The Primary Decomposition of the Factor Group K(G)

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Abstract

The main purpose of this work is to find the primary decomposition of the factor group $K(G) = cf(G,Z)/R(G)$ (The factor group of all Z-valued class functions module the group of Z-valued generalized characters for elementary Abelian group G), where G is a finite Abelian group of type $Z_p^{(n)}$, p=5.

This work depends on finding the rational valued character matrix $\equiv^*\left(Z_5^{(n)}\right)$ from the character table of Z_5 and finding the invariant factors of this matrix, also we found the general solution of this decomposition and prove it by mathematical induction. We have used the MATLAB program to calculate some results of this work.

ايجاد التجزئة الاولية للزمرة الكسرية K(G)

الخلاصة

الهدف الاساسي من هذا البحث هو ايجاد التجزئة الاولية للزمرة الكسرية $K(G) = cf(G,Z)/R(G)$(الزمرة الكسرية لكل دوال الصفوف ذات القيم الصحيحة على زمرة الشواخص العمومية ذات القيم الصحيحة للزمرة الابدالية الاولية Z_p^n , حيث $p = 5$). اعتمد العمل لايجاد هذه التجزئة على حساب المصفوفة $\equiv^*(Z_5^{(n)})$ من جدول الشواخص لـ Z_5 وايجاد العوامل اللامتغيرة لها. كما اوجدنا حل عام لهذه التجزئة والتي تم برهنتها باستخدام الاستقراء الرياضي. استخدم برنامج MATLAB لحساب بعض النتائج لهذا العمل.

1. Introduction

The importance of representation and character theory for the study of the groups stems on the one hand from the fact that should it be necessary to present a concrete description of a group, this can be achieved with a matrix representation .on the other hand, group theory benefits mainly from the use of representations and characters, when these approaches are employed as an additional means to analyze the structure of a group.

Moreover representation and character theory provide application, not only in other branches of mathematics but also in physics and chemistry.

Let G be a finite group, two elements of G are said to be Γ- conjugate if the cyclic subgroups they generate are conjugate in G, and this defines an equivalence relation on G. Its classes

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are called Γ - classes. The Z - valued class function on the group G, which is constant on the Γ - classes forms a finitely generated abelian group $cf(G,Z)$ of a rank equal to the number of Γ - classes.

The intersection of $cf(G,Z)$ with the group of all generalized characters of G, is a normal subgroup of $cf(G,Z)$ denoted by $\bar{R}(G)$,then $cf(G,Z)/\bar{R}(G)$ is a finite abelian factor group which is denoted by K(G).

Each element in R (G) can be written as $u_1\theta_1 + u_2\theta_2 + \cdots \ldots + u_l\theta_l$, where l is the number of Γ- classes, $u_1, u_2, \ldots \ldots, u_l \in Z$ and,

$$\phi_i = \sum_{\sigma \in Gal(Q(\chi_i))} \chi_i^{\sigma}$$

Where χ_i is an irreducible character of the group G and σ is any element in Galios group $Gal(Q(\chi_i)/Q)$.

Let $\equiv^*(G)$ denotes the $l \times l$ matrix which its row corresponds to the $\theta i's$ and its columns correspond to the Γ- classes of G.

The matrix expressing $R(G)$ basis in terms of the $cf(G,Z)$ basis is $\equiv^*(G)$.

We can use the theory of invariant factors to obtain the direct sum of the cyclic Z-module of orders the distinct invariant factors of $\equiv^*(G)$ to find the primary decomposition of K(G) .

In 1982 M. S. Kirdar [1] studied the $K(Z_n)$.In 1994 H.H. Abass [2] studied the $K(D_n)$ and found $\equiv^*(D_n)$ .In1995 N. R. Mahamood [3] studied the factor group $cf(Q_{2m},Z) / R(Q_{2m})$.

