Output-Feedback Stochastic Nonlinear Stabilization and Inverse Optimality

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Abstract

Output-feedback (observer-based) robust and optimal control law which guarantees global (local) asymptotic stability in probability for nonlinear stochastic dynamic system are stated, developed and proved with the help of stochastic Lyapunov function approach supported by necessary theorems and an illustrative example. The inverse optimal stabilization in probability with suitable performance index has also been stated and developed.

Keywords: Backstepping, control Lyapunov functions, inverse optimality, stochastic nonlinear output-feedback systems, stochastic stabilization.

قابلية الاستقرارية الارجاعية-للمخرجات غير الخطية التصادفية ومعكوس الامثلية

الخلاصة

لقد تم عرض وتطوير وبرهان مدعم بالمبرهات بمساعدة دالة ليابونوف التصادفية، النظريات الكافية مع مثال تطبيقي، لايجاد مسيطر مخرج-ارجاعي (مستند على نظام ديناميكي مخمن للاصل) رصين وقانون السيطرة الامثل الذي يضمن الاستقرارية المحاذية-الاحتمالية المطلقة (المحلية) لنظام ديناميكي تصادفي غير خطي. تم عرض وتطوير لقابلية السيطرة المثلى العكسية-الاحتمالية بوجود دالة هدف ملائمة.

Introduction

Little attention until recently. Efforts toward (global) *stabilization* of stochastic nonlinear systems have Despite huge popularity of the linear-quadratic-Gaussian control problem, the stabilization problem for *nonlinear stochastic* systems has been receiving relativelyBeen initiated in the work of Florchinger [5–7] who, among other things,

Extended the concept of control Lyapunov functions to the stochastic setting. A breakthrough toward arriving at *constructive* methods for stabilization of broader classes of stochastic nonlinear systems came with the result of Pan and Basar [16] who derived a backstepping design for strict-feedback systems motivated by a risk-sensitive cost criterion. Deng and Krsti´c [2-4] presented the first result on *global output-feedback* stabilization (in probability) for stochastic nonlinear continuous-time systems. Simpler inverse optimal control laws were designed for strict-feedback systems which guarantee global asymptotic stability in

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probability. The output-feedback problem had received considerable attention in the recent robust and adaptive nonlinear control literature [1], [8-13], and [15]. In this paper, we present two results, first address the *output-feedback* global stabilization problem for stochastic nonlinear systems, second, a robust and optimal control law are designed which guarantees global asymptotic stability in probability for some dynamic systems in the presence of output observer.

The output feedback (observer-based) backstepping control law which guarantees global asymptotic stability in probability has also been discussed supported by some theoretical justification and illustration.

2. Preliminaries on Stability In Probability

Consider the nonlinear stochastic system of the form

$$dx = f(x)dt + g(x)dw \quad (1)$$

where $x \in R^n$ is the state, w is an r-dimensional independent standard Brownian motion, and $f: R^n \to R^n$ and $g: R^n \to R^{n\times r}$ are locally Lipschitz functions and satisfies $f(0) = 0$, $g(0) = 0$, where r < n.

Definition (2.1) [3]

The equilibrium x = 0 of equation (1) is said to be globally asymptotically stable in probability if for any $t_o \geq 0$ and $\in > 0$, $\lim_{x(t_o)\to 0} P\{sup_{t\geq t_o} |x(t)| > \epsilon\} = 0$ and for any initial condition $x(t_o)$, $P\{\lim_{t\to\infty} x(t) = 0\} = 1$.

2.1 "Young's Inequality" [3]

This inequality is mainly used in the simplifications of this work which is formed as follows:

$$xy \leq \frac{\epsilon^P}{P}|x|^P + \frac{1}{q\epsilon^q}|y|^q \quad (2)$$

where $\in > 0$, the constants P > 1, q > 1 which satisfies the relation: (P -1) (q - 1) = 1 and $(x, y) \in R^{2n}$.

Theorem (2.1) [14]

Consider the nonlinear system of equation (1) and suppose that there exist a positive definite, radially unbounded, twice continuously differentiable function V(x) such that the infinitesimal generator

$$\mathcal{L}V(X) = \frac{\partial V}{\partial X}F + \frac{1}{2}Tr\left\{g^T\frac{\partial^2 V}{\partial X^2}g\right\} \quad (3)$$

Is negative definite. Then the equilibrium point $x = 0$ of the above system is globally asymptotically stable in probability, where Tr (.) operator is standing for the trace operation.

