On certain properties of α^{**}-continuous functions

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Abstract

In this work , we introduce and study a new type of continuous functions, which we call α^{**} - continuous function , these are the functions in which the inverse image of α- open set is also α- open. Several properties of these functions are proved .

الخلاصة

في هذا البحث نقدم وندرس نوع جديد من الدوال المستمرة والتي رمزنا لها بالرمز (α^{**} continuous function-) هذه الدوال تحقق ان الصورة العكسية للمجموعات من النوع (α- open) ايضا تكون (α- open). العديد من خواص هذا النوع من الدوال قد تم برهانها .

Introduction:

In 1965, O.Najstad [٢] introduce the concept of α- open set as follows: Let (X,τ) be a topological space, let $A \subseteq X$. We say that A is α- open in X if $A \subseteq \text{Int cl int}A$.
Where IntA means Interior of the set A, and cl A means the closure of A.
It is clear that every open set is α- open, but the converse is not necessarily true. In this work, we introduce and study a new type of continuous functions, which we call α^{**}-continuous functions these are the functions in which the inverse image of α- open set is also α- open we will use the symbol (□) to indicate the end of the proof.

1-Basic definitions:

In this section we recall and introduce the basic definition needed in this work .

1-1 definition:
Let (X,τ) and (Y,F), be two topological spaces, $f{:}X \to Y$ be a function. We say that f is continuous if the inverse image of every open set in Y is an open set in X. Equivalently, $f{:}X \to Y$ is continuous if for every $x \in X$ and for every V open set in Y containing $f(x)$, $\exists$ an open set U in X containing x such that $f(U) \subseteq V$.

1-2 definition:
Let (X,τ) be a topological space and $A \subseteq X$. We say that A is α- open in X if $A \subseteq \text{Int cl int}(A)$.
Every open set is α- open while the converse is not necessarily true as it is shown in the following example.

1-3 Example:

Let X= {a, b, c, d}

$\tau = \{\Phi, X, \{a\}\}$

A= {a, b} $\subset$X

A α- open but not open when IntA={a} , c lint A= cl{a}

τ^{c}={X, Φ, {b,c,d}}

cl intA= cl{a}=X, hence A$\subset$X.

Then A is α- open but not open.

The collection of all α- open sets in X forms a topology on X which is denoted by τ^{α}

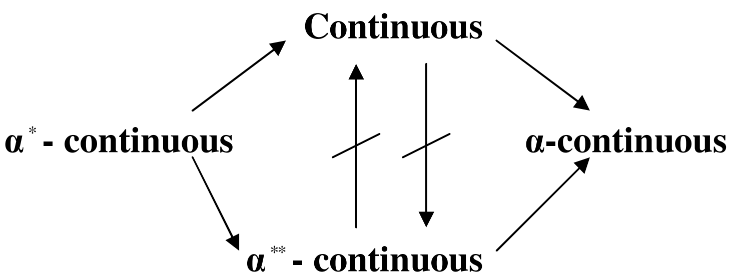
It is clear that $\tau \subseteq \tau^{\alpha}$

1-4 Definitions:

Let f:X$\to$Y be a function. We say that:

1- f is **α-continuous** if the inverse image of every open is α- open .
2- f is α^{*}-continuous if the inverse image of α- open is open .
3- f is α^{**}- continuous if the inverse image of α- open is α- open

The following diagram explain the relations between these types



α^{*}- continuous $\longrightarrow$ Continuous

Define$f$: X$\to$Y which is α^{*}- continuous function and let V an open set in Y by definition (1-2) V is α- open , f is α^{*}- continuous then the inverse image of α- open is Open in X hence f is continuous.

the Proof of the other parts are similar

1-5 Remark:

The concepts of continuous functions and $\boldsymbol{\alpha}^{**}$ **- continuous** functions are independent for example:

1- Let X= { a, b, c, d } , τ_x ={ ϕ, X, {a}, {b, c}, {a, b, c}}

and Y={x, y, z} , τ_y ={ ϕ, Y, {x}}

Define f:X$\rightarrow$Y by

f(a)=x, f(b)=y, f(c)= f(d)=z

Then f is continuous but it is not $\boldsymbol{\alpha}^{**}$ **- continuous**

2- Let X= {a, b, c, d}, τ_x = { ϕ, X, {a}}

And Y={x, y, z}, τ_y = { ϕ, Y, {x}}

Define f:X$\rightarrow$Y by

f(a)= f(b)=x, f(c)=y , f(d)=z

Then f is $\boldsymbol{\alpha}^{**}$ **- continuous** but it is not continuous.

1-6 Definition:

The complement of α- open set is called α-closed.

1-7 Remark:

f: X$\rightarrow$Y is $\boldsymbol{\alpha}^{**}$ **- continuous** iff the inverse image of α-closed is also α-closed

2-Main results:

In this section, we state and prove several properties of $\boldsymbol{\alpha}^{**}$ **- continuous** functions.

2-1 Theorem:

Let f: X$\rightarrow$Y and g: Y$\rightarrow$ Z are α^{**} - continuous functions

Then g$\circ$ f: X $\rightarrow$Z is also α^{**} - continuous.