In 1998 M.N.AL-Harere [4] studies the cyclic decomposition of the factor group $cf(G,Z)/\bar{R}(G)$ for the group of type Z_{p^n} ,$P = 3,5,7.$ In 2005 M.Z.Salman [5]studied the factor group $cf(G,Z)/\bar{R}(G)$ for the group of type $Z_2^{(n)}$. In 2005 N.S.Jasim [6] studied the factor group $cf(G,Z)/\bar{R}(G)$ for the special linear group $SL(2,p)$.In 2006 M.N.AL-Harere [7] studies the cyclic decomposition of the factor group $cf(G,Z_{13^n})/\bar{R}(Z_{13^n})$ .In 2010 M.N.AL-Harere[8] studied the factor group $cf(G,Z)/\bar{R}(G)$ for the group of type $Z_3^{(n)}$.

The aim of this research is to find $\equiv^*(Z_p)$ and determine the primary decomposition of the group K($Z_p^{(n)}$) where $p = 5$.

2. Basic Concepts

In this section, we are going to present an introduction to the concepts representation, characters, characters table of finite group.

2.1 Matrix representation of characters:

Definition (2.1.1),[9]

The set of all non-singular m×m matrices over the field F form a group under matrix multiplication this group is known as the general linear group of degree m over F and denoted by GL(m,F).

Definition (2.1.2),[9]

A matrix representation of G is a homomorphism

$$T: G \rightarrow GL(m, F)$$

m is called the degree of matrix representation T.

Definition (2.1.3),[9]

Two matrices representation $T_{1(X)}$ and $T_{2(X)}$ are said to be equivalent, if they have the same degree, and if there exist a fixed invertible matrix $K \in GL(m,F)$ such that:

$$T_{1(X)} = K^{-1}T_{2(X)}K \quad \forall\ x \in G.$$

Definition (2.1.4) ,[4]

A matrix representation T of a group G is a reducible representation if it is equivalent to a matrix representation of the form

$$\begin{bmatrix} T_{1(x)} & K(x) \\ 0 & T_{2(x)} \end{bmatrix}, \text{ for all } x \in G$$

Where $T_1 \, and \, T_2$ is representation of G whose degree is less than the degree of T.

A reducible representation T is called completely reducible if it is equivalent to a matrix representation of the form

$$\begin{bmatrix} T_{1(x)} & 0 \\ 0 & T_{2(x)} \end{bmatrix}, \text{ For all } x \in G$$

If T is not reducible then T is said to be an irreducible representation.

Definition (2.1.5),[10]

Let G be a finite group and let ρ: $G \rightarrow GL(n, C)$ be a matrix representation of G of degree n given by ρ(x)=A(x) ∀ x∈ G, associated with ρ there is a function χ_ρ: $G \rightarrow$C defined by χ_ρ(x)=tr(A(x)), we call the function χ_ρ the character of the representation ρ .

Proposition (2.1.1),[9]

If χ is the character of a representation ρ of degree n, we have

i) χ(1) = n

ii) $\chi(x^{-1}) = \overline{\chi}(x)$ for x∈G

[where the bar denotes the complex conjugate].

Proposition (2.1.2),[1]

The characters of G are class functions on G, that is, conjugate elements have the same character.

Theorem (2.1.1),[4]

Sum and product of characters are characters.

Definition (2.1.6),[4]

Characters associated with irreducible representations are called irreducible (or simple) characters, and those of reducible representation are compound.

Definition (2.1.7),[11]

A class functions on G of the form

$$\phi = u_1\chi_1 + u_2\chi_2 + \cdots + u_k\chi_k$$

Where $\{\chi_1, \chi_2, \ldots \chi_k\}$ are the complete set of irreducible characters of G, $u_1, u_2, \ldots u_k$ are integers, which is called the generalized characters of G.

Theorem (2.1.2),[9]

The number k of distinct irreducible characters of G is equal to the number of its conjugacy classes.

2.2 character of finite Abelian group

Definition (2.2.1), [1]

For a finite group G of order n, complete information about the irreducible characters of G is displayed in a table called the character table of G.

