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# A New Approach for Finding The Coefficients and Roots of The Ehrhart Polynomial of A Cyclic Polytope With Some Properties

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**Abstract**

The aim of this work is to give a simple description of a cyclic polytope and a new approach for finding the coefficient of its Ehrhart polynomials using Pascal triangles.
Theorem for concluding that the roots of a cyclic polytopes are negative is also given.

**Keywords:** coefficients and roots; cyclic polytopes; Ehrhart polynomial

**طريقة جديدة لايجاد معاملات وجذور متعدد حدود ايرهارت لمتعدد السطوح الدوري مع بعض الصفات**

**الخلاصة**

هدف البحث هو وصف متعدد السطوح الدوري وتقديم طريقة جديدة لحساب معاملاته باستخدام مثلث باسكال.تم تقديم نظرية نستنتج منها ان الجذور لمتعدد السطوح الدوري هي سالبة.

## 1-Introduction

The coefficients of the Ehrhart polynomials have been of interest since Ehrhart first began his study of $L_P(t)$. Recent researches were focused on the roots of Ehrhart polynomial as well, partially in an attempt for better understanding of the coefficients of $L_P(t)$. There have been a veriety of results suggested that the roots of Ehrhart polynomial have interesting and unique behaviors, such as the following: The coefficients and roots are very special [2], for example by B. J. Braun[3]
Showed that the coefficients of their Ehrhart polynomials behave nicely with respect to the free summation operation and J. Pfeile, M.Beck, J.De Loera, M. Develin, and R. P. Stanley[9], showed that the Ehrhart polynomial $L_P(t)$ of d-dimensional lattice polytope is usually written in the lower basis of the vector space of polynomials of degree d, also some of the coefficients in this representation have a nice interpretations and all real roots of Ehrhart polynomials of d-dimensional lattice polytope lie in the half-open interval $[-d, \lfloor d/2 \rfloor)$.

We gave a general formula for the coefficients of Ehrhart polynomials of a cyclic polytope with dimension two, and a new form for a cyclic polytope in three dimensional using Pascal triangle is also given.

In this paper, some properties of the cyclic polytope are presented, theorems for computing Ehrhart polynomials of a cyclic polytope are also given. We showed that, in

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general, the roots of the cyclic polytopes $C_d(n)$ are negative or have negative real part.

**2-The Cyclic Polytope**

**Definition 2.1: [4]**

The moment curve in $R^d$ is defined by $m : R \to R^d$, where

$$m(t) = \begin{pmatrix} t \\ t^2 \\ \vdots \\ t^d \end{pmatrix} \in R^d .$$

**Definition (2.2): [4]**

The cyclic polytope of dimension d with n vertices is the convex hull $C_d(t_1, t_2, ..., t_n) = conv\{m(t_1), m(t_2), ..., m(t_n)\}$ of distinct points $m(t_i)$ where n>d, with $t_1 < t_2 < \ldots < t_n$ on the moment curve, where (the convex mean the convex hull).

Note: the cyclic polytope is sometimes denoted by $C_d(n)$, where n is the number of vertices and d is the dimension of the cyclic polytope.

**Definition (2.3): [10]**

A cyclic polytope $C_d(n)$ in $R^d$ is said to be simple if there are exactly d edges through each vertex, and it is called simplicial if each facet contains exactly d vertices.

**3-Properties of a Cyclic Polytope**

**Theorem (3.1): [6]**

For n>2 and $d \geq 2$, the cyclic polytope $C_d(n) = conv\{m(t_1), ..., m(t_n)\}$, is a simplicial d-polytope.

**Definition (3.1): [11]**

Let $f_i(P)$ be the number of i-faces of a d-polytope P, for $i = 0,1,..., d-1$. The face vector associated with a d-polytope is defined as $f(P) = (f_0(P), f_1(P), ..., f_{d-1}(P))$, where $n = f_0(P)$ is the number of vertices of d-polytope.

