Completion of Fuzzy Normed Spaces

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Received on:29/3/2011
Accepted on:2/6/2011

Abstract

In this paper we recall the definition of fuzzy normed space. Much attention is paid to the concept of completeness a property which a fuzzy normed space may or may not have. We prove that for a fuzzy normed space there is always a competition. Also we study the relation between ordinary normed space and fuzzy normed space.

Keywords; fuzzy normed space, fuzzy continuous, fuzzy isometries, fuzzy sequence.

كمال الفضاءات القياسية الضبابية

الخلاصة

في هذا البحث استخدمنا تعريف الفضاءات القياسية الضبابية ثم اعطينا اهتماماً لمفهوم الكمال والتي هي صفة قد تمتلكها الفضاءات القياسية الضبابية وقد لا تمتلكها وبرهنا لأي فضاء قياسي ضبابي دائماً يوجد له فضاء قياسي كامل وكذلك قمنا بدراسة العلاقة بين الفضاءات القياسية الاعتيادية والفضاءات القياسية الضبابية.

S1: Basic concepts About Fuzzy Sets

Definition 1.1:[1]

Let X be any set of elements. A fuzzy set $\tilde{A}$ in X is characterized by a membership function. $\mu_{\tilde{A}}(x): X \to [0,1]$. Then $\tilde{A}$ can be written by

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x) | x \in X, 0 \leq \mu_{\tilde{A}}(x) \leq 1\}$$

Definition 1.2:[1]

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets of X then

1. $\tilde{A} \subseteq \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$ for all x ∈ X.
2. $\tilde{A} = \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$ for all x ∈ X.
3. The complent of $\tilde{A}$ (denoted by $\tilde{A}^C$) is aslo a fuzzy set with membership function $\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x)$ for all x ∈ X.
4. $\tilde{A} = \emptyset$ if and only if $\mu_{\tilde{A}}(x) = 0$ for all $x \in X$. where $\emptyset$ is the empty fuzzy set.

Definition 1.3:[1]

A fuzzy point P_x in X is a fuzzy set with membership function

$$\mu_{Px}(y) = \begin{cases} \propto & \text{if } y = x. \\ o & \text{otherwise.} \end{cases}$$

for all $y \in X$ where $o < \propto < 1$. We denote this fuzzy point by $x_\propto$ or $(x, \propto)$.

Definition 1.4:[2]

Two fuzzy points $x_\propto$ and y_β are said to be equal if

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https://doi.org/10.30684/etj.29.10.12
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$x = y$ and $\propto = \beta$ where $\propto, \beta \in (0,1]$.

Definition 1.5:[1]

Let $x_\propto$ be a fuzzy point and $\tilde{A}$ a fuzzy set in *X*. Then $x_\propto$ is said to be in $\tilde{A}$ or belongs to $\tilde{A}$ denoted by $x_\propto \in \tilde{A}$ if $\propto \leq \mu_{\tilde{A}}(x)$.

Definition 1.6:[3]

Let f be a function from a nonempty set *X* to a nonempty set *Y*. If $\tilde{B}$ is a fuzzy set in *Y* then $f^{-1}(\tilde{B})$ is a fuzzy set in *X* with membership function, $\mu_{f^{-1}(\tilde{B})} = \mu_{\tilde{B}}$ of. If $\tilde{A}$ is a fuzzy set in *X* then $f(\tilde{A})$ is a fuzzy set in *Y* with membership

$$\mu_{f(\tilde{A})}(y) = \begin{cases} sup\{\mu_{\tilde{A}}(x) | x\epsilon f^{-1}(y)\} & if \ \ f^{-1}(y) \neq \emptyset \\ 0 & other\ wise \end{cases}$$

For all $y \in Y$ where $f^{-1}(y) = \{x \in X | f(x) = y\}$

Proposition 1.7:[2]

Let $f: X \to Y$ be a function then for a fuzzy point $x_\propto$ in $X, f(x_\propto)$ is a fuzzy point in *Y* and $f(x_\propto) = f(x)_\propto$.

Definition 1.8:[4]

Let X and Y be a given nonempty sets. Let $\tilde{A}$ be a fuzzy set in X and $\tilde{B}$ be a fuzzy set in Y. Any subset $\tilde{R}$ of the Cartesian product $\tilde{A} \times \tilde{B}$ is called a fuzzy relation where $\tilde{A} \times \tilde{B}$ is a fuzzy set with membership $\mu_{\tilde{A}\times\tilde{B}} = \min\{\mu_{\tilde{A}}(\mathrm{x}), \mu_{\tilde{B}}(\mathrm{x})\}$.

Equivalence fuzzy relation on $\tilde{A}$ is a fuzzy relation $\tilde{R}$ of $\tilde{A} \times \tilde{A}$ such that

(i) $\tilde{R}(x_\propto, x_\propto)$ for all $x_\propto \in \tilde{A}$.

(ii) $\tilde{R}(x_\propto, y_\beta)$ implies $\tilde{R}(y_\beta, x_\propto)$ for all $x_\propto, y_\beta \in \tilde{A}$.