We list the elements of G in the 1^{st} row, we put $\chi_i(x^j) = \chi_i^j$, 1≤i≤ n, 1≤j≤ n-1. Denoting the number of elements in C_α by $|C_\alpha|$ (α=1,…,n),we have the class equation $|C_1|+\ldots|C_n|$ =$|G|$

$\equiv G =$

C_α	1	x	x^2	…	x^{n-1}
$\|C_\alpha\|$	1	1	1	…	1
χ_1	1	1	1		1
χ_2	1	χ_2^1	χ_2^2		χ_2^{n-1}
. . .	. . .			…	. . .
χ_n	1	χ_n^1	χ_n^2	…	χ_n^{n-1}

If G=Z_n the cyclic group of order n, and let ω=$e^{2\pi i/n}$ be a primitive n-th root of unity, Then the general formula of the character table of Z_n is

$\equiv Z_n =$

C_α	1	z	z^2	…	z^{n-1}
$\|C_\alpha\|$	1	1	1	..	1
χ_1	1	1	1		1
χ_2	1	ω	ω^2	..	ω^{n-1}
χ_2	1	ω^2	ω^4	..	ω^{n-2}
. . .	. . .			..	. . .
χ_n	1	ω^{n-1}	ω^{n-2}	..	ω

3. The factor group K(G)

In this section we study the factor group K(G), for a finite group G. In general, we will be concerned with its order and exponent.

Definition (3.1),[12]

A rational valued character θ of G is a character whose values are in Z, that is θ(x) ∈ Z, for all x∈ G.

Definition (3.2),[10]

Two elements of G are said to be Γ-conjugate if the cyclic subgroups they generate are conjugate in G , this defines an equivalence relation on G, its classes are called Γ-classes of G.

Let G be a finite group and let $\chi_1, \chi_2, \cdots \chi_k$ be its distinct irreducible characters, a class function on G is a character if and only if it is a linear combination of the χ_i with non-negative integer coefficients. We will denote $R^+(G)$ the set of these functions, the group generated by $R^+(G)$ is called the group of the generalized characters of G and is denoted by $R(G)$, we have:

R(G)=$Z\chi_1 \oplus Z\chi_2 \oplus Z\chi_3 \oplus \ldots Z\chi_k$.

An element of R(G) is called a virtual character since the product of two character is a character, R(G) is a subring of the ring cf (G) of C-valued class functions on G.

Let cf(G,Z) be the group of all Z-valued class functions of G which are constant on Q-classes and let $\bar{R}$(G) be the intersection of cf(G,Z) with R(G), $\bar{R}$(G) is a ring of Z-valued generalized characters of G.

Let ε_m be a complex primitive m-th root of unity. We know that the Galois group Gal(F(ε_m)/F) is a subgroup of the multiplicative group $(Z/mZ)^*$ of invertible elements of Z/mZ.

More precisely, if σ∈ Gal (F(ε_m)/F), there exists a unique element t∈ (Z/mZ)* Such that σ(ε_m) $= \varepsilon_m^t$ if $\varepsilon_m = 1$.

We denote by Γ_F for the image of Gal(F(ε_m)/F) in (Z/m Z)*, and if t∈Γ_F, we let σ_t denote the corresponding element of Gal(F(ε_m)/F).

Take as ground field F the field Q of rational numbers the Galois group of Q(ε_m) over Q is the group denoted by Γ.

Theorem (3.1),[4]:
[Gauss-kroncecker]
Γ= (Z/mZ)*.

Proposition (3.1),[10]
The characters $\phi_1, \phi_2, \dots, \phi_m$ form a basis of $\bar{R}$(G) and their number is equal to the number of conjugacy classes of cyclic subgroups of G, where

$$\phi_i = \sum_{\phi \in Gal(Q(\chi_i))} \chi_i^{\sigma}$$

and χ_i are irreducible C-characters of G, for all i=1,2,…m.

Lemma (3.1),[10]
The factor group k (G) has a finite exponent equal to g, where g is the order of G.

Definition (3.3),[12]
Let M be a matrix with entries in principal domain R.A K-minor of M is the determinate of the k×k submatrix preserving row and column order.