**4-The Ehrhart polynomial of a Cyclic Polytope**

The Ehrhart polynomial of a cyclic polytope count the number of lattice points in a dilation of a cyclic polytope by positive integer $t^*$ is equal to its volume plus the number of lattice points in its lower envelope, [2].

The following theorem appears in [8] without proof. F.Ahmed in [1] prove this theorem.

**Theorem (4.1): [8]**

For any integral cyclic polytope $C_d(T)$ where $T = \{t_1, t_2, ..., t_n\}$ for $t_i = 1,2,..., n,$ and $i = 1,..., n.$

$$L(C_d(T), t^*) = vol(t^* C_d(T)) + L(C_{d-1}(T), t^*) \ldots (4.1)$$

Where vol mean the volume of the cyclic polytope

Hence,

$$L(C_d(T), t^*) = \sum_{k=0}^{d} vol(t^* C_k(T)) = \sum_{k=0}^{d} vol_k(C_k(T) t^{*k},$$

Where $vol_k(t^* C_k(T))$ is the volume of $t^* C_k(T)$ in k-dimensional space, and $vol_0(t^* C_0(T)) = 1$.

**Theorem (4.2): [8]**

For any integral set T with $n = |T| = d+1$, which mean that $C_d(n)$ is simplex, then

$$vol(t^* C_d(T)) = \frac{t^{*d}}{d!} \prod_{1 \leq i < j \leq d+1} (t_j - t_i) \ldots (4.2)$$

From the above theorems we get the following theorem.

**Theorem (4.3): [1]**

For any cyclic polytope $C_d(T)$ where $T = \{t_1, t_2, ..., t_n\}$ for $t_i = 1,2,..., n,$ and $i = 1,..., n.$ we get the following:

(i) If $C_d(T)$ is not simplex, one can

decomposed it into simplices, then using equation (4.2) to compute its volume.

(ii) In one dimensional space the volume of cyclic polytope $(t^*C_1(T))$ in the interval $[t_1, t_n]$ is equal to $t_n - t_1$.

## 5- The coefficients and roots of the Ehrhart Polynomial of the cyclic Polytope

In this section a theorem that proves all roots of the Ehrhart polynomials of the cyclic polytope are negative is given. The results are obtained using Matlab software v.6.5

### (5-1) Formulation of the method

In this method we find a general formula that computes the coefficients of Ehrhart polynomial of the cyclic polytope in two and three dimensions with n vertices using Pascal triangle [5].

Comparisons between the coefficients which are obtained by [1] and those obtained by Pascal triangle are the same.

The Ehrhart polynomial of a cyclic polytope of dimension two with n vertices is

$$L(C_2(T, t^*) = \sum_{k=0}^{2} vol_k(C_k(T)t^{*k}.$$

$$= vol_2(C_2(T))t^{2^*} + vol_1(C_1(T))t^* + vol_0(C_0(T))$$

where $vol_2(C_2(T))$ are represented by the second row of the Pascal triangle, $vol_1(C_1(T))$ are the zero row of the Pascal triangle and $vol_0(C_0(T))$ are the constant row of the Pascal triangle which is equal to the number one shown in the figure(5.1), [5].

We get the same results that obtained using the method in reference [1]

For example

$$L(C_2(1,2,3), t^*) = t^{*2} + 2t^* + 1.$$
$$L(C_2(1,2,3,4), t^*) = 4t^{*2} + 3t^* + 1.$$
$$L(C_2(1,2,3,4,5), t^*) = 10t^{*2} + 4t^* + 1.$$
$$L(C_2(1,2,3,4,5,6), t^*) = 20t^{*2} + 5t^* + 1.$$
$$L(C_2(1,2,3,4,5,6,7), t^*) = 35t^{*2} + 6t^* + 1.$$
$$L(C_2(1,2,3,4,5,6,7,8), t^*) = 56t^{*2} + 7t^* + 1.$$
$$L(C_2(1,2,3,4,5,6,7,8,9), t^*) = 84t^{*2} + 8t^* + 1.$$

