On Fuzzy Normed Spaces

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Abstract

In this paper we introduce a new definition of a fuzzy normed space (to the best of our knowledge) then the related concepts such as fuzzy continuous, convergence of sequence of fuzzy points and Cauchy sequence of fuzzy points are discussed in details.

Keywords: fuzzy set, fuzzy normed space, sequence of fuzzy points, fuzzy continuous.

حول الفضاءات القياسية الضبابية

الخلاصة

في هذا البحث قدمنا تعريف جديد للفضاء القياسي الضبابي (حسب علمنا) ثم بعد ذلك قمنا بدراسة المفاهيم المتعلقه بهذا التعريف مثلا الاستمرايه الضبابيه والتقارب للمتتابعات التي عناصر ها نقاط ضبابيه ومتتابعات كوشي بتفاصيل اكثر.

S1: Basic Concepts About Fuzzy Sets

Definition 1.1: [1]

Let X be a nonempty set of elements. A fuzzy set Ã in X is characterized by a membership function, $\mu_{\tilde{A}}: X \to [0.1]$.
Then Ã can be written by $\tilde{A}=\{(x, \mu_{\tilde{A}}(x)) | x \in X, 0 \le \mu_{\tilde{A}}(x) \le 1\}$

Definition 1.2: [1]

Let Ã and Õ be two fuzzy sets in X then

1. $\tilde{A} \subseteq \tilde{O} \Leftrightarrow \mu_{\tilde{A}}(x) \le \mu_{\tilde{O}}(x)$ for all $x \in X$.
2. $\tilde{A} = \tilde{O} \Leftrightarrow \mu_{\tilde{A}}(x) = \mu_{\tilde{O}}(x)$ for all $x \in X$.
3. Then complement of Ã (denoted by $\tilde{A}^c$) is also a fuzzy set with membership function $\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(X)$ for all $x \in X$.
4. $\tilde{A} = \emptyset \Leftrightarrow \mu_{\tilde{A}}(x) = 0$ for all $x \in X$, where $\emptyset$ is the empty fuzzy set.

Definition 1.3: [1]

A fuzzy point P_x in X is a fuzzy set with membership function

$$\mu_{P_x}(y) = \begin{cases} \alpha & if\ y = x \\ 0 & \text{otherwise} \end{cases}$$

For all $y \in X$ where $0 < \alpha < 1$. We denote this fuzzy point by x_α or (x,α).

Definition 1.4: [2]

Two fuzzy poins x_α and y_β are said to be equal if x= y and $\alpha = \beta$ where $\alpha, \beta \in (0,1]$.

Definition 1.5: [1]

Let x_α be a fuzzy point and Ã a fuzzy set in X. then x_α is said to be in Ã or belongs to Ã denoted by $x_\alpha \in \tilde{A}$ if $\alpha \le \mu_{\tilde{A}}(x)$.

Definition 1.6: [3]

Let f be a function from a

Proposition 2.2:

Let (X, ||.||) be an ordinary normed

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nonempty set X into a nonempty set Y. if Õ is a fuzzy set in Y then f^{-1} (Õ) is a fuzzy set in X with membership function $\mu_{f^{-1}(\tilde{O})} = \mu_{\tilde{O}} \circ f$.

If Ã is a fuzzy set in X then f (Ã) is a fuzzy set in Y with membership

$$\mu_{f(\tilde{A})}(y) = \begin{cases} \sup \{\mu_{\tilde{A}}(x) \mid x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 \text{ otherwise} \end{cases}$$

For all y ∈ Y where $f^{-1}(y) = \{x \in X \mid f(x) = y\}$

Proposition 1.7: [4]

Let f: X → Y be a function then for a fuzzy point x_α in X, $f(x_\alpha)$ is a fuzzy point in Y and $f(x_\alpha) = f(x)_\alpha$.

Definition 1.8: [2]

Let X be a vector space over field $\mathbb{K}$ and let Ã be a fuzzy set in X. then Ã is called a fuzzy subspace of X if for all x, y ∈ X and $\lambda \in \mathbb{K}$.