Theorem (3.2),[11]
Let M, P, Q be matrices with entries in a principal domain R. Let P and Q be invertible matrices.
Then D_k(QMP) = $D_k(M)$.

Theorem (3.3),[12]
Let M be an m×n matrix with entries in a principal domain R.
Then there exist matrices P, Q, D such that:
1. P and Q are invertible.
2. QMP=D.
3. D is diagonal matrix.
4. if we denote D_{ii} by d_i then there exists a natural number r, o≤ r ≤ min(m,n) such that j>r implies d_j =0 and j≤r implies d_j ≠0 and 1≤j< r implies d_j divides d_{j+1}.

Definition (3.4),[12]
Let M, be a matrix with entries in a principal R, equivalent to a matrix D=daig$\{d_1, d_2, \dots d_r, 0, \dots ,0\}$ such that d_j/d_{j+1} for 1≤ j < r ,we call D the invariant factor matrix of M and $d_1, d_2, \dots, d_r$ the invariant factors of M.

Theorem (3.4),[12]
Let M be a finitely generated module over a principal domain R, then M is the direct sum of cyclic submodules with annihilating ideals $\langle d_1\rangle, \langle d_2\rangle, \dots \langle d_r\rangle$, d_j/d_{j+1} for j=1,2,…,m-1.

Theorem (3.5),[10]
$K(G) = \oplus_{i=1}^{r} Z_{di}$
$d_{i=\pm} D_i(\equiv^* G)/D_{i-1}(\equiv^* G)$.

Theorem (3.6),[1]
$|K(G)| = \det(\equiv^* G)$

Theorem (3.7),[10]

Let $\{x_i\}$, $1\leq i \leq t$ be the set of the representatives of Γ-classes of G and assume each x_i contains n_i classes of G.

Then $|K(G)| = \left[\prod_{i=1}^{t} \frac{n_i g}{(x_i)}\right]^{1/2}$.

Lemma (3.2), [1]

Let A and B be two non-singular matrices of degree n and m respectively, over a principal domain R, and let

$P_1 A Q_1 = D(A) = diag\{d_1(A), d_2(A), \ldots, d_n(A)\}$,

$P_2 B Q_2 = D(B) = diag\{d_1(B), d_2(B), \ldots, d_n(B)\}$.

Be the invariant factor matrices of A and B. Then

$(P_1 \otimes P_2)(A \otimes B)(Q_1 \otimes Q_2) = D(A) \otimes D(B)$,

And from this lemma, the invariant factor matrix of $A \otimes B$ can be written as follow:

Let H and L be P_1 and P_2 groups respectively, where P_1 and P_2 are distinct primes, we know that

$\equiv (H \times L) = \equiv (H) \otimes \equiv (L)$

Since gcd $(p_1, p_2) = 1$, we have the next proposition.

Proposition (3.2), [2]

$$\equiv^* (H \times L) = \equiv^* (H) \otimes \equiv^* (L).$$

We consider the case when the order is prime ; all the non principal irreducible characters are Γ-conjugate. We have

$$\equiv^* G = \begin{bmatrix} 1 & 1 \\ P-1 & 1 \end{bmatrix}$$

Theorem (3.8),[1]

Let G be a cyclic p-group, then K (G) $= Z_p$.

Theorem (3.9),[1]: let G be a cyclic group of order p^n then K (G) $= \oplus_1^n Z_{p^i}$.

Lemma (3.3),[1]

$\equiv^* (Z_p^{(n)})$ has rank $= \sum_{i=1}^{n-1} p^i + 2$.

4. K (G), For G cyclic & elementary abelian group

This section is devoted to study the rational valued characters table of the group Z_5 and to find the primary decomposition of the finite abelian group K(G) where G is cyclic group and the case when G is elementary abelian group.