In a similar way the results are obtained for d=3

The Ehrhart polynomial of the cyclic polytope of dimension three with n vertices is

$$L(C_3(T), t^*) = \sum_{k=0}^{3} vol_k(C_k(1,2,3,4)t^{*k}.$$

$$= vol_3(C_3(T))t^{3^*} + vol_2(C_2(T))t^{2^*} + vol_1(C_1(T))t^* + vol_0(C_0(T))$$

where the $vol_3(C_3(T))$ are computed by the above theorems. There are nice relations between coefficients of Ehrhart polynomials of the cyclic polytope for n=4, 5, 6, 7, 8 such that

$$2 = 2$$
$$16 = 2^4$$
$$70 = 2 \times 5 \times 7$$
$$224 = 2^5 \times 7$$
$$588 = 2^2 \times 3 \times 7^2$$

For $vol_2(C_2(T))$ are representations by the second row of Pascal triangle, $vol_1(C_1(T))$ are the zero row of the Pascal triangle and $vol_0(C_0(T))$ are the constant row of the Pascal triangle which is equal to the number one see [figure(5.1)]

We get the same results that obtained using the method in reference [1] for example

$$L(C_3(1,2,3,4),t^*) = 2t^{*3} + 4t^{*2} + 3t^* + 1.$$

$$L(C_3(1,2,3,4,5),t^*) = 16t^{*3} + 10t^{*2} + 4t^* + 1.$$

$$L(C_3(1,2,3,4,5,6),t^*) = 70t^{*3} + 20t^{*2} + 5t^* + 1.$$

$$L(C_3(1,2,3,4,5,6,7),t^*) = 224t^{*3} + 35t^{*2} + 6t^* + 1.$$

$$L(C_3(1,2,3,4,5,6,7,8),t^*) = 588t^{*3} + 56t^{*2} + 7t^* + 1.$$

**Theorem (5.2):**

The roots of a cyclic polytope of dimension d with n-vertices are negative.

**Proof**:

The Ehrhart polynomial of cyclic polytope are given by, [8]

$$L(C_d(T),t^*) = \sum_{k=0}^{d} vol(t^* C_k(T)) = \sum_{k=0}^{d} vol_k(C_k(T)t^{*k})$$

where $vol_k(t^* C_k(T))$ is the volume of $t^* C_k(T)$ in k-dimensional space.

The Ehrhart polynomial of cyclic polytope is a polynomial in $t^*$ of degree d.Now consider

$$p(t^*) = a_d t^{*d} + a_{d-1} t^{*d-1} + ... + a_1 t^* + a_0 .......(5.2.1)$$

where $p(t^*)$ is a polynomial of degree d and $a_d, a_{d-1}, ..., a_1, a_0$ are the constant coefficients of the polynomial $p(t^*)$.

we can rewrite Equation (5.2.1) in its factored form, as shown in Equation (5.2.2)

$$p(t^*) = a_n(t^* - r_1)(t^* - r_2)...(t^* - r_d) ........(5.2.2)$$

Where $r_1, r_2 \ldots, r_d$ are the roots of the polynomial $p(t^*)$

If a polynomial $p(t^*)$ of degree d has repeated roots $r_1, r_2 \ldots, r_s$ with respective multiplicities $k_1, k_2 \ldots, k_s$ , $(k_1 + k_2 + ...k_s = d)$ the n the Equation (5.2.3) is true.

$$p(t^*) = a_n(t^* - r_1)^{k_1}(t^* - r_2)^{k_2}...(t^* - r_s)^{k_s} ....(5.2.3)$$