(iii) $\tilde{R}(x_\propto, y_\beta)$ and $\tilde{R}(y_\beta, z_\sigma)$ implies $\tilde{R}(x_\propto, z_\sigma)$ for all $x_\propto, y_\beta, z_\sigma \in \tilde{A}$.

When $\tilde{R}$ is equivalence fuzzy relation on $\tilde{A}$ then $\tilde{R}(x_\propto, y_\beta)$ written $x_\propto \sim y_\beta$ in this case (i), (ii) and (iii) because

(i) $x_\propto \sim x_\propto$ for all $x_\propto \in \tilde{A}$.

(ii) if $x_\propto \sim y_\beta$ then $y_\beta \sim x_\propto$ for all $x_\propto, y_\beta \in \tilde{A}$.

(iii) if $x_\propto \sim y_\beta$ and $y_\beta \sim z_\sigma$ then $x_\propto \sim z_\sigma$ for all $x_\propto, y_\beta, z_\sigma \in \tilde{A}$

The equivalence class of $x_\propto \in \tilde{A}$ denoted by $\hat{x}_\propto = \{y_\beta \in \tilde{A} | y_\beta \sim x_\propto\}$

Definition 1.9:[5]

Let *X* be a vector space over $\mathbb{K}$ and let $\tilde{A}$ be a fuzzy set in *X*. Then $\tilde{A}$ is called a fuzzy subspace of *X* if for all $x, y \in X$ and $\lambda \in \mathbb{K}$

(i) $\mu_{\tilde{A}}(x + y) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$

(ii) $\mu_{\tilde{A}}(\lambda x) \geq \mu_{\tilde{A}}(x)$

S2: Fuzzy Normed Space

Definition 2.1:[6]

Let *X* be a vector space over field $\mathbb{K}[\mathbb{K} = \mathbb{R}\ or\ \mathbb{K} = \mathbb{C}]$. Let $\|.\|_f: X \to [0, \infty)$ be a function which assigns to each $x_\propto \epsilon X, \propto \in (0,1]$ a nonnegative real number $\|x_\alpha\|_f$ such that

$(FN_1) \|x_\propto\|_f = 0 \iff x = 0$

$(FN_2) \|\lambda x_\propto\|_f = |\lambda| \|x_\propto\|_f$ for all $\lambda \in K$

$(FN_3) \|x_\propto + y_\beta\|_f \leq \|x_\propto\|_f + \|y_\beta\|_f$

(FN_4) if $\|x_\sigma\|_f < r$ where $r > 0$ then there exists $0 < \sigma \leq \propto < 1$ such that $\|x_\propto\|_f < r$.

Then $\|.\|_f$ is called fuzzy norm and $(X, \|.\|_f)$ is called fuzzy normed spaces.

Proposition 2.2:[6]

If $(X, \|.\|)$ is an ordinary normed space then $(X, \|.\|_f)$ is a fuzzy normed space with $\|x_\propto\|_f = \frac{1}{\propto}\|x\|$ for every $x \in X, \propto \in (0,1]$

Definition 2.3:[6]
A fuzzy subspace $\tilde{Y}$ of a fuzzy normed space $(X, \|.\|_f)$ is a fuzzy subspace of X considered as a vector space with the fuzzy norm obtained by restricting the fuzzy norm on X to $\tilde{Y}$.

Definition 2.4:[6]
Let $(X, \|.\|_f)$ be a fuzzy normed space. Given $x_\propto \in X$ and a real number $r > 0, \tilde{B}(x_\propto, r) = \left\{y_\beta \in X : \left\|x_\propto - y_\beta\right\|_f < r\right\}$ is open fuzzy ball with center $x_\propto$ and radius r.

Definition 2.5:[6]
A fuzzy set $\tilde{A}$ in a fuzzy normed space $(X, \|.\|_f)$is said to be open if it contains a fuzzy ball about each of its fuzzy element. A fuzzy set $\tilde{B}$ is closed if its complement is open fuzzy set.

Definition 2.6:[6]
Let $(X, \|.\|_f)$ and $(Y, \|.\|_{f2})$ be fuzzy normed spaces. A mapping $T: X \to Y$ is said to be fuzzy continuous at $x_\propto \in X$ is for every $\varepsilon > 0$ there is $\delta > 0$ such that $\left\|T(y)_\beta - T(x)_\propto\right\|_{f2} < \varepsilon$ for all $y_\beta \in X$ satisfying $\|y_\beta - x_\propto\|_{f1} < \delta$.
T is fuzzy continuous if it is fuzzy continuous at every fuzzy point $x_\propto \in X$.

Theorem 2.7:[6]
A mapping T from a fuzzy normed space $(X, \|.\|_{f1})$ into a fuzzy normed space $(Y, \|.\|_{f2})$ is fuzzy continuous if and only if the inverse image of any open fuzzy set in Y is open fuzzy set in X.

Definition 2.8:[6]
Let $\tilde{A}$ be a fuzzy set in a fuzzy normed space $(X, \|.\|_f)$. The closure of $\tilde{A}$ denoted by $CL(\tilde{A}) = \tilde{A} \cup \{\text{limit fuzzy points of } \tilde{A}\}$. $\tilde{A}$ is closed if $\mathrm{CL}(\tilde{A}) = \tilde{A}$.