3. Output-Feedback Stochastic Nonlinear Stabilization In Probability

In this section we deal with nonlinear output-feedback systems driven by Brownian motion and some of its theoretical results. This class of systems is given by the following nonlinear stochastic differential equations.

Consider the stochastic nonlinear system described by:

$$dx_i = x_{i+1}dt + f_i(\bar{x}_i)dt + \phi_i(y)^T dw + \psi_i(\bar{x}_i)^T dw$$

$$i = 1,2,\dots,n-1$$

$$dx_n = udt + f_n(x_n)dt + \phi_n(y)^T dw + \psi_n(x_n)^T dw$$

$$y = \sum_{i=1}^{n} c_i x_i \quad (4)$$

where

1. $X \in R^n$ is the state, $\bar{x}_i = [x_1, x_2, \dots x_n]^T$
2. w is an r- dimensional independent standard Brownian motion
3. $f=(f_1, f_2, \dots, f_n)^T$, f is a vector valued function which satisfies:

- $f: R^n \to R^n,\ f(0) = 0.$
- $f_i(\bar{x}_i) = f_i(x_1, x_2, \dots, x_i)$
- $\|f(x)\| \le (x^T Q_1 x) \le \lambda_{max}(Q_1)|x|$ (5)

where Q_1 is a positive definite matrix, and $\lambda_{max}(Q_1)$ is the largest eigenvalue of Q_1.

4. $\phi_i(y)$ are r-vector-valued smooth functions with $\phi = (\phi_1, \phi_2 \dots \phi_n)^T$, ϕ: $R^n \to R^n$ and $\phi_i(0) = 0.$

5. $\psi_i(\bar{x}_i)$ are r-vector-valued smooth functions with

$\psi = (\psi_1, \psi_2, \dots \psi_n)^T, \psi: R^n \to R^{nxr}$, with $\psi_i(0) = 0.$

6. f_i, ϕ_i, ψ_i are assumed to satisfy Lipschitz condition.
7. The dynamic observer system is suggested as follows:

$$d\hat{x}_i = \hat{x}_{i+1}dt + L_i\left(y - \sum_{i=1}^{n} c_i \hat{x}_i\right)dt \qquad i = 1, \dots, n$$

$$= \hat{x}_{i+1}\, dt + L_i\left(\sum_{i=1}^{n} c_i x_i - \sum_{i=1}^{n} c_i \hat{x}_i\right)dt$$

$$d\hat{x}_i = \hat{x}_{i+1}\, dt + L_i \sum_{i=1}^{n} c_i \tilde{x}_i\, dt \dots\dots (6)$$

8. The observation error $e_i = x_i - \hat{x}_i = \tilde{x}_i$ satisfies:

$$d\tilde{x}_i = dx_i - d\hat{x}_i = x_{i+1}\, dt + f_i(\bar{x}_i)dt + \phi_i(y)^T dw + \psi_i(\bar{x}_i)^T - \hat{x}_{i+1}dt - L_i \sum_{i=1}^{n} c_i \tilde{x}_i dt$$

$$d\tilde{x}_i = \tilde{x}_{i+1}dt - L_i \sum_{i=1}^{n} c_i \tilde{x}_i dt + f_i(\bar{x}_i)dt + \phi_i(y)^T dw + \psi_i(\bar{x}_i)^T \quad \dots\dots(7)$$

Or in vector form, we can write:

$$d\tilde{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \dots & \dots & 0 & u \end{bmatrix} \tilde{x}dt - \begin{bmatrix} L_1C_1 & L_1C_2 & \cdots & \cdots & L_1C_n \\ L_2C_1 & L_2C_2 & \cdots & \cdots & L_2C_n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ L_nC_1 & \dots & \dots & \dots & L_nC_n \end{bmatrix} \tilde{x}dt + \begin{bmatrix} f_1(\bar{x}_1) \\ f_2(\bar{x}_1, \bar{x}_2) \\ \vdots \\ f_n(\bar{x}_1, \dots, \bar{x}_n) \end{bmatrix} dt + \begin{bmatrix} \phi_1(y)^T \\ \phi_2(y)^T \\ \vdots \\ \phi_n(y)^T \end{bmatrix} dw + \begin{bmatrix} \psi_1(\bar{x}_1) \\ \psi_2(\bar{x}_1, \bar{x}_2) \\ \vdots \\ \psi_n(\bar{x}_1, \dots, \bar{x}_n) \end{bmatrix} dw \quad \dots\dots(8)$$