(i) $\mu_{\tilde{A}}(x+y) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$

(ii) $\mu_{\tilde{A}}(\lambda x) \geq \mu_{\tilde{A}}(x)$

S2: Fuzzy Normed Spaces

Definition 2.1:

let X be a vector space over field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Let $\|.\|_f : X \to [0.\infty)$ be a function which assigns to each point x_α in X, $\alpha \in (0,1]$ a nonnegative real number $\| x_\alpha \|_f$ such that

(FN_1) $\| x_\alpha \|_f = 0$ if and only if x = 0

(FN_2) $\| \lambda x_\alpha \|_f = |\lambda| \, \| x_\alpha \|_f$ for all $\lambda \in \mathbb{K}$

(FN_3) $\| x_\alpha + y_\beta \|_f \leq \| x_\alpha \|_f + \| y_\beta \|_f$

(FN_4) if $\| x_\sigma \|_f < r$ where r >0 then there exists $0 < \sigma \leq \alpha < 1$ such that $\| x_\alpha \|_f < r$.

Then $\|.\|_f$ is called fuzzy norm and $(X, \|.\|_f)$ is called fuzzy normed space.

space, define $\| x_\alpha \|_f = \frac{1}{\alpha} \| x \|$ for every $x_\alpha \in X$ where $\alpha \in (0,1]$. Then $(X, \|.\|_f)$ is a fuzzy normed space

Proof: let x_α, $y_\beta \in X$ where $\alpha, \beta \in (0,1]$ and $\gamma \in$ **K**. Then

(FN_1) $\| x_\alpha \|_f = 0 \Leftrightarrow \frac{1}{\alpha} \| x \| = 0 \Leftrightarrow \| x \| = 0 \Leftrightarrow x = 0$

(FN_2) $\| \gamma x_\alpha \|_f = \frac{1}{\alpha} \| \gamma x \| = \frac{|\gamma|}{\alpha} \| x \| = |\gamma| \, \| x_\alpha \|_f$

(FN_3) $\| x_\alpha + y_\beta \| = \|(x+y)_\lambda\|_f$ [$\lambda = \max\{\alpha, \beta\}$]

$$= \frac{1}{\lambda} \| x+y \|$$
$$\leq \frac{1}{\lambda} \| x \| + \frac{1}{\lambda} \| y \|$$
$$\leq \frac{1}{\alpha} \| x \| + \frac{1}{\beta} \| y \|$$
$$\leq \| x_\alpha \|_f + \| y_\beta \|_f$$

(FN_4) if $\| x_\alpha \|_f < r$ where r>0 then for $\sigma \in (0,1]$ with $\alpha \leq \sigma$, we here $\frac{||x||}{\sigma} \leq \frac{||x||}{\alpha} < r$ that is $\| x_\sigma \|_f < r$

The proof of the following result is clear. Hence is omilted

Proposition 2.3:

Let $(X, \|.\|_f)$ be a fuzzy normed space by defining $\| x \| = \| (x,1) \|_f$, then $(X, \|.\|)$ is a normed spaces .

Example 2.4:

Let X=$\mathbb{R}$, then $\| x_\alpha \|_f = \frac{1}{\alpha} |x|$ is a fuzzy norm on $\mathbb{R}$ by proposition 2.2 called the usual fuzzy norm.

Remark 2.5:

From the definition 2.1 we obtain by induction the generalized of (FN_3)

$\|(x_1,\alpha_1) - (x_n,\alpha_n)\|_f \leq \|(x_1,\alpha_1) - (x_2,\alpha_2)\|_f + \|(x_2,\alpha_2) - (x_3,\alpha_3)\|_f + \ldots + \|(x_{n-1},\alpha_{n-1}) - (x_n, \alpha_n)\|_f$

Where $(x_2, \alpha_2), (x_3, \alpha_3), \ldots, (x_{n-1}, \alpha_{n-1}) \in X$

Definition 2.6:
A fuzzy subspace Ŷ of a fuzzy normed space $(X, \|.\|_f)$ is a fuzzy subspace of X considered as a vector space with the fuzzy norm obtained by restricting the fuzzy norm on X to Ŷ.