Definition 2.9:[6]
A fuzzy set $\tilde{A}$ in a fuzzy normed space $(X, \|.\|_f)$ is said to be dense in X if $CL(\tilde{A}) = X$.

Definition 2.10:[6]
A fuzzy sequence $\{(x_n, \propto_n)\}$ in a fuzzy normed space $(X, \|.\|_f)$ is said to be convergent to $x_\propto$ in X if $\lim_{n\to\infty} \|(x_n, \propto_n) - x_\propto\|_f = 0$ $x_\propto$ is called the limit of $\{(x_n, \propto_n)\}$ and we write $\lim_{n\to\infty}(x_n, \propto_n) = x_\propto$ or simply $(x_n, \propto_n) \to x_\propto$.

Definition 2.11:[6]
A fuzzy set $\tilde{A}$ in $(X, \|.\|_f)$ is bounded if its fuzzy diameter $\delta(\tilde{A}) = sup\{\|x_\propto - y_\beta\|_f : x_\propto, y_\beta \in \tilde{A}\}$ is finite.

Definition 2.12:[6]
In a fuzzy normed space $(X, \|.\|_f)$ we call a sequence $\{(x_n, \propto_n)\}$bounded if the corresponding fuzzy set is bounded.

Theorem 2.13:[6]
Let $(X, \|.\|_f)$ be a fuzzy normed space. Then

(i) A convergent sequence of fuzzy points in X is bounded and its limit is unique.

(ii) If $(x_n, \propto_n) \to x_\propto$ and $(y_m, \beta_m) \to y_\beta$ in X then $\|(x_n, \propto_n) - (y_m, \beta_m)\|_f \to \left\|x_\propto - y_\beta\right\|_f$.

Definition 2.14:[6]
A sequence of fuzzy points $\{(x_n, \propto_n)\}$ in a fuzzy normed space is said to be Cauchy if for every $\varepsilon > 0$there is an integer $N > 0$ such that$\|(x_m, \propto_m) - (x_n, \propto_n)\|_f < \varepsilon$ for every $m, n > N$.

Theorem 2.15:[6]
Every convergent sequence of fuzzy points in a fuzzy normed space $(X. \|.\|_f)$ is Cauchy.

Theorem 2.16:[6]

Let $\tilde{A}$ be a nonempty fuzzy set in a fuzzy normed space $(X, \|.\|_f)$. Then $x_\propto \in CL(\tilde{A})$ if and only if there is a sequence of fuzzy points $\{(x_n, \propto_n)\}$ in $\tilde{A}$ such that $(x_n, \propto_n) \to x_\propto$.

Theorem 2.17:[6]

A mapping $T, X \to Y$ from the fuzzy normed space $(X, \|.\|_{f1})$ into the fuzzy normed space $(Y, \|.\|_{f2})$ is fuzzy continuous if and only if $(x_n, \propto_n) \to x_\propto$ implies $(T(x_n), \propto_n) \to T(x)_\propto$.

S3: Completion of Fuzzy Normed Spaces

Definition 3.1:

A fuzzy normed space $(X, \|.\|_f)$ is said to be complete if every Cauchy sequence $\{(x_n, \propto_n)\}$ in X converge to $x_\propto$ in X.

Theorem 3.2:

A fuzzy subspace $\tilde{A}$ of a complete fuzzy normed space $(X, \|.\|_f)$ is complete if and only if $\tilde{A}$ is closed.

Proof:

Let $\tilde{A}$ be a complete fuzzy subspace in $(X, \|.\|_f)$. By Theorem 2.16 for every $x_\propto \in CL(\tilde{A})$ there is a sequence $\{(x_n, \propto_n)\}$ in $\tilde{A}$ which is converges to $x_\propto$. Since $\{(x_n, \propto_n)\}$is Cauchy by Theorem 2.15 and $\tilde{A}$ is complete $\{(x_n, \propto_n)\}$converge in $\tilde{A}$ the limit being unique by Theorem 2.13. Hence $x_\propto \in \tilde{A}$, this proves $\tilde{A}$ is closed because $x_\propto \in CL(\tilde{A})$ was arbitrary.

Conversely, let $\tilde{A}$ be closed and $\{(x_n, \propto_n)\}$ be Cauchy in $\tilde{A}$. Then$(x_n, \propto_n) \to x_\propto \in X$ which implies that $x_\propto \in CL(\tilde{A})$ by Theorem 2.16 and $x_\propto \in \tilde{A}$since $\mathrm{cL}(\tilde{A}) = \tilde{A}$ by assumption. Hence the arbitrary Cauchy sequence$\{(x_n, \propto_n)\}$converges in $\tilde{A}$, which proves completeness of $\tilde{A}$.

We give now an example of incomplete fuzzy normed space.

Example 3.3

Let $\mathbb{Q}$ be the set of all rational numbers with fuzzy norm $\|x_\propto\|_f = \frac{1}{\propto}|x|$ for all $x_\propto \in \mathbb{Q}$. Then by proposition 2.2 $(\mathbb{Q}, \|.\|_f)$ is a fuzzy normed space but it is not complete since the sequence of fuzzy points(0.1,1) $\left(0.101\frac{1}{2}\right)$, $\left(0.101001, \frac{1}{3}\right)$, $\left(0.101, \frac{1}{4}\right)$, $\left(0.010001, \frac{1}{5}\right)$,… is Cauchy in $\mathbb{Q}$ but whose limit dose not belongs to $\mathbb{Q}$.

