*
*$\alpha \in [0,1]$ and $\sup T(\mu(e), \alpha) \geq a$ are called α-cut subgroup of*

***Theorem(4-6)***:

*Let (G,+),(H,+)be two groups μ,v a t-fuzzy subgroup of G and H, respectively ,then the α-cut subset* $(\mu+v)_\alpha{}^T$ *for α ϵ[0,1],is subgroup of G+H where e and e are identities of G+H, respectively.*

***Proof:***

$$(\mu+v)_\alpha{}^T=\{(x,y)\in G+H: sup_{\substack{x\in G\\ y\in H}} T((\mu+v)(x+y),\alpha)\ \geq\ \alpha\}$$

*let* $(x_1,x_2),(y_1,y_2)\ \in(\mu+v)_\alpha{}^T$ *,then*

$sup\ T((\mu+v)(x_1+y_1),\alpha)\ \geq \alpha$

$sup\ T((\mu+v)(x_1+y_1),\alpha)\ \geq \alpha$ *.since(μ+v) is a t-fuzzy group of G+H ,*

*We get*

$$\begin{aligned} sup\ T((\mu+v)((x_1+x_2)+(y_1+y_2),\alpha) &\geq sup\ T(sup\ T(\mu(x_1+x_2),v(y_1+y_2)),\alpha)\\ &= sup\ T(\mu(x_1+x_2),\ sup\ T(v(y_1+y_2)))\\ &\geq sup\ T(\mu(x_1+x_2),\alpha)\geq\alpha \end{aligned}$$

*Hence* $(x_1+x_2),(y_1+y_2)\in(\mu+v)_\alpha{}^T$

*A gain* $(x,y)\in(\mu+v)_\alpha{}^T$ *implies*

$$\begin{aligned} sup\ T((\mu+v)\ -(x+y),\alpha) &= sup\ T((\mu+v)((-x)+(-y)),\alpha)\\ &= sup\ T(sup\ T(\mu(-x),(v(-y)),\alpha)\\ &= sup\ T(\mu(-x),\ sup\ T(v(-y)),\alpha))\\ &\geq sup\ T(\mu(-x)),\alpha)\geq\alpha \end{aligned}$$

*This means that* $(-x,-y)\ \in(\mu+v)_\alpha{}^T$ *therefore* $(\mu+v)_\alpha{}^T$ *is a subgroup of G+H*

***Theorem(4-7):***

*Let* $\mu_1, \mu_2,\ldots\ldots,\mu_n$ *be a fuzzy subgroup of the groups* $G_1, G_2,\ldots..,G_n$ *respectively, and let α ϵ[0,1],then*

$$(\mu_1+\mu_2+\ldots\ldots+\mu_n)_\alpha{}^T=\mu_{1\alpha}{}^T+\mu_{2\alpha}{}^T+\ldots\ldots+\mu_{n\alpha}{}^T$$

***Proof:***

*Let* $(x_1,x_2,\ldots,x_n)$*be an element of* $(\mu_1+\mu_2+\ldots\ldots+\mu_n)_\alpha{}^T$

*Then using definition (4-1)and definition (2-4) we can write*

$$supT\left((\mu_1+\mu_2+\ldots+\mu_n)(x_1,x_2,\ldots,x_n),\alpha\right)=$$

$$SupT\ \left(min\left(\mu_1(x_1),\mu_2(x_2),\ldots.,\mu_n(x_n)\right),\alpha\right)$$

*by theorem(3-1)*

*For all i=1,…,n* $min\left(\mu_1(x_1),\mu_2(x_2),\ldots.,\mu_n(x_n)\right)=\mu_i(x_i)$

*This given us*

$SupT\ \left(min\left(\mu_1(x_1),\mu_2(x_2),\ldots.,\mu_n(x_n)\right),\alpha\right)=SupT(\mu_i(x_i),\alpha)\geq\alpha$*.Thus we have* $x_i\in\mu_{i\alpha}{}^T$ *.That is* $(x_1,x_2,\ldots,x_n)\in\ \mu_{1\alpha}{}^T+\mu_{2\alpha}{}^T+\cdots+\mu_{n\alpha}{}^T$

*Similarly , let* $(x_1,x_2,\ldots,x_n)$*be an element of* $\mu_{1\alpha}{}^T+\mu_{2\alpha}{}^T+\cdots+\mu_{n\alpha}{}^T$

*Then for all i=1,2,…,n ,* $x_i \in \mu_{i\alpha}{}^T$*.that is* $SupT(\mu_i(x_i),\alpha) \geq \alpha$.
*Since* $min\left(\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n)\right) = \mu_i(x_i)$ *and* $SupT(\mu_i(x_i),\alpha) \geq \alpha$ *,we get*

$$supT\left((\mu_1 + \mu_2 + \ldots + \mu_n)(x_1, x_2, \ldots, x_n), \alpha\right) =$$
$$SupT\left(min\left(\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n)\right), \alpha\right) = SupT(\mu_i(x_i), \alpha) \geq \alpha$$

*Thus* $(x_1, x_2, \ldots, x_n) \in (\mu_1 + \mu_2 + \ldots\ldots + \mu_n)_\alpha{}^T$.

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## الخلاصة:-

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