S3: Open Fuzzy Sets, Closed Fuzzy Sets, Fuzzy Continuity of Functions

Definition 3.1:
Let $(X, \|.\|_f)$ be a fuzzy normed space. Given $x_\alpha \in X$, where $\alpha \in (0,1]$ and a real number r>0

(i) $\tilde{O}(x_\alpha, r) = \{y_\beta \in X: \|x_\alpha - y_\beta\| < r\}$ is open fuzzy ball, where $\beta \in (0, 1]$

(ii) $\tilde{O}(x_\alpha, r) = \{y_\beta \in X: \|x_\alpha - y_\beta\| \leq r\}$ is closed fuzzy ball, where $\beta \in (0, 1]$

(iii) $S(x_\alpha, r) = \{y_\beta \in X : \|x_\alpha - y_\beta\| = r\}$is fuzzy sphere, where $\beta \in (0, 1]$

In all three cases, x_α is called the center and r is radius.

Definition 3.2:
A fuzzy set Ã in fuzzy normed space $(X, \|.\|_f)$ is said to be open if it contains a fuzzy ball about each of it's, fuzzy element.
A fuzzy set Õ is said to be closed if it's complement is open fuzzy set.

Definition 3.3:
Let $(X, \|.\|_f)$ be a fuzzy normed space, an open fuzzy ball $\tilde{O}(x_\alpha, \varepsilon)$ of radius ε is often called an ε - neighborhood of x_α (here ε >0).
By a neighborhood of x_α we mean a fuzzy set of X which contains an ε-neighborhood of x_α.

Definition 3.4:
We call x_α an interior fuzzy point of the the fuzzy set Ã if Ã is a neighborhood of x_α.
The interior of Ã is the set of all interior fuzzy points of Ã and is denoted by int (Ã).
Int (Ã) is open fuzzy set and is the largest open fuzzy set contained in Ã.

Definition 3.5:
Let $(X, \|.\|_{f_1})$ and $(Y, \|.\|_{f_2})$ be a fuzzy normed spaces. A mapping $T:X \rightarrow Y$ is said to be fuzzy continuous at the fuzzy point $x_\alpha \in X$ where $\alpha \in (0,1]$ if for every ε >0 there is δ >0 such that
$\|T(y)_\beta - T(x)_\alpha\|_{f_2} < \varepsilon$ for all $y_\beta \in X$ satisfying $\| y_\beta - x_\alpha\|_{f_1} < \delta$, where $\beta \in (0,1]$
T is said to be fuzzy continuous if it is fuzzy continuous at every fuzzy point $x_\alpha \in X$.

Theorem 3.6:
A mapping T of a fuzzy normed space $(X, \|.\|_{f_1})$ into a fuzzy normed space $(Y, \|.\|_{f_2})$ is fuzzy continuous if and only if the inverse image of any open fuzzy set in Y is open fuzzy set in X.

Proof:
Suppose T is fuzzy continuous. Let Õ be open fuzzy set in Y and $\tilde{U}$ is the inverse image of S i.e T^{-1} (Õ) $=\tilde{U}$. If $\tilde{U} = \emptyset$ it is open fuzzy set. Let $\tilde{U} \neq \emptyset$, for any $x_\alpha \in \tilde{U}$ where $\alpha \in (0,1]$. Let $y_\alpha = T(x_\alpha) = T(x)_\alpha$ since Õ is open, it contains as ε-neighborhood $\tilde{N}_2$ of y_α. Since T is fuzzy continuous, x_α has an δ-neighborhood $\tilde{N}_1$ which is mapped into $\tilde{N}_2$. Since $\tilde{N}_2 \subset$ Õ we have $\tilde{N}_1 \subset \tilde{U}$ so that $\tilde{U}$ is open fuzzy set because $x_\alpha \in \tilde{U}$ was arbitrary.
Conversely, assume that the inverse image of every open fuzzy set in Y is open fuzzy set in X. Then for each $x_\alpha \in X$ where $\alpha \in (0,1]$ and any ε- neighborhood $\tilde{N}_2$ of $T(x)_\alpha$ the inverse image of $\tilde{N}_1$ is open since $\tilde{N}_2$ is open and $\tilde{N}_1$ contains x_α. Hence $\tilde{N}_1$ also contains a δ-neiboughood of x_α which is mapped

