Eng. & Tech. Journal, Vo.29, No .8 , 2011

Solvability of Semilinear Initial Value Perturbed Control Problems with Unbounded Control Operators

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Received on: 7/4/2011
Accepted on: 7/4/2011

Abstract

In this paper the local existence and uniqueness of the mild solution to some operator semi-linear initial value control problem were studied and developed by using the theory of perturbation, composite, admissibility and "Banach contraction principle", in arbitrary Hilbert space H via perturbation composite semigroup approach.

قابلية الحل لمسائل سيطرة مقلقلة شبه الخطية ذات قيم ابتدائية بمؤثرات سيطرة غير مقيدة

الخلاصة

لقد تم في هذا البحث , دراسة وتطوير الوجودية المحلية والوجدانية للحلول الضعيفة لبعض مؤثرات السيطرة شبه الخطية ذات القيم الابتدائية ضمن فضاءات هلبرت ملائمة باستخدام نظرية القلقلة , التركيب , القبولية ومبدأ الانكماش لبناخ في فضاء هلبرت بأستخدام شبه الزمرة المركبة المقلقلة.

1. Introduction

The theory of one parameter semigroup of linear operator on Banach spaces started in the first half of this century, acquired its core in 1948 with the Hill-Yoside generation theorem, and attained its first apex with the 1957 edition of semigroup and functional analysis by E. Hille and R. S. Phillips, in 1970's and 1980's.The theory reached a certain state of perfection, which is well represented in the monograph by [2], [4], [9] and others.

Bahuguna.D in 1997 [1], had studied the local existence without uniqueness of the mild solution to the semi linear initial value problem:

$$\frac{dx}{dt}+Ax(t)=f(t,x(t))+\int_{s=0}^{t} h(t-s)g(s,x(s))ds, t>0 \quad (1)$$
$$x(0)=x_0$$

where A is the infinitesimal generator of a C_o-semigroup defined from $D(A) \subset X$ into X and f and g are a nonlinear continuous ap define from $[0,r) \times X$ into X and h is the real valued continuous function defined from [0,r) into R where R is the real number.

Pavel in 1999[5] studied the uniqueness of the miled solution to the semi linear initial value problem given by (1).

Radhi A.Zboon and Manaf A.Salah in 2007[6] studied the local existence and uniqueness of the mild solution and also studied the controllability to the semilinear intial value problem:

$$\frac{dx}{dt}+Ax(t)=f(t,x(t))+\int_{s=0}^{t} h(t-s)g(s,x(s))ds+Bu(t)dt, t>0 \quad (2)$$
$$x(0)=x_0$$

where A is the infinitesimal generator of a C_o-semigroup defined from $D(A) \subset X$ into X and f and g are a nonlinear continuous map define from $[0,r) \times X$ into X and h is the real valued continuous function defined

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https://doi.org/10.30684/ etj.29.8.6
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from [0,r) into R where R is the real number and B is a bounded linear operator definded from U into X. Where U is a Banach space and u(.)be arbitrary control function is given in $L^p([0,r):U)$, a Banach space of control functions with $\|u(t)\|_U \le k_1 \quad for\ 0 \le t < r.$

Our work is concerned the following semi linear initial value perturbed control problems with unbounded control operators by using perturbation composite semigroup , As one can see this in the main problem formulation with improving some necessary and sufficient conditions of solvability as well as its admissibility.

2. Some mathematical concepts

The following definitions are adapted in this work.

Definition(2.1) :

Let L(H) be a Banach space, a one-parameter family

$(t)\}_{t\ge 0} \subset L(L((H)),\ t\in[0,\infty)$ of bounded linear operators defined by:

$(t) = S_1(t)ZS_2(t)$, (3)

for generator $+ \Delta$, for any $Z\in L(H)$ and $t \in[0,\infty)$ is called composite perturbation semigroup, where $S_1(t),S_2(t)$ are two perturbation semigroups defined from H into H for $(A_1+\Delta A_1)$ and$(A_2 + \Delta A_2)$ respectively.

The infinitesimal generator $+ \Delta$ of (t) of problem formulation on a uniform operator topology defined as the limit:$(\mathbb{A} + \Delta\mathbb{A})Z = \tau\text{-}\lim_{t\downarrow 0}\left\{\frac{\mathbb{S}(t)Zh-Z}{t}\right\} \in D(\mathbb{A} + \Delta\mathbb{A})$, where $D(\mathbb{A} + \Delta\mathbb{A}) \subset L(H)$ is the domain of $A+\Delta A$ defined as follows: $D(A + \Delta A) = \left\{ Z\in L(H):\ \tau\text{- exist in } \{L(H),\tau\} \right\}$.where $\{L(H), \tau\}$ stands for L(H) equipped with the strong operator topology τ, i.e., topology induced by family of seminorms $\rho = \{\rho_h\}_{h\in H}$, where siminorms $\rho_h(Z) = \|Zh\|_H,\ Z \in L(H)$.

Lemma(2.3),[7]:

Consider the problem formulation $= S_1(t)ZS_2(t),\ t \ge 0$ be a composite perturbation semigroup defined on L(L(H)); $S_1(t)$ and $S_2(t)$ are, perturbation semigroups defined on L(H) then

a- The family $\{\ (t)\}_{t\ge 0} \subseteq L(H),\ t \ge 0$ is a semigroup, i.e.,

1. $(0)Z = Z,\ \forall\ Z \in L(H)$
2. $S(t + s)Z = S(t)(S(s)z) = S(s)(S(t))Z$ for $Z \in L(H)$, and $t, s \in [0, \infty)$.

b- $\|S(t)\|_{L(H)} \le M_1M_2 e^{t(w_1+w_2)+M_1\|\Delta A_1\|_{L(H)}+M_2\|\Delta A_2\|_{L(H)}}$, for $t \in [0, \infty)$.

c- $(t) \in L(L(H))$ is a strong-operator and continuous at the origin, i.e., $\tau\text{-}\lim_{t\downarrow 0} \|(S(t)Z)h - (S(0)Z)h\|_H = 0,\ h \in H,\ Z \in L(H)$.