4.1The Rational Valued Characters Table of the Group Z_5

The character table of Z_5 is

$\equiv Z_5$ =

C_α	1	z	z^2	z^3	z^4
$\lvert C_\alpha$	1	1	1	1	1
χ_1	1	1	1	1	1
χ_2	1	ω	ω^2	ω^3	ω^4
χ_3	1	ω^2	ω^4	ω	ω^3
χ_4	1	ω^3	ω	ω^4	ω^2
χ_5	1	ω^4	ω^3	ω^2	ω

Where $\omega = e^{2\pi i/5}$.

To calculate the rational valued characters table of Z_5, the elements of

Gal(Q(χ_2/Q), are:{ $\sigma_1, \sigma_2, \sigma_3, \sigma_4$} where

$\sigma_1(\chi_2)= \chi_2$

$\sigma_2(\chi_2)= \chi_3$

$\sigma_3(\chi_2)= \chi_4$

$\sigma_4(\chi_2)= \chi_5$

By proposition (3.1)

$\theta_2 = \sigma_1(\chi_2)+ \sigma_2(\chi_2)+ \sigma_3(\chi_2)+ \sigma_4(\chi_2)$

=1+1+1+1=4

Since

w+$w^2 + w^3 + \cdots + w^{n-1} = -1$

$\theta_2(r) = w + w^2 + w^3 + w^4 = -1$

$\theta_2(r^2) = w^2 + w + w^4 + w^3 = -1$

$\theta_2(r^3) = w^3 + w^4 + w^1 + w^2 = -1$

$\theta_2(r^4) = w^4 + w^3 + w^2 + w = -1$

And

$\theta_1=\chi_1$

Since the elements r, r^2, r^3 and r^4 generate the same cyclic group Z_5 then, they are in the same $\Gamma-conjugate[r]$.

$\equiv^* G$=

Γ –classes	[1]	[r]
θ_1	1	1
θ_2	4	- 1

Then

$$\equiv^* Z_5= \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}.$$

4.2 The primary decomposition of K ($Z_p^{(n)}$), p=5:

In this section we will determine the primary decomposition of K($Z_5^{(n)}$) depending on Lemma(3.2).

Let P $=\begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix}$, Q=$\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ and $\equiv^* Z_5=\begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}$

When n=1

Then

$$P\equiv^* Z_5 Q = \begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix}$$

, by theorem (3.8)

K (Z_5) = Z_5

When n=2

$$P \otimes P= \begin{bmatrix} 16 & -4 & -4 & 1 \\ -4 & 0 & 1 & 0 \\ -4 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$Q \otimes Q= \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\equiv^* (Z_5^2) = \equiv^* Z_5 \otimes \equiv^* Z_5= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & -1 & 4 & -1 \\ 4 & 4 & -1 & -1 \\ 16 & -4 & -4 & 1 \end{bmatrix}$$

By Lemma (3.2) we obtain

$P \otimes P \equiv (Z_5^2)\, Q \otimes Q$

$$= \begin{bmatrix} 25 & 0 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\equiv (Z_5^2)$~ diag {25,-5,-5,1}

Hence K (Z_5^2) = $Z_{25} \oplus Z_5^2$

$= Z_{5^2} \oplus Z_5^2$

When n=3

Let

P□P□P=

$$\begin{bmatrix} -64 & 16 & 16 & -4 & 16 & -4 & -4 & 1 \\ 16 & 0 & -4 & 0 & -4 & 0 & 1 & 0 \\ 16 & -4 & 0 & 0 & -4 & 1 & 0 & 0 \\ -4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 16 & -4 & -4 & 1 & 0 & 0 & 0 & 0 \\ -4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Q□Q□Q=

$$\begin{bmatrix} -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\equiv^* (Z_5^3) = \equiv^* Z_5 \square \equiv^* Z_5 \square \equiv^* Z_5 =$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & -1 & 4 & -1 & 4 & -1 & 4 & -1 \\ 4 & 4 & -1 & -1 & 4 & 4 & -1 & -1 \\ 16 & -4 & -4 & 1 & 16 & -4 & -4 & 1 \\ 4 & 4 & 4 & 4 & -1 & -1 & -1 & -1 \\ 16 & -4 & 16 & -4 & -4 & 1 & -4 & 1 \\ 16 & 16 & -4 & -4 & -4 & -4 & 1 & 1 \\ 64 & -16 & -16 & 4 & -16 & 4 & 4 & -1 \end{bmatrix}$$

And we obtain

P□P□P $\equiv^* (Z_5^3)$ Q□Q□Q =

$$\begin{bmatrix} 125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -25 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$\equiv^* (Z_5^3)$~diag {125,-25,-25,5,-25,5,5,1}

Hence

K (Z_5^3) = $Z_{125} \oplus Z_{25}^3 \oplus Z_5^3$.