$\tilde{N}_2$. Consequently T is fuzzy continuous at x_α. Since $x_\alpha \in X$ was arbitrary T is fuzzy continuous.

Definition 3.7:

Let $\tilde{A}$ be a fuzzy set in a fuzzy normed space $(X, \|.\|_f)$. Then a fuzzy point $x_\alpha \in X$ where $\alpha \in (0,1]$ (which may or not be a fuzzy element of $\tilde{A}$) is called a limit of $\tilde{A}$ if every neighborhood of x_α contains at least one fuzzy element $y_\beta \in \tilde{A}$ distinct from x_α. The fuzzy set consisting of $\tilde{A}$ and its limit fuzzy points is called closure of $\tilde{A}$ and is denoted by cl $(\tilde{A})$. It is the smallest closed fuzzy set containing $\tilde{A}$.

Definition 3.8:

A fuzzy set $\tilde{A}$ of a fuzzy normed space $(X, \|.\|_f)$ is said to be dense in X if cl $(\tilde{A}) = X$.

S4: Convergence, Cauchy Fuzzy Sequences

Definition4.1:

A fuzzy sequence $\{(x_n, \alpha_n)\}$ in a fuzzy normed space $(X, \|.\|_f)$ is said to be convergent to x_α in X where $\alpha_1, \alpha_2 \in (0,1]$ for i=1, 2, … if

$\lim_{n\to\infty} \| (x_n, \alpha_n) - x_\alpha\|_f = 0$

x_α is called the limit if $\{(x_n, \alpha_n)\}$and we write

$\lim_{n\to\infty} (x_n, \alpha_n) = x_\alpha$ or simply $(x_n, \alpha_n) \to x_\alpha$

if $\{(x_n, \alpha_n)\}$ is not convergent then it is called divergent.

Remark 4.2:

If $(x_n, \alpha_n) \to x_\alpha$, an $\varepsilon > 0$ being given, there is a positive integer N such that (x_n, α_n) with n>N lie in ε-neighborhood B (x_α, ε) of x_α that is:

$\| (x_n, \alpha_n) - x_\alpha\|_f < \varepsilon$ for all n>N

Definition4.3:

We call a nonempty fuzzy set $\tilde{A}$ in $(X, \|.\|_f)$ bounded if its fuzzy diameter

$\delta(\tilde{A}) = \sup \{\| x_\alpha - y_\beta\|_f ; x_\alpha, y_\beta \in \tilde{A},$ into $\tilde{N}_2$ because $\tilde{N}_1$ is mapped into

Definition4.4:

In a fuzzy normed space $(X, \|.\|_f)$ we call a sequence $\{(x_n, \alpha_n)\}$ is bounded if the corresponding fuzzy set is bounded.

Remark 4.5:

If $\tilde{A}$ is a bounded fuzzy set then $\tilde{A} \subset \tilde{O}(x_\alpha, r)$ where $x_\alpha \in \tilde{A}$ is any fuzzy element and r>0 is a (sufficiently large) real number.

Theorem 4.6:

Let $(X, \|.\|_f)$ be a fuzzy normed space. Then (i) a convergent fuzzy sequence in X is bounded and its limit is unique.