Lemma(2.4),[7]:

The operator $A + \Delta A$ of problem formulation is infinitesimal generator for (t) defined on its domain $D(A+ \Delta A)$ satisfying the following properties:

(a) $D(A + \Delta A)$ is strong-operator dense in L(H).

(b) $A + \Delta A$ is uniform-operator closed on L(H).

(c) For $Z \in L(H)$:

$\int_0^t (S(r)Z)\, dr \in D(A + \Delta A)$, and $(A + \Delta A)\ \left(\int_0^t \mathbb{S}(=\ (t)Z - Z.$

(d) For $Z \in D(A)$,$S(t)Z \in D(A + \Delta A)$, the function $[0,\infty) \ni t \longmapsto (t)Z\in L(H)$ is continuously differentiable in $\{L(H),\tau\}$ and

$\frac{d}{dt}(S(t)Z) = (A + \Delta A)\ (S(t)Z)= S(t)((A + \Delta A)Z)$

(d) For $Z \in D(A + \Delta A)$ and $h \in D(A_1 + \Delta A_1)$ $((A + \Delta A)Z)h = (A_1 + \Delta A_1)Zh + Z(A_2 + \Delta A_2)h$

Definition (2.5):

Let U be a Hilbert space and $\Delta B \in L(U, H_0)$, then $B \in L(U, H_{-1})$ is said to be admissible perturbed control operator for $\{\ S(t)\}_{t\ge 0}$ for the problem if for some $\tau > 0$ and any $u \in L^2([0, \infty], U)$, we have that $\phi_\tau u \in H_0$, and

$$\phi_\tau u = \int_0^t S_{-1}(\tau - r)(B + \Delta B)u(r)\, dr \quad (4)$$

Remarks (2.6):

If B is admissible perturbed control operator, then for any $\tau > 0$, ϕ_τ defined above is a bounded linear operator from $L^2([0, \infty], U)$ to H_0 (this follows from the closed graph theorem). In the other hand: $\|\phi_\tau u\|_{H_0} \le k_\tau \|u\|_{L^2([0,\infty),U)}$, $\forall$ u $\in L^2(0, \infty)$, U). (5)

Theorem (2.7), [7]:

Let $\{S(t)\}_{t\ge0}$ be a family of a C_0-composite perturbation semigroup generated by unbounded linear operator A + ΔA satisfies:

$\|S(t)\|_{L(L(H))} \le M_1M_2 e^{((w_1+w_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)t}$, for $M_1,M_2\ge1$, $w_1,w_2\ge0$ and $\Delta A_1, \Delta A_2 \in L(H)$.Then the resolvent set $\rho(A + \Delta A)$ contains the ray $(w_1+w_2 +M_1\|\Delta A_1\| + M_2\|\Delta A_2\|, \infty)$such that the resolvent operator of A+ΔA is estimated as:

$$\|R(\lambda : A + \Delta A)\| \le \frac{M_1M_2}{\mathrm{Re}\lambda - [(w_1+w_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|]}$$

,for $\mathrm{Re}\lambda > (w_1+w_2)+M_1\|\Delta A_1\|_{L(H)}+M_2\|\Delta A_2\|_{L(H)}$

The following necessary conditions (1-12) are required to prove the following theorem of problem formulation. $\dot{Z}(t) = (A + \Delta A)Z(t) + f(t, Z(t)) + \int_0^t h(t - s)g(s, Z(s))\, ds + ((B + \Delta B)u)(t), t > 0$ (6)

$Z(0) = Z_0$.

The local existence and uniqueness of a mild solution to the semilinear initial value perturbed unbounded control problem (problem formulation) will be developed by assuming the following assumptions:

1. Let H_0=L(H) is a Banach space with the norm $\|Z\|_{H_0} = \|R(\lambda; (A+\Delta A))Z\|_{H1}$ for $Z \in H_0$ and $\lambda \in \rho(A+\Delta A)$;we define a family of seminorm $P_*=\{P_*h\}_{h\in H}$, where $P_*h(Z)=\|R(\lambda;(A+\Delta A))Zh\|$,for $Z \in H_0$, and H_0 with the strong operator topology τ_* induced by p_* is denoted by $\{H_0, \tau_*\}$. It is clear that:
$\|Z\|_{H_0} = \|R(\lambda; (A+ \Delta A))Z\|_{D(A+\Delta A)} = \sup \frac{P_*h(Z)}{\|h\|_H}$, for $Z \in H_0$ and $\lambda \in \rho(A+\Delta A)$ (H_0 does not depend on λ).
2. $(A + \Delta A)Zh = AZh + \Delta AZh = (A_1Zh + ZA_2h) + (\Delta A_1Zh + Z\Delta A_2h)= (A_1 + \Delta A_1)Zh + Z(A_2 + \Delta A_2)h$,for any $Z \in H_0$ and $h \in H$, is infinitesimal generator of a C_o composite perturbation semigroup $\{S(t)\}_{t\ge0}$ and with domain $D(A + \Delta A) = D(A) \subseteq H_0$.
3. A_1, A_2 are linear unbounded operators on H generating C_o- semigroup $\{T_1(t)\}_{t\ge0} \subset H_0$ and $\{T_2(t)\}_{t\ge0} \subset H_0$, respectively.
4. $A_1 + \Delta A_1$ with domain $D(A_1 + \Delta A_1) = D(A_1) \subseteq H$ and $A_2 + \Delta A_2$ with the domain $D(A_2 + \Delta A_2) = D(A_2) \subset H$ are linear perturbation unbounded linear operator on H generating C_o-perturbation semigroups $\{S_1(t)\}_{t\ge0} \subset H_0$ and $\{S_2(t)\}_{t\ge0} \subset H_0$, respectively.
5. Let H_{-1} is the set of all equivalence classes of norm bounded Cauchy sequences in $\{H_0, \tau^*\}$ and an element of H_{-1} is denoted by $\tilde{Z} = [\{Z_n\}]$, where $[\{Z_n\}]$ denotes the equivalence class containing $\{Z_n\}_{n\in N}$. We define a family of seminorm $p_{-1} = \{p_{-1}h\}_{h\in H}$, where
$$p_{-1}h(\tilde{Z}) = p_{-1}([\{Z_n\}]) = \lim_{n\to\infty} p_*h(Z_n),$$
for $\tilde{Z} = [\{Z_n\}] \in H_{-1}$ and a norm $\|.\|_{-1}$ as follows:
$$\|\tilde{Z}\|_{-1} = \sup_{\substack{h\ne0\\ h\in H}} \frac{p_{-1}([Z_n])}{\|h\|} = \sup_{\substack{h\ne0\\ h\in H}} \frac{p_*h(Z_n)}{\|h\|_H} =$$