thus

$$d\tilde{x} = (A - LC)\tilde{x}dt + f(\tilde{x})dt + \phi(y)^T dw + \psi(\tilde{x})^T dw$$

where A= $\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \dots & \dots & 0 & u \end{bmatrix}$,

LC = $\begin{bmatrix} L_1C_1 & L_1C_2 & \cdots & \cdots & L_1C_n \\ L_2C_1 & L_2C_2 & \cdots & \cdots & L_2C_n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ L_nC_1 & \dots & \dots & \dots & L_nC_n \end{bmatrix}$

and thus

$$\tilde{x} = A_0\tilde{x}dt + f(\tilde{x})dt + \phi(y)^T dw + \psi(\tilde{x})^T dw \qquad (9)$$

where $A_0 = (A - LC)$ is designed to be asymptotically stable, the coefficients L_i, i=1,…,n are computed in a way that guarantee asymptotic stability of A_0 (if possible).

Now the entire system can be expressed as:

$$d\tilde{x} = A_0\tilde{x}dt + f(\tilde{x})dt + \phi(y)^T dw + \psi(\tilde{x})^T dw$$

And

$$dy = \sum_{i=1}^{n} c_i\, dx_i$$

$$= \sum_{i=1}^{n} c_i\, x_{i+1} dt + \sum_{i=1}^{n} c_i f_i(\bar{x}_i)dt + \sum_{i=1}^{n} c_i \phi_i(y)^T dw + \sum_{i=1}^{n} c_i \psi_i(\bar{x}_i)^T dw$$

$$d\hat{x}_i = \hat{x}_{i+1}\, dt + L_i \textstyle\sum_{i=1}^{n} c_i\tilde{x}_i dt$$

$$d\hat{x}_n = u\ dt + L_n \sum_{i=1}^{n} c_i\tilde{x}_i dt \qquad (10)$$

where $\hat{x}_{n+1} = u$.

9. The r-vector-valued smooth functions $\phi(y)$ and $\psi(\tilde{x})$ satisfies the following imposed conditions, respectively:

$$\phi(y) \le \|\phi(y)\| \le \lambda_{max}(Q_2)|\tilde{x}| \quad (11)$$

$$\psi(\tilde{x}) \le \|\psi(\tilde{x})\| \le \lambda_{max}(Q_3)|\tilde{x}| \quad (12)$$

where Q_2, Q_3 are positive definite matrices, and $\lambda_{max}(Q_2)$, $\lambda_{max}(Q_3)$ are the largest eigenvalues of Q_2 and Q_3 respectively.

10. Since $\phi_i(0) = 0$, $\psi_i(0) = 0$, $f_i(0) = 0$, the $\propto_i$'s will vanish at $\bar{\hat{x}}_{i-1} = 0, y = 0$, as well as at $\bar{z}_i = 0$ where $\bar{z}_i = (z_1, \dots, z_i)^T$.

Thus, by the mean value theorem $\propto_i\ (\bar{\hat{x}}_i, y)$ can be expressed as:

$$\propto_i\ (\bar{\hat{x}}_i, y) - \sum_{l-1}^{i} z_l \propto_{il} (\bar{\hat{x}}_i, y) \quad \dots\dots\dots (13)$$

where $\propto_{il}\ (\bar{\hat{x}}_i, y)$ are smooth functions.

On depending on the conditions of dynamic system (4), the following main theorem is stated and proved to guarantee the global asymptotic stability in probability to the stochastic dynamic control system defined by equation (4).