$= Z_{5^3} \oplus Z_{5^2}^3 \oplus Z_5^3$.

When n=4

we repeated the same method, and we obtain

P□P□P□P $\equiv^* (Z_5^4)$Q□Q□Q□=

$$\begin{bmatrix}
625 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -152 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 25 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 25 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 25 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Hence

K $(Z_5^4) = Z_{625} \oplus Z_{125}^4 \oplus Z_{25}^6 \oplus Z_5^4$

$= Z_{5^4} \oplus Z_{5^3}^4 \oplus Z_{5^2}^6 \oplus Z_5^4.$

The general case for p=5 is given by the following.

5. Conclusions

This is as result the primary decomposition of the factor group K(G) for Z_5^n is given by the next theorem:

Theorem (5.1)

$$K(Z_p^n) = Z_{p^n}^{\binom{n}{n}} \oplus Z_{p^{n-1}}^{\binom{n}{n-1}} \oplus Z_{p^{n-2}}^{\binom{n}{n-2}} \oplus Z_{p^{n-3}}^{\binom{n}{n-3}} \oplus \ldots \oplus Z_p^{\binom{n}{1}}$$

$$K(Z_p^n) = \bigoplus_{i=0}^{(n-1)} Z_{p^{n-i}}^{\binom{n}{n-i}} .$$

Where p=5, and $\binom{n}{i} = \frac{n!}{i!(n-i)}$.

Proof: assume the statement holds for k factors

Then

$$\equiv^* (Z_p^k) \sim \text{diag} \{\pm p^k; \underbrace{\pm p^{k-1}, \ldots, \pm p^{k-1}}_{\binom{k}{k-1}}; \underbrace{\pm p^{k-2}, \ldots, \pm p^{k-2}}_{\binom{k}{k-2}}; \ldots; \underbrace{\pm p^2, \ldots, \pm p^2}_{\binom{k}{2}}; \underbrace{\pm p, \ldots, \pm p}_{\binom{k}{1}}; \mp 1\}$$

Now by proposition (3.2)

$$\equiv^* (Z_p^{k+1}) = \equiv^* (Z_p^k) \square \equiv^* (Z_p).$$

Hence , by lemma (3.2),

$$\equiv^* (Z_p^{k+1}) \sim \text{diag} \{\pm p^{k+1}; \underbrace{\pm p^k, \ldots, \pm p^k}_{\binom{k}{k-1}}; \underbrace{\pm p^{k-1}, \ldots, \pm p^{k-1}}_{\binom{k}{k-2}}; \ldots; \underbrace{\pm p^3, \ldots, \pm p^3}_{\binom{k}{2}}; \underbrace{\pm p^2, \ldots, \pm p^2}_{\binom{k}{1}}; \pm p; \pm p^k; \underbrace{\pm p^{k-1}, \ldots, \pm p^{k-1}}_{\binom{k}{k-1}}; \ldots; \underbrace{p^2, \ldots, p^2}_{\binom{k}{2}}; \underbrace{p, \ldots, p}_{\binom{k}{1}}; \mp 1\}$$

$\binom{k}{i} + \binom{k}{i-1} = \binom{k+1}{i}$ when $0 \le i \le k$, which means p^i appears $\binom{k+1}{i}$ times in the matrix above.

Hence K $(Z_p^{k+1}) = \bigoplus_{i=1}^{(k+1)} Z_{p^i}^{\binom{k+1}{i}}$, p=5 and we have the result stated.

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