The following relation between the roots of algebraic equation (taking into account their multiplicities) and its coefficients hold true, [7]:

$$r_1 + r_2 + ... + r_d = \sum_{i=1}^{d} r_i = {}^{-a_{d-1}}\!/\!_{a_d}$$

$$r_1 r_2 + r_1 r_3 + ... + r_{d-1} r_d = \sum_{1 \le i \prec j}^{d} r_i r_j = {}^{-a_{d-2}}\!/\!_{a_d}$$

$$r_1 r_2 r_3 + r_1 r_2 r_4 + ... + r_{d-2} r_{d-1} r_d = \sum_{1 \le i \prec j \prec k}^{d} r_i r_j r_k = {}^{-a_{d-3}}\!/\!_{a_d}$$

Now for d=2 then

$$p(t^*) = a_2 t^{*2} + a_1 t^* + a_0 ...........(5.2.1)$$

$$p(t^*) = a_2(t^{*2} + {}^{a_1}\!/\!_{a_2}\, t^* + {}^{a_0}\!/\!_{a_2})$$

We can write the equation (5.2.1) as the equation (5.2.2)

$$p(t^*) = a_2(t^* - r_1)$$

$$(t^* - r_2).........(5.2.2)$$

$$= t^{*2} - (r_1 + r_2)t^* + r_1 r_2$$

To ensure that the coefficients are all positive (they are the volume of the cyclic polytope) the roots must all be negative. By the properties of the polynomial $p(t^*)$ of degree d, and from the above profness we get contradiction, then the roots of the Ehrhart polynomial for a cyclic polytopes are all negative.

For d=3

$$p(t^*) = a_3 t^* + a_2 t^{*2} + a_1 t^* + a_0 ...........(5.2.1)$$

$$p(t^*) = a_3(t^* + {}^{a_2}\!/\!_{a_3}\, t^{*2} + {}^{a_1}\!/\!_{a_3}\, t^* + {}^{a_0}\!/\!_{a_3})$$

We can write the equation (5.2.1) as the equation (5.2.2)

$$p(t^*) = a_3(t^* - a_1)$$

$$(t^* - r_2)(t^* - r_3).......(5.2.2)$$

$$= t^{*3} - (r_1 + r_2 + r_3)t^{*2} + (r_1 r_2 + r_1 r_3 + r_2 r_3)t^* - r_1 r_2 r_3$$

The coefficients obtained are negative which is contradiction because these coefficients are the volume of the cyclic polytope.

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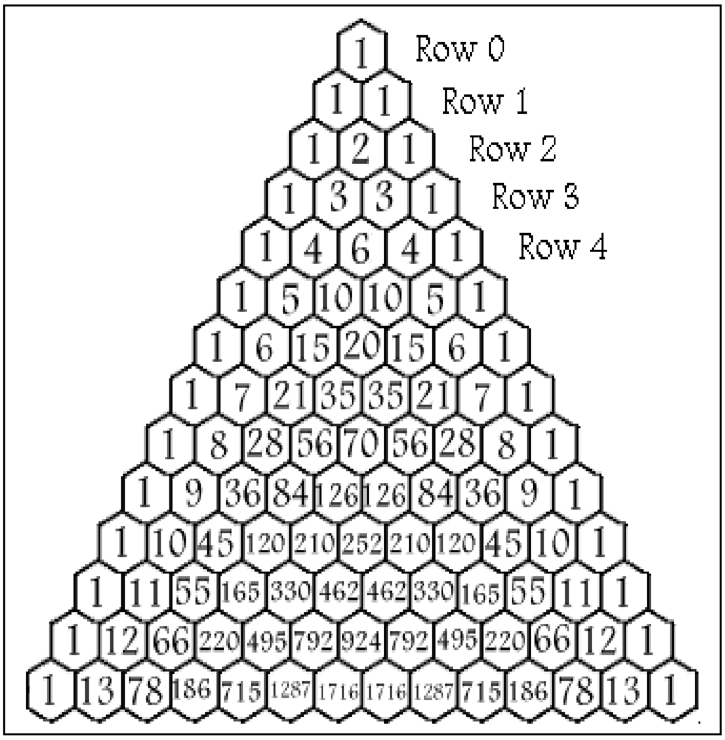


**Figure (5.1) Pascal triangles.**