THEOREM (3.1)

Consider the stochastic dynamic control system defined by equation (4), and assume that the dynamic observer system is designed to be

$$d\tilde{x} = A_0\tilde{x}dt + f(\tilde{x})dt + \phi(y)^T dw + \psi(\tilde{x})^T dw$$

a sequence of stabilizing functions $\propto_i\ (\bar{\hat{x}}_i, y)$, where $\bar{\hat{x}}_i = [\hat{x}_1, \hat{x}_2, \dots, \hat{x}_i]^T$, will be constructed recursively to build the Lyapunov function of the form

$$V(z, \tilde{x}) = \frac{1}{4}\sum_{i=1}^{n} z_i^4 + \frac{b}{2}(\tilde{x}^T P\tilde{x})^2 \ (14)$$

where P is a positive definite matrix which satisfies the following algebraic equation:

$$A_o{}^T P + PA_0 = -I$$

where

$$z_i = \hat{x}_i - \propto_{i-1}\ (\bar{\hat{x}}_{i-1}, y) \qquad i = 1, \dots, n \ (15)$$

and if the following are satisfied:

$$\propto_i = -S_i z_i - L_i \sum_{i=1}^{n-1} c_i\tilde{x}_i + \sum_{l=1}^{i-1} \frac{\partial \propto_{i-1}}{\partial \hat{x}l}\left(\hat{x}_{l+1} + L_l \sum_{i=1}^{n-1} c_i\tilde{x}_i\right) + \frac{\partial \propto_{i-1}}{\partial y}\sum_{i=1}^{n-1} c_i\hat{x}_{i+1} + \frac{\partial \propto_{i-1}}{\partial y}\sum_{i=1}^{n-1} c_i f_i(\bar{x}_i)$$
$$+ \frac{1}{2}\left(\frac{\partial^2 \propto_{i-1}}{\partial y^2}\right)\left(\sum_{i=1}^{n-1} c_i\phi_i(\bar{x}_i)\right)^T \left(\sum_{i=1}^{n-1} c_i\phi_i(\bar{x}_i)\right)$$

$$+\frac{1}{2}\left(\frac{\partial^2 \alpha_{i-1}}{\partial y^2}\right)\left(\sum_{i=1}^{n-1} c_i\psi_i(\tilde{x}_i)\right)^T\left(\sum_{i=1}^{n-1} c_i\psi_i(\tilde{x}_i)\right)-\frac{3}{4}\sigma_i^{4/3}z_i-\frac{1}{4\sigma_{i-1}^4}z_i-\frac{3}{4}\eta_i^{4/3}\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^{4/3}z_i$$
$$-\frac{3}{4\xi_i^2}\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^4 z_i-\frac{3}{4\delta_i^2}\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^4 z_i$$

And the control is designed as:

$$u=\Bigg[-S_n z_n-L_n c_n\tilde{x}_n+\sum_{l=1}^{n-1}\frac{\partial \alpha_{n-1}}{\partial \hat{x}_l}(\hat{x}_{l+1}+L_l C_n\tilde{x}_n)+\frac{\partial \alpha_{n-1}}{\partial y}c_n f_n(\tilde{x}_n)$$
$$+\frac{1}{2}\left(\frac{\partial^2 \alpha_{n-1}}{\partial y^2}\right)(c_n\phi_n(y))^T(c_n\phi_n(y))+\frac{1}{2}\left(\frac{\partial^2 \alpha_{n-1}}{\partial y^2}\right)(c_n\psi_n(\tilde{x}_n))^T(c_n\psi_n(\tilde{x}_n))$$
$$-\frac{3}{4}\sigma_n^{4/3}z_n-\frac{1}{4\sigma_{n-1}^4}z_n-\frac{3}{4}\eta_n^{4/3}\left(\frac{\partial \alpha_{n-1}}{\partial y}\right)^{4/3}z_n-\frac{3}{4\xi_n^2}\left(\frac{\partial \alpha_{n-1}}{\partial y}\right)^4 z_n$$
$$-\frac{3}{4\delta_n^2}\left(\frac{\partial \alpha_{n-1}}{\partial y}\right)^4 z_n\Bigg]$$

Where u is standing for x_{n+1} as discussed in equations (4) and (10) of the previous section (3). Then the equilibrium point $x = 0$ of the closed-loop nonlinear stochastic system (10) is globally asymptotically stable in probability.