Definition 3.4:

Let $(X, \|.\|_{f1})$and (Y, $\|.\|_{f2}$) be fuzzy normed space.

(i) Then a mapping $T: X \to Y$ is said to be a fuzzy isometric or a fuzzy isometry if $\|T(x)_\propto\|_{f2} = \|x_\propto\|_{f1}$for all $x_\propto \in X$.

(ii) The space X is said to be fuzzy isometric with the space Y if there exists a bijective fuzzy isometry of X onto Y.

The spaces X and Y are then called fuzzy isometric spaces.

Theorem 3.5:

Let $(X, \|.\|_{f1})$ be a fuzzy normed space. Then there is a complete fuzzy normed space $(\hat{X}, \|.\|_{f2})$ and a fuzzy isometry T from X onto a fuzzy subspace $\widetilde{W}$of $\hat{X}$ which is dense in $\hat{X}$. The space $\hat{X}$ is unique except for fuzzy isometrics.

Proof :

The proof is somewhat lengthy but straight forward. We subdivide into five steps (a) to (e). We constract

(a) $(\hat{X}, \|.\|_{f2})$

(b) A fuzzy isometry$T$ of X onto $\widetilde{W}$ where $CL(\widetilde{W}) = \hat{X}$

Then we prove

(c) Completeness of $\hat{X}$

(d) Uniqueness of $\hat{X}$, except for fuzzy isometries.

(e) Define addition and scalar multiplication on X.

(a) Construction of $(\hat{X}, \|.\|_{f2})$.

Let $\{(x_n, \propto)\}$and $\{(x'_n, \propto'_n)\}$ be Cauchy fuzzy sequences in X. Define $\{(x_n, \propto_n)\}$ to be equivalent to

$\{(x'_n, \propto'_n)\}$, written $\{(x_n, \propto_n)\} \sim \{(x'_n, \propto'_n)\}$ if
$\lim_{n\to\infty} \|(x_n, \propto_n) - (x'_n, \propto'_n)\|_{f1} = 0$ ………………………(1)
Let $\hat{X}$ be the set of all equivalence classes $\hat{x}_\propto, \hat{y}_\beta, \ldots$ of Cauchy fuzzy sequences thus obtained we write $\{(x_n, \propto_n)\} \in \hat{x}_\propto$ to mean that $\{(x_n, \propto_n)\}$ is a member of $\hat{x}_\propto$ (a representative of the class $\hat{x}_\propto$). We now set

$$\|\hat{x}_\propto\|_{f2} = \lim_{n\to\infty} \|(x_n, \propto_n\|_{f1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Where $\{(x_n, \propto_n)\} \in \hat{x}_\propto$.We show that this limit exists.
We have

$$\left|\|(x_n, \propto_{n)}\|_{f1} - \|(x_m, \propto_m)\|_{f1}\right| \le \|(x_n, \propto_{n)} - (x_m, \propto_m)\|_{f1} \quad \ldots\ldots\ldots (3)$$

Since $\{(x_n, \propto_n)\}$ is Cauchy we can make the right side as small as we please. This implies that the limit in (2) exists because $\mathbb{R}$ is complete.
We must show that the limit in (2) is independent of the particular choice of representative. In fact if $\{(x_n, \propto_n)\} \sim \{(x'_n, \propto'_n)\}$ then by (1)

$$\left|\|(x_n, \propto_n)\|_{f1} - \|(x'_n, \propto'_n\|_{f1}\right| \le \|(x_n, \propto_n) - (x'_n, \propto'_n)\|_f \to 0 \quad \text{as} \quad n \to \infty$$

Which implies the assertion

$$\lim_{n\to\infty} \|(x_n, \propto_n)\|_{f1} = \lim_{n\to\infty} \|(x'_n, \propto'_n)\|_f$$

We prove that $\|.\|_{f2}$ in (2) is a fuzzy norm on $\hat{X}$

(FN_1) $\|\hat{x}_\propto\|_{f2} = 0$ if and only if $\lim_{n\to\infty} \|(x_n, \propto_n)\|_{f1} = 0 \Leftrightarrow \hat{x}_\propto = 0_\propto$

(FN_2) $\|\lambda\hat{x}_\propto\|_{f2} = \lim_{n\to\infty} \|(\lambda x_n, x_n)\|_{f1} = |\lambda| \lim_{n\to\infty} \|(x_n, \propto_n)\|_{f1} = |\lambda| \|\hat{x}_\propto\|_{f2}$

(FN_3)

$$\left\|\hat{x}_\propto + \hat{y}_\beta\right\|_{f2} = \lim_{n\to\infty} \|(x_n, \propto_n) + (y_n, \beta_n)\|_{f1}$$
$$\le \lim_{n\to\infty} \left[\|(x_n, \propto_n)\|_{f1} + \|(y_n, \beta_n)\|_{f1}\right]$$
$$\le \lim_{n\to\infty} \|(x_n, \propto_n)\|_{f1} + \lim_{n\to\infty} \|(y_n, \beta_n)\|_{f1}$$
$$\le \|\hat{x}_\propto\|_{f2} + \left\|\hat{y}_\beta\right\|_{f2}$$

(FN_4)if $\|\hat{x}_\sigma\|_{f2} < r$ where $r > 0$ *then* $\lim_{n\to\infty} \|(x_n, \sigma_n)\|_{f1} < r$ then there exists $\sigma_n \le \propto_n$ such that $\lim_{n\to\infty} \|(x_n, \propto_n)\|_{f1} < r$ so $\|\hat{x}_\propto\|_{f2} < r$ with $\sigma \le \propto$.