Proof:

Let V be an α-open set in Z. Since g is α^{**} - continuous therefore the inverse image g^{-1}(v) is α-open in Y, and f is α^{**} - continuous therefore the inverse image $f^{-1}(g^{-1}(v))=(g\circ f)^{-1}(v)$ is α- open in X. This implies that g$\circ f$ is an α^{**} - continuous □

2-2 theorem:

Let f :X$\rightarrow$Y be a function of topological spaces. Then the following statement are equivalent:

1- f is $\boldsymbol{\alpha}^{**}$ **- continuous**
2- if x$\in$X , V is α- open in Y containing f(x), then $\exists$ α- open U in X containing x such that f(U) $\subseteq$V.

Proof:

(1)$\Rightarrow$(2)

Let V be α- open in Y and f(x)$\in$ V. Since f is $\boldsymbol{\alpha}^{**}$ **- continuous,** f^{-1}(V) is α- open in X and x$\in$ f^{-1}(V). Put U= f^{-1}(V) therefore x$\in$ U and f (U) = f (f^{-1}(V))$\subseteq$V.

(2)$\Rightarrow$(1)

Let V be α- open in Y and x$\in f^{-1}(V)$. Then f(x) $\in$ V therefore, $\exists$ a U_x α- open in X such that x$\in U_x$ and $f(U_x)\subseteq$V. Therefore x$\in U_x \subseteq f^{-1}(V)$.

This implies that $f^{-1}(V)$ is a union of α- open sets , hence $f^{-1}(V)$ is α- open in X, so f is **α^{**}- continuous** □

Before, we state the theorem (2-5) we introduce and recall the following definition and remark.

2-3 Definition:

An α- open set which is closed is termed C- α- open.

2-4 Remark:

Let B $\subseteq$ A$\subseteq$X. If A is closed in X and B is α- open in X, then B is α- open in A.

2-5 Theorem:

If f: X$\rightarrow$Y is **α^{**}- continuous** and A is C- α- open in X , then the restriction g= f/A : A$\rightarrow$Y is **α^{**}- continuous** .

Proof:

Let V be α- open in Y. Since f is **α^{**}- continuous** therefore $f^{-1}(V)$ is α- open in X.
Now A is α- open in X, then $f^{-1}(V)\cap A$ is α- open in X but A is closed , hence $f^{-1}(V)\cap A$ is α- open in A, but $g^{-1}(V)=(f/A)^{-1}(V)= f^{-1}(V)\cap A$.
So $g^{-1}(V)$ is α- open in A which means that g is **α^{**}- continuous □.**

2-6 Remark:

If A is closed only, then f/A is not always **α^{**}- continuous**.

For if we take:

X={a, b, c, d} , τ_x = { ϕ, X, {a}}

Y={x, y, z} , τ_y = { ϕ, Y, {x}}

A= {b, c, d}

Define f:X$\rightarrow$Y by

f(a)= f(b)=x

f(c)=y , f(d)=z

Then f is **α^{**}- continuous** but f/A is not **α^{**}- continuous.**

Before, we state the next theorem; we introduce and recall the following definition and remark.

2-7 Remark:

If A is α- open in X and B is α- open in Y then A×B is α- open in X×Y.

2-8 definition:

A space X is said to be α-Hausdorf (α-T_2) if for any two distinct points x, y of X, $\exists$ disjoint α- open sets U, V of X such that $x\in U$, $y\in V$.

2-9 Definition:

Let f: $X\rightarrow Y$ be a function. The subset $\{(x, f(x)| x\in X\}$ of $X\times Y$ is called the graph of f and is denoted by G(f)

It is well known that G(f) is a closed set of $X\times Y$ whenever f is continuous and Y is T_2 .

2-10 Theorem:

If f: $X\rightarrow Y$ is **α^{**} - continuous** and Y is α-T_2 then G(f) is α-closed in $X\times Y$.

Proof:

Let $(x, y) \in X\times Y - G(f)$, then $y\neq f(x)$.

But Y is α-T_2 , so $\exists$ disjoint α- open sets W and V in Y $\ni$ $f(x) \in W$ and $y\in V$.

Since f is **α^{**} - continuous** therefore $\exists$ U α- open in X such that $x\in U$ and $f(U) \subseteq W$.Now $(x, y) \in U\times V \subseteq X\times Y - G(f)$.

Since $U\times V$ is α- open in $X\times Y$, hence $X\times Y$-G (f) is a union of α- open sets.

Therefore, $X\times Y$-G (f) is α- open.

Consequently, G (f) is α-closed in $X\times Y$.

Let f: $X\rightarrow Y$ and g: $X\rightarrow Y$ be tow functions of topological spaces. It is well known that the set (called difference kernel) $A=\{x\in X \mid f(x) = g(x)\}$ is closed in X whenever f and g are continuous and Y is T_2 .

An analogous result can be given as follows:

2-11 Theorems:

If f and g are two **α^{**} - continuous** functions from a space X into an α-T_2 space Y then the set $A=\{x\in X| f(x) = g(x)\}$ is α-closed in X.

Proof:

Let $x\in X$-A then $f(x)\neq g(x)$.

Since Y is α-T_2 $\exists$ disjoint α- open sets U and V in Y $\ni$ $f(x) \in U$ and $g(x)\in V$.

Therefore $f^{-1}(U)$ and $g^{-1}(V)$ are α- open sets in X.

Let $B= f^{-1}(U) \cap g^{-1}(V)$, therefore $x\in B$ and B is α- open in X.

Moreover $B\cap A=\phi$.

For otherwise $U \cap V\neq \phi$. Consequently, $x \in B\subseteq X-A$.

So X-A is a union of α- open sets in X , and thus X-A is α- open , which means that A is α- closed in X □.

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