$\sup_{\substack{h\neq 0\\ h\in H}} \lim_{n\to\infty}$ ||R(λ;A+ΔA) Z_n h||_(H_1)/〖||h||〗_H $= \tau_* - \lim_{n\to\infty} \|R(\lambda;A+\Delta A)Z_n\|_H = \|R(\lambda;A+\Delta A)Z\|_H$, for $\lambda\in\rho(A+\Delta A)$ and $\{Z_n\}\in H_0$, whre $Z\in H_{-1}$.

6. Let O be an open subset of $[0, r)\times H_0$ for $0 < r \le \infty$, where H_0 is densely and continuously embedded in a Banach space H_{-1}.

7. For every $(t, Z)\in O$, there exists a neighborhood $G\subset O$ of (t, Z); the nonlinear maps $f, g: [0, r)\times H_0 \longrightarrow H_{-1}$ satisfy the locally Lipschitize condition: $\|f(t, Z) - f(s, Z_1)\|_{H_{-1}} \le L_0\|Z - Z_1\|_{H_0}$, $\|g(t, Z) - g(s, Z_1)\|_{H_{-1}} \le L_1\|Z - Z_1\|_{H_0}$, $L_1>0$, $L_0>0$ for all (t, Z) and $(s, Z_1)\in G$.

8. For $t' > 0$, $\|f(t, v)\|_{H_{-1}} \le B_1$, $\|g(t, v)\|_{H_{-1}} \le B_2$, for $0 \le t \le t'$ and for every $v\in H_0$.

9. For $t'' > 0$, $\| R(\lambda; A\ A)(S + \Delta_{-1}(t- I)) Z_0|_{H_0}| \le \delta'$, $0 \le t \le t''$ and $\delta'<\delta$.

10. h is a continuous function which at least $h\in L^1(0, r: \mathbb{R})$, where $\mathbb{R}$ is the real numbers, such that $h_t = \int_0^r |h(s)|ds$.

11. u(.) be an arbitrary the control function is defined in $L^2_{loc}([0,\infty): U)$,with the norm:

$$\|u\|_{L^2_{loc}([0,\infty):U)} = t^{1/2}\|u\|_{L^2([0,\infty):U)} \le t^{1/2}k_1$$

for $0 \le t < r$ As frechet space of the control functions with U as a Hilbert space and B is a perturbed unbounded control linear operator (admissible) [see definition(2.5)] such that $B+\Delta B\in L(U, H_{-1})$.

12. Let $t_1 > 0$such that $t_1 = \min\{t', t'', r\}$, satisfies the condition:

i. $t_1 \le$

$$\frac{1}{(W_1+W_2)+M_1\|\Delta A_1\|_{H_0}+M_2\|\Delta A_2\|_{H_0}}.\delta A_x$$

where:

$$A_x = (-\log(k_{t_1}t_1^{1/2}k_1 + A_y).A_z.(\delta-\delta'),$$

suchthat

$$A_y = \frac{M_1M_2(\delta h_{t_1}L_1+\delta L_0+B_1+h_{t_1}B_2)}{(W_1+W_2)+M_1\|\Delta A_1\|_{H_0}+M_2\|\Delta A_2\|_{H_0}},$$ and

$$A_z = \frac{\mathrm{Re}\lambda-((W_1+W_2)+M_1\|\Delta A_1\|_{H_0}+M_2\|\Delta A_2\|_{H_0}).}{M_1M_2}.$$

The condition of t_1 is necessary for local solution of (6) and to ensure equation (15) satisfied and then , contraction principle will be guaranteed and hence the local solvability is then ensured. Also the mild solution of equation (6) have been adapted as follows:

Definition (3.1):

A continuous function Z_u is said to be a mild solution to the semilinear initial value problem If it satisfies the following:

$$Z_u(t) = S_{-1}(t)Z_0+\int_0^t S_{-1}(t-s)[((B+\Delta B))(s)+f(s, Z(s))+\int_0^t h(s-\tau).g(s, Zu(\tau))d\tau]\,ds\ \forall u\in L^2_{loc}([0,\infty): U) \text{ and } Z\in L(H). \quad (7)$$

We proved the following lemma as a necessary result for main theorem.