(b) Construction of a fuzzy isometry $T: X \to \widetilde{W}$ where $\widetilde{W} \subset \hat{X}$.
With each $b_\sigma \in X$ we associate the class $\hat{b}_\sigma \in \hat{X}$ which contains the constant Cauchy fuzzy sequence $b_\sigma, b_\sigma, \ldots$
This defines a mapping $T: X \to \widetilde{W}$ onto the fuzzy set $\widetilde{W} = T(X) \subset \hat{X}$. The mapping T is given by $b_\sigma \mapsto \hat{b}_\sigma = T(b)_\sigma$ where $b_\sigma, b_\sigma, \ldots \in \hat{b}_\sigma$. We see that T is a fuzzy isometry since (2) becomes simply $\left\|\hat{b}_\sigma\right\|_{f2} = \|b_\sigma\|_{f1}$.
Any fuzzy isometry is injective and $T: X \to \widetilde{W}$ is surjective since $T(X) \to \widetilde{W}$. Hence $\widetilde{W}$ and X are fuzzy isometric.
We show that $\widetilde{W}$ is dense in $\hat{X}$. We consider any $\hat{x}_\propto \in \hat{X}$. Let $\{(x_n, \propto_n)\} \in \hat{x}_\propto$. For every $\varepsilon > 0$ there is an N such that

$$\|(x_n, \propto_n) - (x_N, \propto_N)\|_{f1} \le \frac{\varepsilon}{2} \qquad (n > N)$$

Let $x_N, x_{?N}, \ldots \in (\hat{x}_N, \propto_N)$. Then $(\hat{x}_N, \propto_N) \in \widetilde{W}$.
By (2)

$$\|\hat{x}_\propto - (\hat{x}_N, \propto_N)\|_{f2} = \lim_{n\to\infty} \|(x_n, \propto_n) - (x_N, \propto_N)\|_{f1} \le \frac{\varepsilon}{2} < \varepsilon$$

This shows that every ε-neighborhood of arbitrary $\hat{x}_\propto \in \hat{X}$ contains an element of $\widetilde{W}$.
Hence $\widetilde{W}$ is dense in $\hat{X}$.

(c) Completeness of $\hat{X}$. Let $\{(\hat{x}_n, \propto_n)\}$ be any Cauchy sequence in $\hat{X}$.

Since $\widetilde{W}$ is dense in $\hat{X}$, for every $(\hat{x}_n, \propto_n)$ there is a $(\hat{z}_n, \sigma_n) \in \widetilde{W}$ such that

$$\|(\hat{x}_n, \propto_n) - (\hat{z}_n, \sigma_n\|_{f2} < \frac{1}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Hence by (FN_3) we have

$$\begin{aligned}\|(\hat{z}_m, \sigma_m) - (\hat{z}_n, \sigma_n)\|_{f2} &= \|(\hat{z}_m, \sigma_m) - (\hat{x}_m, \propto_m) + (\hat{x}_m, \propto_m) - (\hat{x}_n, \propto_n) + (\hat{x}_n, \propto_n) - (\hat{z}_n, \sigma_n\|_{f2} \\ &\le \|(\hat{z}_m, \sigma_m) - (\hat{x}_m, \propto_m)\|_{f2} + \|(\hat{x}_m, \propto_m) - (\hat{x}_n, \propto_n)\|_{f2} + \|(\hat{x}_n, \propto_n) - (\hat{z}_n, \sigma_n)\|_{f2}\end{aligned}$$

$$< \frac{1}{m} + \|(\hat{x}_m, \propto_m) - (\hat{x}_n, \propto_n)\|_{f2} + \frac{1}{n}$$

And this is less than any given $\varepsilon > 0$ for sufficiently large m and n because $\{(\hat{x}_m, \propto_m)\}$ is Cauchy. Hence $\{(\hat{z}_m, \sigma_m)\}$ is Cauchy. Since $T: X \to \widetilde{W}$ is fuzzy isometric and $(\hat{z}_m, \sigma_m) \in \widetilde{W}$, the sequence $\{(\hat{z}_m, \sigma_m)\}$ where $(z_m, \sigma_m) = T^{-1}(\hat{z}_m)_{\sigma_m}$ is Cauchy in X. Let $\hat{x}_\propto \in \hat{X}$ be the class to which $\{(z_m, \sigma_m)\}$ belongs. We show that $\hat{x}_\propto$ is the limit of $\{(\hat{x}_n, \propto_n)\}$. By (4)

$$\|(\hat{x}_n, \propto_n) - \hat{x}_\propto\|_{f2} \le \|(\hat{x}_n, \propto_n) - (\hat{z}_n, \sigma_n)\|_{f2} + \|(\hat{z}_n, \sigma_n) - \hat{x}_\propto\|_{f2}$$

$$< \frac{1}{n} + \|(\hat{z}_n, \sigma_n) - \hat{x}_\propto\|_{f2} \quad \ldots\ldots\ldots\ldots (5)$$