(ii) if $(x_n, \alpha_n) \to x_\alpha$ and $(y_m, \beta_m) \to y_\beta$ in X

There $\alpha, \alpha_i, \beta, \beta_i \in (0.1]$ i= 1, 2, …

Then $\|(x_n, \alpha_n) - (y_m, \beta_m)\|_f \to \| x_\alpha - y_\beta\|_f$

Proof:

(i) Suppose that $(x_n, \alpha_n) \to x_\alpha$ then taking $\varepsilon = 1$ we can find N > 0 such that $\| (x_n, \alpha_n) - x_\alpha \|_f < 1$ for all n > N.

Hence by remark 2.5 for all n we have $\|(x_n, \alpha_n) - x_\alpha \|_f < 1 + a$, where $a = \max\{\|(x_1, \alpha_1) - x_\alpha \|_f, \|(x_2, \alpha_2) - x_\alpha \|_f, \ldots, \| (x_n, \alpha_n) - x_\alpha \|_f\}$

This shows that $\{(x_n, \alpha_n)\}$ is bounded. Now, assuming that$(x_n, \alpha_n) \to x_\alpha$ and

$(x_n, \alpha_n) \to z_\beta$, we obtain from ($FN_3$)

$0 \leq \| x_\alpha - z_\beta \|_f \leq \| x_\alpha - (x_n, \alpha_n) \|_f + \|(x_n, \alpha_n) - z_\beta\|_f \to 0+0$

Thus $\| x_\alpha - z_\beta \|_f = 0$ which implies that $x_\alpha = z_\beta$.

(ii) By remark 2.5 we have $\|(x_n, \alpha_n) - (y_m, \beta_m) \|_f \leq \|(x_n, \alpha_n) - x_\alpha\|_f + \| x_\alpha - y_\beta\|_f + \|y_\beta - (y_m, \beta_m)\|_f$

Hence we obtain

$\| (x_n, \alpha_n) - (y_m, \beta_m) \|_f - \| x_\alpha - y_\beta\|_f \leq \| (x_n, \alpha_n) - x_\alpha\|_f + \|(y_m, \beta_m) - y_\beta\|_f$

And a similar inequality by interchanging (x_n, α_n) and x_α as well

$\alpha, \beta \in (0,1]\}$ is finite.

by -1. Together $|\ \|(x_n, \alpha_n) - (y_m, \beta_m)\|_f - \|x_\alpha - y_\beta\|_f\ | \leq \|(x_n, \alpha_n) - x_\alpha\|_f + \|(y_m, \beta_m) - y_\beta\|_f \to 0$ as $n \to \infty$.

Definition 4.7:

A sequence $\{(x_n, \alpha_n)\}$ in a fuzzy normed space $(x,\|.\|_f)$ is said to be Cauchy if for every $\varepsilon > 0$ there is integer $N>0$ such that $\|(x_m, \alpha_m) - (x_n, \alpha_n)\|_f < \varepsilon$ for every $m,n > N$.

Theorem 4.8:

Every convergent fuzzy sequence in a fuzzy normed space $(X,\|.\|_f)$ is Cauchy.

Proof:

Let $\{(x_n, \alpha_n)\}$ be a sequence of fuzzy points in X such that $(x_n, \alpha_n) \to x_\alpha$ then for every $\varepsilon>0$ there is an integer $N>0$ such that $\|(x_n, \alpha_n) - x_\alpha\| < \frac{\varepsilon}{2}$ for all $n>N$.

Hence by (FN_3) we obtain for $m,n > N$.

$\|(x_m, \alpha_m) - (x_n, \alpha_m)\|_f \leq \|(x_m, \alpha_m) - x_\alpha\|_f + \|x_\alpha - (x_n, \alpha_n)\|_f \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

This shows that $\{(x_n, \alpha_n)\}$ is Cauchy.

Therome 4.9:

Let $\tilde{A}$ be a nonempty fuzzy set in a fuzzy normed space $(X,\|.\|_f)$ and cl $(\tilde{A})$ its closure.

Then

(i) $x_\alpha \in$ cl $(\tilde{A})$ if and only if there is a fuzzy sequence $\{(x_n, \alpha_n)\}$ in $\tilde{A}$ such that $(x_n, \alpha_n) \to x_\alpha$.