Lemma (3.2):

Consider te problem formulation with assumption(1-12),let $S_{-1}(t)$ be a continuous extension of the perturbation composite semigroup S(t) on H_{-1} and $R(\lambda; A + \Delta A)$ is the resolvent operator of the infinitesimal generator $A + \Delta A$ of a perturbation composite semigroup S(t),$t \ge 0$ such that: $\mathrm{Re}\lambda > (W_1 + W_2) + M_1\|\Delta A_1\|_{H_0} + M_2\|\Delta A_2\|_{H_0}$, where

$W_1, W_2 \geq 0$, $M_1, M_2 \geq 1$ and bounded operators $\Delta A_1, \Delta A_2 \in H_0$. Then:

$$\|S_{-1}(t)\|_{H_{-1}} \leq M_1M_2 e^{t((W_1+W_2)+M_1\|\Delta A_1\|_{H_0}+\|\Delta A_2\|_{H_0})},$$

for $M_1, M_2 \geq 1$, $W_1, W_2 \geq 0$ and bounded operators $\Delta A_1, \Delta A_2 \in H_0$.

Proof:

Let $Z \in H_{-1}$. From condition(6),H_0 is dense in H_{-1}, so there exists a sequence $\{Z_n\} \in H_0$ such that $Z_n \longrightarrow Z \in H_{-1}$.

By using remark(2.6), we get:

$$\|S_{-1}(t)\|_{H_{-1}} = \|\tau_*\text{-}\lim_{n\to\infty} S(t)Z_n\|_{H_0}.$$

From lemma(2.3)(b),we have that:

$$\|\tau_*\text{-}\lim_{n\to\infty} S(t)Z_n\| \leq M_1M_2 e^{t((W_1+W_2)+M_1\|\Delta A_1\|_{H_0}+\|\Delta A_2\|_{H_0})} \|\tau_*\text{-}\lim_{n\to\infty} Z_n\|_{H_0},$$

where $M_1, M_2 \geq 1$, $W_1, W_2 \geq 0$. Now

$$\|\tau_*\text{-}\lim_{n\to\infty} S(t)Z_n\| \leq M_1M_2 e^{t((W_1+W_2)+M_1\|\Delta A_1\|_{H_0}+\|\Delta A_2\|_{H_0})} \|Z\|_{H_{-1}}$$

Hence:

$$\|{}_{-1}(t)\|_{H_{-1}} \leq M_1M_2 e^{t((W_1+W_2)+M_1\|\Delta A_1\|_{H_0}+\|\Delta A_2\|_{H_0})} \|Z\|_{H_{-1}}. \quad \blacksquare$$

Our interest will focus on the linear control operator B, it is called unbounded if it is a bounded operator from U to some larger Banach space H_{-1}, such that $H_0 \subseteq H_{-1}$; but not from U to H_0.

The work of [3,6, 8] which induce the usual generator and input control operator which is bounded linear operator on certain Banach spaces while the following theorem generalized to induce semi-linear dynamical control system with perturbed generator and unbounded perturbed control input on a certain Hilbert space.

Theorem (3.3):

Assume that hypothesis of problem formulation (1-12) are hold, then for every $Z_0 \in H_0$, there exists a fixed number t_1, $0 < t_1 < r$, such that the initial value perturbation control problem has a unique local continuous strong solution $Z_u \in C((0, t_1]:H_0)$, for every control function $u(.) \in L^2_{loc}((0, \infty):U)$.

Proof:

Without losing of generality, we may suppose that $r < \infty$, because we are concerned here with the local existence only.For a fixed point $(0,Z_0)$ in the open subset O of $[0, r)\times H_0$, we choose $\delta > 0$ such that the neighborhood G of the point $(0, Z_0)$ is defined as follows:

$$G = \{(t, Z) \in O: 0 \leq t \leq t', \|Z - Z_0\|_{H_{-1}} \leq \delta\} \subset O,$$

since O is an open subset of $[0, r)\times H_0$ .Set $Y = C([0, t_1]: H_{-1})$, then Y is a Banach space with the supremum norm:

$$\|y\|_Y = \sup_{0\leq t\leq t_1} \|y(t)\|_{H_{-1}} = \sup_{0\leq t\leq t_1} \|R(\lambda;A + \Delta A)y(t)\|_{H_0}.$$

Let S_u be the nonempty subset of Y, defined as follows:

$$S_u = \{Z_u \in Y: Z_u(0) = Z_0, \|Z_u(t)-Z_0\|_{H_{-1}} = \|R(\lambda;A + \Delta A)(Z_u(t)-Z_0)\|_{H_0} \leq \delta, 0 \leq t \leq t_1\}. \quad (8)$$

To prove the closedness of S_u as a subset of Y , Let $Z_u^n \in S_u$, such that $Z_u^n \xrightarrow{p.c.} Z_u$ as $n \longrightarrow \infty$, we must prove that $Z_u \in S_u$ where (P.C) stands for point wise convergence.

Since $Z_u^n \in S_u$, then we have $Z_u^n(0) = Z_0$ and:

$\|R(\lambda;A + \Delta A)(Z_u^n(0) - Z_0)\|_{H_0} \leq \delta,\ 0 \leq t \leq t_1$.

Since $Z_u^n \xrightarrow{u.c} Z_n$, hence $Z_u \in Y$, where (u.c) stands for the uniform convergence, and also since $Z_u^n \xrightarrow{u.c} Z_u$, then $\| Z_u^n - Z_u\|_Y \longrightarrow 0$ and therefore:

$$\sup_{0\leq t\leq t_1} \|R(\lambda;A + \Delta A)(Z_u^n(t) - Z_u(t))\|_{H_0} \longrightarrow 0 \text{ as } n \longrightarrow \infty ,$$

which implies that: $\|R(\lambda;A + \Delta A)(Z_u^n(t) - Z_u(t))\|_{H_0} \longrightarrow 0$ as $n \longrightarrow \infty$,

for every $0 \leq t \leq t_1$, i.e., andτ_*- $\lim_{n\to\infty} R(\lambda;A + \Delta A) Z_u^n(t) = R(\lambda;A + \Delta A)Z_u(t),\ \forall\ 0 \leq t \leq t_1$, (9)

hence:

τ_*- $\lim_{n\to\infty} R(\lambda;A + \Delta A) Z_u^n(0) = R(\lambda;A + \Delta A)Z_u(0)$ (by (10)) .

since $Z_u^n \in S_u$ and $Z_u^n(0) = Z_0$ for any $n\in$ □ , that yields: τ_*- $\lim_{n\to\infty} R(\lambda;A + \Delta A) Z_u^n(0) = R(\lambda;A + \Delta A)Z_0 = R(\lambda;A + \Delta A)Z_u(0)$.