Since $\{(z_m, \sigma_m)\} \in \hat{x}_\propto$ and $(\hat{z}_n, \sigma_n) \in \widehat{W}$, so that $(z_n, \sigma_n), (z_n, \sigma_n), \ldots \in (\hat{z}_n, \sigma_n)$, the inequality (5) becomes $\|(\hat{x}_n, \propto_n) - \hat{x}_\propto\|_{f2} < \frac{1}{n} + \lim_{n\to\infty} \|(z_n, \sigma_n) - (z_m, \sigma_m)\|_{f1}$ and the right side is smaller than any given $\varepsilon > 0$ for sufficiently large n. Hence the arbitrary Cauchy sequences $\{(\hat{x}_n, \propto_n)\}$ in $\hat{X}$ has the limit $\hat{x}_\propto \in \hat{X}$.

(d) Uniqueness of $\hat{X}$ except for fuzzy isometries.

If $(Z, \|.\|_{f3})$ s another complete fuzzy normed and fuzzy isometric with X then for any $\hat{y}_\beta \in$ Y we have sequence $\{(y_n, \beta_n)\}$ in $\tilde{V}$ such that $(y_n, \beta_n) \to y_\beta$, hence $[T': X \to Z, T'(X) = \tilde{V}]$

$$\|\hat{y}_\beta\|_{f3} = \lim_{n\to\infty} \|(y_n, \beta_n)\|_{f1}$$

Follows from

$$\big|\|(y_n, \beta_n)\|_{f1} - \|(y_m, \beta_m)\|_{f1}\big| \le \|(y_n, \beta_n)\|_{f1} - \|(y_m, \beta_m)\|_{f1} \to 0$$

[the inequality being similar to (3)]. Since $\tilde{V}$ is fuzzy isometric with $\widetilde{W} \subset \hat{X}$ and $CL(\widetilde{W}) = \hat{X}$ the fuzzy distance on $\hat{X}$ and Z must be the same. Hence Z and $\hat{X}$ are fuzzy isometric.

(e) To define on $\hat{X}$ the two algebraic operations addition and scalar multiplication, we consider any $\hat{x}_\propto$ and $\hat{y}_\beta \in \hat{X}$ and any representatives $\{(x_n, \propto_n)\} \in \hat{x}_\propto$ and $\{(y_n, \beta_n)\} \in \hat{y}_\beta$.

Remember that $\hat{x}_\propto$ and $\hat{y}_\beta$ are equivalence classes of Cauchy fuzzy sequences in X. We set $(z_n, \sigma_n) = (x_n, \propto_n) + (y_n, \beta_n)$ where $\sigma_n = max\{\propto_n, \beta_n\}$.

Then $\{(z_n, \sigma_n)\}$ is Cauchy in X since

$$\begin{aligned}\|(z_n, \sigma_n) - (z_m, \sigma_m)\|_{f1} &= \|(x_n, \propto_n) + (y_n, \beta_n) - (x_m, \propto_m) - (y_m, \beta_m)\|_{f1} \\ &\le \|(x_n, \propto_n) - (x_m, \propto_m)\|_{f1} + \|(y_n, \beta_n) - (y_m, \beta_m)\|_{f1}\end{aligned}$$

We define the sum $\hat{z}_\sigma = \hat{x}_\propto + \hat{y}_\beta$ where $\sigma = \max\{\propto, \beta\}$ to be the equivalence class for which

$\{(z_n, \sigma_n)\}$ is a representative, thus $\{(z_n, \sigma_n)\} \in \hat{z}_\sigma$.

This definition is independent of the particular choice of Cauchy sequences belonging to $\hat{x}_\propto$ and $\hat{y}_\beta$.

In fact (1) shows that if $\{(x_n, \propto_n)\} \sim \{(x'_n, \propto'_n)\}$ and $\{(y_n, \beta_n)\} \sim \{(y'_n, \beta'_n)\}$ then $\{(x_n, \propto_n) + (y_n, \beta_n)\} \sim \{(x'_n, \propto'_n) + (y'_n, \beta'_n)\}$ because

$$\|(x_n, \propto_n) + (y_n, \beta_n) - (x'_n, \propto'_n) - (y'_n, \beta'_n)\|_{f1} \le \|(x_n, \propto_n) - (x'_n, \propto'_n)\|_{f1} + \|(y_n, \beta_n) - (y'_n, \beta'_n)\|_{f1}$$

Similarly, we define the product $\lambda\hat{x}_\propto \in \hat{X}$ of a scalar λ and $\hat{x}_\propto$ to be the equivalence class for which $\{(\lambda x_n, \propto_n)\}$ is a representative. Again this definition is independent of the particular choice of a representative of $\hat{x}_\propto$.