(ii) $\tilde{A}$ is closed if and only if the situation $(x_n, \alpha_n) \in \tilde{A}$ and $(x_n, \alpha_n) \to x_\alpha$ implies $x_\alpha \in \tilde{A}$.

Proof:

(i) Let $x_\alpha \in$ cl $(\tilde{A})$. If $x_\alpha \in \tilde{A}$ a fuzzy sequence of that type is $x_\alpha, x_\alpha, \ldots$

If $x_\alpha \notin \tilde{A}$ it is a limit of $\tilde{A}$. Hence as (y_m, β_m) and y_β and multiplying

$(x_n, \alpha_n) \to x_\alpha$ because $\frac{1}{n} \to 0$ as $n \to \infty$.

Conversely, if $\{(x_n, \alpha_n)\}$ is in $\tilde{A}$ and $(x_n, \alpha_n) \to x_\alpha$ then $x_\alpha \in \tilde{A}$ or every neighborhood of x_α contains a fuzzy point $(x_n, \alpha_n) \neq x_\alpha$ so that is x_α a limit of $\tilde{A}$. Hence $x_\alpha \in$ cl $(\tilde{A})$. It is clear that $\tilde{A}$ = cl $(\tilde{A})$.

(ii) $\tilde{A}$ is closed if and only if $\tilde{A}$ = cl $(\tilde{A})$ so that (ii) follows readily from (i).

Theorem 4.10:

A mapping $T:X \to Y$ of a fuzzy normed space $(x, ||.||_{f_1})$ into a fuzzy normed space $(Y, ||.||_{f_2})$ is fuzzy continuous at a fuzzy point $x_\alpha, \in X$ if and only if $(x_n, \alpha_n) \to x_\alpha$ implies $(T(x_n), \alpha_n) \to T(x)_\alpha$.

Proof:

Assume that T is fuzzy continuous at x_α. Then given $\varepsilon > 0$ there is $\delta > 0$ such that $\|y_\beta - x_\alpha||_{f_1} < \delta$ implies $\|T(y)_\beta - T(x)_\alpha||_{f_2} < \varepsilon$. Let $(x_n, \alpha_n) \to x_\alpha$. Then there is $N > 0$ such that $\|(x_n, \alpha_n) - x_\alpha||_{f_1} < \varepsilon$ for all $n > N$.

Hence for all $n > N$.

$\|(T(x_n), \alpha_n) - T(x)_\alpha||_{f_2} < \varepsilon$ this means that $(T(x_n), \alpha_n) \to T(x)_\alpha$.

Conversely, we assume that $(x_n, \alpha_n) \to x_\alpha$ implies $(T(x_n), \alpha_n) \to T(x)_\alpha$, and prove that T is fuzzy continuous at x_α. Suppose this is false. Then there is an $\varepsilon > 0$ such that for every $\delta > 0$ there is $y_\beta \neq x_\alpha$ satisfying $\|y_\beta - x_\alpha||_1 < \delta$ but $||_{f_2}$ $T(y)_\beta - T(x)_\alpha\| \geq \varepsilon$

In particular for $\delta = \frac{1}{n}$ there is $\{(x_n, \alpha_n)\}$ satisfying $\|(x_n, \alpha_n) - x_\alpha||_{f_1} < \frac{1}{n}$ but $\|(T(x_n), \alpha_n) - T(x)_\alpha||_{f_2} \geq \varepsilon$. Clearly $(x_n, \alpha_n) \to x_\alpha$ but $\{T(x_n),$

for each n=1,2,… the fuzzy ball B $(x_\alpha, \frac{1}{n})$ contains an $(x_n, \alpha_n) \in \tilde{A}$ and $\alpha_n)\}$ does not converse to $T(x)_\alpha$. This contradicts $(T(x_n), \alpha_n) \rightarrow T(x)_\alpha$ and proves the theorem.

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