Hence $Z_0 = Z_u(0)$. Now: $\|R(\lambda;A+\Delta A)(Z_u(t)-Z_0)\|_{H_0} = \|R(\lambda;A+\Delta A)\tau_{-1}$- $\lim_{n\to\infty} Z_u^n(t)-Z_0\|_{H_0} = \tau_*$- $\lim_{n\to\infty} \|R(\lambda;A + \Delta A)\ (Z_u^n(t) - Z_0)\| \leq \tau_*$- $\lim_{n\to\infty} \delta$.

Thus $Z_u(t) \in S_u$ .Since $Z_u(t)$ arbitrary element in S_u, hence S_u is a closed subset of Y .

Now, define a map $F_u: S_u \longrightarrow Y$, given by: $(F_uZ_u)(t) = S_{-1}(t)Z_0 + \int_0^t S_{-1}(t - s)(B + \Delta B)u(s)\, ds + \int_0^t [S_{-1}(t - s) f(s, Z(s)) + \int_0^t h(t - \tau)g(\tau, Z(\tau))\, d\tau]\, ds$, for arbitrary $u(.) \in L^2_{loc}([0, \infty): U)$.To show that $F_u(S_u) \subseteq S_u$,let Z_u be an arbitrary element in S_u and let $F_uZ_u \in F_u(S_u)$, to prove that $F_uZ_u \in S_u$ for an arbitrary element Z_u in S_u.From (8), notice that $F_uZ_u \in Y$ (by the definition of the map F_u) and $(F_uZ_u)(0) = Z_0$ (by (10)). Notice also that

$$\|(F_uZ_u)(t) - Z_0\|_{H_{-1}} = \|\ _{-1}(t)Z_0 - Z_0 + \int_0^t S_{-1}(t - s)(B + \Delta B)u(s)\, ds + \int_0^t S_{-1}(t - s)[f(s, Z(s)) + \int_0^t h(t - \tau)g(\tau, Z(\tau))\, d\tau]\, ds\|_{H_{-1}} = \|R(\lambda;A + \Delta A)(S_{-1}(t)Z_0 - Z_0 + \int_0^t S_{-1}(t - s)\, (B + \Delta B)u(s)ds + \int_0^t S_{-1}(t-s)[f(s,Z(s)) + \int_0^t h(t-\tau)g(\tau,Z(\tau))d\tau]ds)\|_{H_0}$$

(11)

By adding and subtracting the following terms

$\int_0^t S_{-1}(t-s)f(s, Z_0)\, ds + \int_0^t S_{-1}(t-s)\left(\int_0^t h(s-\tau)g(\tau, Z_0)\, d\tau\right) ds$, we obtain:

$$\|(F_uZ_u)(t) - Z_0\|_{H_{-1}} = \|R(\lambda; A + \Delta A)(S_{-1}(t)Z_0 - Z_0 + \int_0^t S_{-1}(t-s)(B + \Delta B)u(s)\, ds + \int_0^t S_{-1}(t-s)[f(s, Z_u(s)) + \int_0^t h(t-\tau)\, g(\tau,Z_u(\tau))\, d\tau]ds + \int_0^t S_{-1}(t-s) f(s, Z_0)\, ds - \int_0^t S_{-1}(t-s) f(s, Z_0)\, ds + \int_0^t S_{-1}(t-s)\left(\int_0^t h(s-\tau)g(\tau, Z_0)\, d\tau\right) ds - \int_{s=0}^t S_{-1}(t-s)\left(\int_0^t h(s-\tau)\, g(\tau, Z_0)\, d\tau\right) ds)\|_{H_0} \le \| R(\lambda;A + \Delta A)(S_{-1}(t)Z_0 - Z_0)\|_{H_0} + \|R(\lambda;A + \Delta A)\| (\|\int_0^t S_{-1}(t-s)(B + \Delta B)u(s)\|_{H_{-1}}\, ds + \int_0^t \| S_{-1}(t-s)\|_{H_0} \|f(s, Z_u(s)) - f(s, Z_0)\|\, ds + \int_0^t \| S_{-1}(t-s)\|_{H_0}\left(\int_0^t |h(s-\tau)|\, \|g(\tau, Z_0) - g(\tau, Z_u(\tau)\|_{H_{-1}}\, d\tau\right) ds + \int_0^t \| S_{-1}(t-s)\|_{H_{-1}}\left(\|f(s, Z_0)\|_{H_{-1}} + \int_0^t |h(s-\tau)|\|g(\tau, Z_0)\|_{H_{-1}}\, d\tau\right) ds).$$

From condition (9), we get:

$$\|(F_uZ_u)(t) - Z_0\|_{H_{-1}} \le \delta' + \delta_x + \delta_y ,$$

Where:

$$\delta_x = \frac{M_1M_2}{\mathrm{Re}\lambda - ((W_1+W_2)+M_1\|\Delta A_1\|_{H_0} + M_2\|\Delta A_2\|_{H_0})},$$

and

$$\delta_y = (\|\int_0^t S_{-1}(t-s)(B+\Delta B)u(s)\|_{H_{-1}}\, ds + \int_0^t M_1M\, e^{(t-s)((W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)} \|f(s,Z_u(s)) - f(s,Z_0)\|_{H_{-1}}\, ds + \int_0^t M_1M_2\, e^{t((W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)} + \left(\int_0^t |h(s-\tau)|\, \|g(\tau, Z_u(\tau)) - g(\tau, Z_0\|_{H_{-1}}\, d\tau\right) ds + \int_0^t M_1M_2\, e^{t((W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)} \left(\|f(\right.$$

$s,Z_0)\|+\int_0^t |h(s-\tau)|\ \|g(\tau, Z_0)\|\ d\tau \Big)$

$ds\ \Big)$.