S4: Relation Between Ordinary Normed Spaces and Fuzzy Normed Spaces

Lemma 4.1:

Let $(X, \|.\|)$ be an ordinary normed space then $\{x_n\}$ $coverge\ to\ x \in X\ in\ (X, \|.\|)$ If and only if $\{(x_n, \alpha_n\}$ coverage to $x_\propto$ in $(X, \|.\|_f)$ with $\|x_\propto\|_f = \frac{1}{\propto}\|x\|$ for evrey $x \in X, \propto \in (0,1]$.

Proof:

Let $\{x_n\}$ be a sequence in the ordinary normal space $(X, \|.\|)$ converges to $x \in X$.

Let $\{\propto_n\}$ be an increasing sequence in (0,1] converges to $\propto$ [This is possible by setting $\propto_n = \left(1 - \frac{1}{n}\right)\propto$]. Now consider the fuzzy sequence $\{(x_n, \propto_n)\}$ in $(X, \|.\|_f)$.

Since $x_n \to x$ then $0 = \lim_{n\to\infty}\|x_n - x\|$

$$= \frac{1}{\lambda}\lim_{n\to\infty}\|x_n - x\| \ [\lambda = max\{x, x_n\}]$$

$$0 = \lim_{n\to\infty}\|(x_n, \propto_n) - x_\propto\|_f$$

Thus $\{(x_n, \propto_n)\}$ converges to $x_\propto$ in $(X, \|.\|_f)$.

Conversely, assume that $\{(x_n, \propto_n)\}$ converges to $x_\propto$ in $(X, \|.\|_f)$ that is

$$0 = \lim_{n\to\infty}\|(x_n, \propto_n) - x_\propto\|_f = \frac{1}{\lambda}\lim_{n\to\infty}\|x_n - x\| \ [\text{where } \lambda = \max\{\propto, \propto_n\}]$$

Therefore, $0 = \lim_{n\to\infty}\|x_n - x\|$

Thus $\{x_n\}$ converges to $x_\propto$ in $(X, \|.\|)$

Lemma 4.2:

Let $(X, \|.\|)$ be an ordinary normed space then $\{x_n\}$ is Cauchy in $(X, \|.\|)$ if and only if $\{(x_n, \propto_n)\}$ is Cauchy in $(X, \|.\|_f)$ with $\|x_\propto\|_f = \frac{1}{\propto}\|x\|$ for every $x \in X, \propto \in (0,1]$.

Proof:

Let $\{x_n\}$ be a Cauchy sequence in the ordinary normed space $(X, \|.\|)$. Let $\{\propto_n\}$ be an increasing sequence in (0,1]. Consider the sequence $\{(x_n, \propto_n)\}$ of fuzzy points in $(X, \|.\|_f)$. For any given $\varepsilon > 0$ there is an integer $N > 0$ such that $\|x_m - x_n\| < \varepsilon$ for all $m, n > N$. Now for all $m, n > N$

$\|(x_m, \propto_m) - (x_n, \propto_n)\|_f = \frac{1}{\lambda}\|x_m - x_n\| < \frac{\varepsilon}{\lambda} = \varepsilon_1$ [where $\lambda = max\{\propto_m, \propto_n\}$].

Thus $\{(x_n, \propto_n)\}$ is Cauchy in $(X, \|.\|_f)$.

Conversely, assume that $\{(x_n, \propto_n)\}$ is Cauchy in $(X, \|.\|_f)$ then for any given $\varepsilon > 0$ there is an integer $N > 0$ such that

$\|(x_m, \propto_m) - (x_n, \propto_n)\|_f < \varepsilon$ for all $m, n > N$.

Now for all $m, n > N$

$\frac{1}{\lambda}\|x_m - x_n\| < \varepsilon$ or $\|x_m - x_n\| < \lambda\varepsilon = \varepsilon_1$ where $\lambda = max\{\propto_m, \propto_n\}$. Thus $\{x_n\}$ is Cauchy in $(X, \|.\|)$.

Proposition 4.3:

If $(X, \|.\|_f)$ is a fuzzy normed space then $(X, \|.\|)$ is an ordinary normed space by defining $\|x\| = \|x_\propto\|_f$ for all $x \in X, \propto \in (0,1]$

Proof:

$(N_1)\|x\| > 0$ since $\|x_\propto\|_f > o$ for every $x \in X$ and $\|x\| = 0 \Leftrightarrow \|x_\propto\|_f = 0 \Leftrightarrow x = 0$

$(N_2)\|rx\| = \|rx_\propto\|_f = |r|\|x_\propto\|_f = |r|\|x\|$ for all $r \in K$.

$(N_3)\|x+y\| = \|(x+y)_\propto\|_f = \|x_\propto + y_\propto\|_f \le \|x_\propto\|_f + \|y_\propto\|_f = \|x\| + \|y\|$

Thus $(X, \|.\|)$ is a normed space.

The proof of the following two lemma's is easy. Hence is omitted.