From conditions (10),(11) and remark(2.6),we have that:

$\|(F_uZ_u)(t)-Z_0\|_{H_{-1}} \le \delta' + \delta_x$, where

$$\delta_x=\frac{M_1M_2}{\mathrm{Re}\lambda-((W_1+W_2)+M_1\|\Delta A_1\|_{H_0}+M_2\|\Delta A_2\|_{H_0})}$$

$(K_t t^{1/2}\|u(s)\|_{L^2([0,\infty),U)}$ +

$\int_0^t M_1M_2\, e^{(t-s)((W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)} L_0\|Z_u(s)-Z_0\|_{H_{-1}}\, ds$

$+\int_0^t M_1M_2\, e^{t((W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)}\, h_{t_1} L_1\|Z_u(s)-Z_0\|_{H_0}\, ds$ +

$\int_0^t M_1M_2\, e^{(t-s)((W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)} (B_1 + h_{t_1} B_2)\, ds)$ .Hence:

$\|(F_uZ_u)(t)-Z_0\|_{H_{-1}} \le \delta' + \delta_x$,

where:

$$\delta_x=\frac{M_1M_2}{\mathrm{Re}\lambda-((W_1+W_2)+M_1\|\Delta A_1\|_{H_0}+M_2\|\Delta A_2\|_{H_0})}$$

$$(t_1^{1/2}k_{t_1}k_1+(\frac{M_1M_2\delta L_0 e^{t_1((W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)}}{(W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|}+$$

$$\frac{M_1M_2h_{t_1}\delta L_1 e^{t_1((W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)}}{(W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|}+M_1M_2\delta_y),$$

with:

$\delta_y = e^{t_1((W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)}$.

$$\frac{B_1+h_{t_1}B_2}{(W_1+W_2)+M\|\Delta A_1\|_{H_0}+M_2\|\Delta A_2\|_{H_0}}$$

$\|(F_uZ_u)(t)-Z_0\|_{H_{-1}} \le \delta' + \delta_x$, Where:

$$\delta_x=\frac{M_1M_2}{\mathrm{Re}\lambda-((W_1+W_2)+M_1\|\Delta A_1\|_{H_0}+M_2\|\Delta A_2\|_{H_0})}.$$

$$(k_{t_1}t_1^{1/2}k_1+\frac{M_1M_2(\delta h_{t_1}L_1+\delta L_0+B_1+h_{t_1}B_2)}{(W_1+W_2)+M_1\|\Delta A_1\|_{H_0}+M_2\|\Delta A_2\|_{H_0}}).\delta_y$$

With

$\delta_y = e^{t_1((W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)}$

By using condition (12i),we get $\|(F_uZ_u)(t)-Z_0\|_{H_{-1}} \le \delta$, for $0 \le t \le t_1$.

So one can select $t_1 > 0$, such that:

$$t_1=\min\{t',t'',r,\frac{1}{(W_1+W_2)+M_1\|\Delta A_1\|_{H_0}+M_2\|\Delta A_2\|_{H_0}}.\delta_x.\delta_y\},$$

where

$$\delta_x=(-\log(k_{t_1}t_1^{1/2}k_1+\frac{M_1M_2(\delta h_{t_1}L_1+\delta L_0+B_1+h_{t_1}B_2)}{(W_1+W_2)+M_1\|\Delta A_1\|_{H_0}+M_2\|\Delta A_2\|_{H_0}}),$$

and

$$\delta_y = \frac{\text{Re}\,\lambda - ((W_1 + W_2) + M_1 \| \Delta A_1 \|_{H_0} + M_2 \| \Delta A_2 \|_{H_0})}{M_1 M_2}$$

(δ-δ').

Thus, we have that F_u: $S_u \longrightarrow S_u$.

Now, we need to show that F_u is a strict contraction on S_u, this will ensure the existence of a unique mild solution to the semilinear initial value perturbation control problem. Let $\overline{\overline{Z}}_u$, $\overline{Z}_u \in S_u$, then:

$$\|(F_u \overline{\overline{Z}}_u)(t) - (F_u \overline{Z}_u)(t)\|_{H_{-1}} = \Big\| R(\lambda;A + \Delta A)\Big(S(t)Z_0 + \int_0^t S_{-1}(t-s).(B + \Delta B)u(s)\,ds + \int_0^t S_{-1}(t-s)\Big[f(s,\overline{\overline{Z}}_u)) + \int_0^t h(s-\tau)\, g(\tau, \overline{\overline{Z}}_u)\,d\tau \Big] ds - S_{-1}(t)Z_0 - \int_0^t {}_{-1}(t-s)(B+\Delta B)u(s)\,ds - \int_0^t S_{-1}(t-s)\Big[f(s, \overline{Z}_u) + \int_0^t h(s-\tau).g(\tau,\overline{Z}_u)d\tau \Big] ds \Big) \Big\|_{H_0}.$$

Hence:

$$\|(F_u \overline{\overline{Z}}_u)(t) - (F_u \overline{Z}_u)(t)\|_{H_{-1}} = \Big\| R(\lambda;A+\Delta A)\Big(\int_0^t S_{-1}(t-s)\Big[f(s, \overline{\overline{Z}}_u) + \int_0^t h(s-\tau)g(\tau,\overline{\overline{Z}}_u)\,d\tau \Big] ds - \int_0^t S_{-1}(t-s)\Big[f(s, \overline{Z}_u) +, \int_0^t h(s-\tau)\, g(\tau, \overline{Z}_u)\,d\tau \Big] ds \Big) \Big\|_{H_0}$$

$$\le \int_0^t \|R(\lambda;A + \Delta A)\|\, \| S_{-1}(t-s)\|\, \|f(s, \overline{\overline{Z}}_u) - f(s, \overline{Z}_u)\|_{H_{-1}}\, ds + \int_0^t \|R(\lambda;A + \Delta A)\|\, \| S_{-1}(t-s)\| \Big[\int_0^t |h(s-\tau)|\, \|g(\tau, \overline{\overline{Z}}_u) - g(\tau, \overline{Z}_u)\|_{H_{-1}}\, d\tau \Big] ds$$

From theorem (2.7) and lemma (3.2), we have that:

$$\|(F_u \overline{\overline{Z}}_u)(t) - (F_u \overline{Z}_u)(t)\|_{H_{-1}} \le \int_0^t \frac{M_1M_2}{\text{Re}\lambda - ((W_1+W_2)+M_1 \| \Delta A_1 \|_{H_0} + M_2 \| \Delta A_2 \|_{H_0})} (M_1M_2 e^{(t-s)((W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)} \|f(s, \overline{\overline{Z}}_u) - f(s, \overline{Z}_u)\|_{H_{-1}}\, ds$$

$$\int_0^t \frac{M_1M_2}{\text{Re}\lambda - ((W_1+W_2)+M_1 \| \Delta A_1 \|_{H_0} + M_2 \| \Delta A_2 \|_{H_0})} M_1M\, e^{(t-s)((W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)} \Big[\int_0^t |h(s-\tau)|\, \|g(\tau, \overline{\overline{Z}}_u) - g(\tau, \overline{Z}_u)\|_{H_{-1}}\, d\tau \Big] ds \quad (12)$$

From(12),we get

$\|(F_u \overline{\overline{Z}}_u)(t) - (F_u \overline{Z}_u)(t)\|_{H_{-1}}$

$\le$

$$\int_0^t \frac{M_1^2 M_2^2}{\operatorname{Re}\lambda-((W_1+W_2)+M_1\|\Delta A_1\|_{H_0}+M_2\|\Delta A_2\|_{H_0})} e^{(t-s)(W_1+W_2+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)} L_0 . \|\bar{\bar{Z}}_u(s)-\bar{Z}_u(s)\|_{H_0}\, ds$$

$$+ \int_0^t \frac{M_1^2 M_2^2}{\operatorname{Re}\lambda-((W_1+W_2)+M_1\|\Delta A_1\|_{H_0}+M_2\|\Delta A_2\|_{H_0})} e^{(t-s)(W_1+W_2+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)} h_{t_1} L_1 \|\bar{\bar{Z}}_u(\tau)-\bar{Z}_u(\tau)\|_{H_0}\, ds ,$$

$$\le \frac{M_1^2 M_2^2}{\operatorname{Re}\lambda-((W_1+W_2)+M_1\|\Delta A_1\|_{H_0}+M_2\|\Delta A_2\|_{H_0})} . \left(\frac{L_0 e^{t_1((W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)}}{W_1+W_2+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|} \|\bar{\bar{Z}}_u(s)-\bar{Z}_u(s)\|_{H_0} + \frac{h_{t_1} L_1 e^{t_1((W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)}}{W_1+W_2+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|} \|\bar{\bar{Z}}_u(\tau)-\bar{Z}_u(\tau)\|_{H_0}\right)$$

$$\le \frac{M_1^2 M_2^2}{\operatorname{Re}\lambda-((W_1+W_2)+M_1\|\Delta A_1\|_{H_0}+M_2\|\Delta A_2\|_{H_0})} . \Big(M_1 M_2 L_0 e^{t_1((W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)} \sup_{0\le t\le t_1} \|\bar{\bar{Z}}_u(s)-\bar{Z}_u(s)\|_{H_0}$$

$$+ \frac{M_1 M_2 h_{t_1} L_1 e^{t_1((W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)}}{W_1+W_2+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|} \sup_{0\le t\le t_1} \|\bar{\bar{Z}}_u(\tau)-\bar{Z}_u(\tau)\|_{H_0}\Big)$$

$$\le \frac{M_1^2 M_2^2}{\operatorname{Re}\lambda-((W_1+W_2)+M_1\|\Delta A_1\|_{H_0}+M_2\|\Delta A_2\|_{H_0})} \left(\frac{L_0 e^{t_1((W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)}}{W_1+W_2+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|} e^{t_1((W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)} \|\bar{\bar{Z}}_u-\bar{Z}_u\|_Y + \frac{h_{t_1} L_1 e^{t_1((W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)}}{W_1+W_2+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|} \|\bar{\bar{Z}}_u-\bar{Z}_u\|_Y\right). \quad (13)$$

From (13), we have

$$\|(F_u\bar{\bar{Z}}_u)(t) - (F_u\bar{Z}_u)(t)\|_{H_{-1}} \le C_x . C_y \|\bar{\bar{Z}}_u-\bar{Z}_u\|_Y, \quad (14)$$

where:

$$C_x = \frac{M_1^2 M_2^2}{\operatorname{Re}\lambda-((W_1+W_2)+M_1\|\Delta A_1\|_{H_0}+M_2\|\Delta A_2\|_{H_0})},$$

$$C_y = \frac{e^{t_1((W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)}(L_0+h_{t_1}L_1)}{W_1+W_2+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|}.$$