Lemma 4.4:

Let $(X, \|.\|_f)$ be a fuzzy normed space then $\{(x_n, \propto_n)\}$ converges to $x_\propto$ in $(X, \|.\|_f)$ if and only if $\{x_n\}$ converge to x in $(X, \|.\|)$ with

$$\|x\| = \|x_\propto\|_f \text{for every } x \in X, \propto \in (0,1]$$

Lemma 4.5:

Let $(X, \|.\|_f)$ be a fuzzy normed space then $\{(x_n, \propto_n)\}$ is Cauchy in $(X, \|.\|_f)$ if and only if $\{x_n\}$ is Cauchy in $(X, \|.\|)$ with $\|x\| = \|x_\propto\|_f$ for every $x \in X, \propto \in (0,1]$.

Theorem 4.6:

Let $(X, \|.\|)$ be an ordinary normed space then $(X, \|.\|)$ is complete if only if $(X, \|.\|_f)$ is a complete fuzzy normed space with $\|x_\propto\|_f = \frac{1}{\propto}\|x\|$ for every $x_\propto \in X, \propto \in (0,1]$.

Proof:

Suppose that $(X, \|.\|)$ is a complete normed space. Let $\{(x_n, \propto_n)\}$ be a Cauchy sequence in $(X, \|.\|_f)$ then by Lemma 4.2 $\{x_n\}$ is Cauchy in $(X, \|.\|)$. But $(X, \|.\|)$ is complete so $\{x_n\}$ converge to $x \in X$.

Now by Lemma 4.1 $\{(x_n, \propto_n)\}$ converge to $x_\propto$ in $(X, \|.\|_f)$. Therefore $(X, \|.\|_f)$ is complete.

Conversely assume that $(X, \|.\|_f)$ is complete let $\{x_n\}$ be a Cauchy sequence in $(X, \|.\|)$. By lemma 4.1 $\{(x_n, \propto_n)\}$ is Cauchy in $(X, \|.\|_f)$. But $(X, \|.\|_f)$ is complete so $\{(x_n, \propto_n)\}$ converge to $x_\propto \in X$ in $(X, \|.\|_f)$. Therefore $\{x_n\}$ converge to $x \in X$ in $(X, \|.\|)$ by lemma 4.1. Thus $(X, \|.\|)$ is complete.

The proof of the next result is similar to the proof of Theorem 4.6. Hence is omitted.

Theorem 4.7:

Let $(X, \|.\|_f)$ be a fuzzy normed space then $(X, \|.\|_f)$is complete if and only if $(X, \|.\|)$ is complete with $\|x\| = \|x_\propto\|_f$ for every $x \in X, \propto \in (0,1]$.

Theorem 4.8:

Let $(X, \|.\|_1)$ and $(Y, \|.\|_2)$ be a normed spaces then the mapping $T: (X, \|.\|_1) \to (Y, \|.\|_2)$ is continuous if and only if $T: (X, \|.\|_{f1}) \to (Y, \|.\|_{f2})$ is fuzzy continuous with $\|x_\propto\|_{f1} = \frac{1}{\propto}\|x\|_1$ and $\|y_\beta\|_{f2} = \frac{1}{\beta}\|y\|_2$ for every $x_\propto \in X, y_\beta \in Y, \propto, \beta \in (0,1]$.

Proof:

Assume that $T: (X, \|.\|_1) \to (Y, \|.\|_2)$ is continuous. Let $\{(x_n, \propto_n)\}$ be a sequence of fuzzy points in $(X, \|.\|_f)$ converges to $x_\propto \in X$. Then $\{x_n\}$ is a sequence in $(X, \|.\|_1)$ converges to x by Lemma 4.1. By continuity of T $\{T(x_n)\}$ converges to $T(x)$ in $(Y, \|.\|_2)$. Again by Lemma 4.1 $\{(T(x_n), \propto_n)\}$ converges to $T(x)_\propto$in $(Y, \|.\|_{f2})$. Thus T is fuzzy continuous by Theorem 2.17.

Conversely assume that $T: (X, \|.\|_{f1}) \to Y, (\|.\|_{f2})$ is fuzzy continuous. Let $\{x_n\}$ be a sequence in $(X, \|.\|_1)$ converges to $x \in X$ then by Lemma 4.1 $\{(x_n, \propto_n)\}$ is a sequence in $(X, \|.\|_{f1})$ converge to $x_\propto \in X$. By Theorem 2.17 $\{(T(x_n), \propto_n)\}$ converges to $T(x)_\propto$. Again by lemma 4.1 $\{T(x_n)\}$ converges to $T(x)$in $(Y, \|.\|_2)$. Thus T is continuous.

The proof of the next theorem is similar to the proof of theorem 4.8. Hence is omitted.

Theorem 4.9:
Let $(X, \|.\|_{f1})$and $(Y, \|.\|_{f2})$ be two fuzzy normed spaces. The mapping $T: (X, \|.\|_{f1}) \to (Y, \|.\|_{f2})$ is fuzzy continuous if and only if $T: (X, \|.\|_1) \to (Y, \|.\|_2)$ is continuous with $\|x\|_1 = \|x_\propto\|_{f1}$ and $\|y\|_2 = \|y_\beta\|_{f2}$ for every $x \in X$ and $y \in Y$.

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