From (17), we have:

$$\|(F_u\bar{\bar{Z}}_u)(t) - (F_u\bar{Z}_u)(t)\|_{H_{-1}} \le \frac{1}{\delta} D_x\, D_y \|\bar{\bar{Z}}_u-\bar{Z}_u\|_Y ,$$ where

$$D_x = \frac{M_1^2 M_2^2}{\operatorname{Re}\lambda-((W_1+W_2)+M_1\|\Delta A_1\|_{H_0}+M_2\|\Delta A_2\|_{H_0})} (\delta L_0+\delta h_{t_1} L_1)$$ and

$D_y = \dfrac{e^{t_1((W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)}}{W_1+W_2+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|}$

$\|(F_u\bar{\bar{Z}}_u)(t) - (F_u\bar{Z}_u)(t)\|_{H_{-1}}$

$$\le \frac{1}{\delta} E_x . E_y . E_z \|\bar{\bar{Z}}_u - \bar{Z}_u\|_Y$$

,where (15)

$E_x = \dfrac{M^2_1 M^2_2}{\mathrm{Re}\lambda - ((W_1+W_2)+M_1\|\Delta A_1\|_{H_0} + M_2\|\Delta A_2\|_{H_0})}$,

$E_y = (k_{t_1} t_1^{1/2} k_1 + \dfrac{(\delta h_{t_1} L_1 + \delta L_0 + B_1 + h_{t_1} B_2)}{(W_1+W_2)+M_1\|\Delta A_1\|_{H_0} + M_2\|\Delta A_2\|_{H_0}})$

and

$E_z = e^{t_1((W_1+W_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)}$.

By using condition (12i), (13), implies to

$\|(F_u\bar{\bar{Z}}_u)(t) - (F_u\bar{Z}_u)(t)\|_{H_{-1}} \le \dfrac{1}{\delta} E_x (k_{t_1} t_1^{1/2} k_1 + E_y) E_z E_w \|\bar{\bar{Z}}_u - \bar{Z}_u\|_Y$,

where:

$E_x = \dfrac{M^2_1 M^2_2}{\mathrm{Re}\lambda - ((W_1+W_2)+M_1\|\Delta A_1\|_{H_0} + M_2\|\Delta A_2\|_{H_0})}$,

$E_y = \dfrac{(\delta h_{t_1} L_1 + \delta L_0 + B_1 + h_{t_1} B_2)}{(W_1+W_2)+M_1\|\Delta A_1\|_{H_0} + M_2\|\Delta A_2\|_{H_0}}$),

$E_z = (k_{t_1} t_1^{1/2} k_1 + \dfrac{M_1 M_2 (\delta h_{t_1} L_1 + \delta L_0 + B_1 + h_{t_1} B_2)}{(W_1+W_2)+M_1\|\Delta A_1\|_{H_0} + M_2\|\Delta A_2\|_{H_0}})^{-1}$,

and

$E_w = \dfrac{\mathrm{Re}\lambda - ((W_1+W_2)+M_1\|\Delta A_1\|_{H_0} + M_2\|\Delta A_2\|_{H_0})}{M_1 M_2}$

$(\delta - \delta')$ .Hence:

$\|(F_u\bar{\bar{Z}}_u)(t) - (F_u\bar{Z}_u)(t)\|_{H_{-1}} \le (1 - \dfrac{\delta'}{\delta})\|\bar{\bar{Z}}_u - \bar{Z}_u\|_Y$. (16)

Taking the supremum over $[0, t_1]$ of both sides to (16), we get:

$$\sup_{0\le t\le t_1} \|(F_u\bar{\bar{Z}}_u)(t) - (F_u\bar{Z}_u)(t)\|_{H_{-1}} \le (1 - \frac{\delta'}{\delta})\|\bar{\bar{Z}}_u - \bar{Z}_u\|_Y.$$

We obtain: $\|(F_u\bar{\bar{Z}}_u)(t) - (F_u\bar{Z}_u)(t)\|_Y \le (1 - \dfrac{\delta'}{\delta})\|\bar{\bar{Z}}_u - \bar{Z}_u\|_Y$

Hence from condition (9) that $\delta' < \delta$, then F_u is a strict contraction map from S_u into S_u and therefore by Banach contraction principle, there exists a unique fixed point Z_u of Fu in S_u, i.e., there is a unique $Z_u \in S_u$, such that $F_u Z_u = Z_u$.

The fixed point satisfies the integral equation: $Z_u(t) = S_{-1}(t)Z_0 + \int_0^t S_{-1}(t-s)\left[f(s, u(s)) + \int_0^s h(s-\tau)\, g(\tau, Z(\tau))\, d\tau \right] ds + \int_0^t S_{-1}(t-s)$ $(B + \Delta B)u(s)\, ds$, for $0 \le t \le t_1$ and $\forall\, u(.) \in L^2_{loc}([0, \infty): U)$.

Conclusions

1- he necessary and sufficient conditions for the unbounded control operator of to be admissible can be showed as follows:

a-

he unbounded control operator is admissible that need to be defined from control space into the extension space H_{-1} where the extension spaces defined as the set of all equivalence classes of norm bounded Cauchy sequences in $\{H_0, \tau_*\}$ and an element of H_{-1} is denoted by $\tilde{Z} = [\{Z_n\}]$, where $[\{Z_n\}]$ denotes the equivalence class containing $\{Z_n\}_{n\in}$ □ .

b-

he unbounded control operator is admissible if and only if the adjoint of the unbounded control operator is admissible in a Hilbert space or reflexive Banach space.One can conclude that, on using the definitions of step (a), the adjoint of some linear operator L* for L is defined to a bounded operator if the linear operator L is bounded defined on some domain or L* has a densely defined domain so that L* is extended to be bounded. This fact is very important in this approach.

2-

he solvability of perturbed semilinear problem with unbounded input can be studied for many solutions like mild solution, strong solution and week solution in the extension space H_{-1}.

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