Proof:

Since we have by equation (15) that:

$$z_i = \hat{x}_i - \alpha_{i-1}\left(\bar{\hat{x}}_{i-1}, y\right)$$

According to Itô differentiation we have:

$$dz_i = d\hat{x}_i - d\alpha_{i-1}\left(\bar{\hat{x}}_{i-1}, y\right) \qquad i = 1,2,\dots,n$$

where the second part of the above equation (16) is computed as follows:

$$d\alpha_{i-1}\left(\bar{\hat{x}}_{i-1}, y\right)$$
$$=\sum_{l=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{x}_l}\left(\hat{x}_{l+1}+L_l\sum_{i=1}^{n}c_i\tilde{x}_i\right)dt+\frac{\partial \alpha_{i-1}}{\partial y}\left(\sum_{i=1}^{n}c_i x_{i+1}+\sum_{i=1}^{n}c_i f_i(\tilde{x}_i)\right)$$
$$+\frac{1}{2}\left(\frac{\partial^2 \alpha_{i-1}}{\partial y^2}\right)\left(\sum_{i=1}^{n}c_i\phi_i(y)\right)^T\left(\sum_{i=1}^{n}c_i\phi_i(y)\right)dt$$
$$+\frac{1}{2}\left(\frac{\partial^2 \alpha_{i-1}}{\partial y^2}\right)\left(\sum_{i=1}^{n}c_i\psi_i(\tilde{x}_i)\right)^T\left(\sum_{i=1}^{n}c_i\psi_i(\tilde{x}_i)\right)dt+\frac{\partial \alpha_{i-1}}{\partial y}\sum_{i=1}^{n}c_i\phi_i(y)^T dw$$
$$+\frac{\partial \alpha_{i-1}}{\partial y}\sum_{i=1}^{n}c_i\psi_i(\tilde{x}_i)^T dw \qquad (17)$$

Set the Lyapunov function as follows:

$$V(z,\tilde{x}) = \frac{1}{4}\sum_{i=1}^{n} z_i^4 + \frac{b}{2}(\tilde{x}^T P\tilde{x})^2$$

where P is a suitable positive definite matrix will be designed later on and the above form indicates that the first term constitutes a Lyapunov function for the $(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n) -$ system, while the second term is a Lyapunov function for the $\tilde{x}$ $-$ system.

Now, we start the process of selecting the functions α_i $(\bar{\hat{x}}_i, y)$ to make $\mathcal{L}V$ negative definite. Along the solution of equations (9) and (16), from definition of $\mathcal{L}V$ (equation (3)), we have that:

$$\mathcal{L}V=\frac{4}{4}\sum_{i=1}^{n}z_i^3\Bigg[\hat{x}_{i+1}+L_i\sum_{i=1}^{n}c_i\tilde{x}_i-\sum_{l=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{x}_l}\left(\hat{x}_{l+1}+L_l\sum_{i=1}^{n}c_i\tilde{x}_i\right)$$
$$-\frac{\partial \alpha_{i-1}}{\partial y}\left(\sum_{i=1}^{n}c_i x_{i+1}+\sum_{i=1}^{n}c_i f_i(\tilde{x}_i)\right)-\frac{1}{2}\left(\frac{\partial^2 \alpha_{i-1}}{\partial y^2}\right)\left(\sum_{i=1}^{n}c_i\phi_i(y)\right)^T\left(\sum_{i=1}^{n}c_i\phi_i(y)\right)$$
$$-\frac{1}{2}\left(\frac{\partial^2 \alpha_{i-1}}{\partial y^2}\right)\left(\sum_{i=1}^{n}c_i\psi_i(\tilde{x}_i)\right)^T\left(\sum_{i=1}^{n}c_i\psi_i(\tilde{x}_i)\right)\Bigg]$$
$$+\frac{3}{2}\sum_{i=1}^{n}z_i^2\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2\left(\sum_{i=1}^{n}c_i\psi_i(\tilde{x}_i)\right)^T\left(\sum_{i=1}^{n}c_i\psi_i(\tilde{x}_i)\right)$$
$$+\frac{3}{2}\sum_{i=1}^{n}z_i^2\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2\left(\sum_{i=1}^{n}c_i\phi_i(y)\right)^T\left(\sum_{i=1}^{n}c_i\phi_i(y)\right)$$
$$+\frac{2}{2}b(\tilde{x}^T P\tilde{x})[\tilde{x}^T P d\tilde{x}+P\tilde{x}d\tilde{x}^T]+2b\,Tr\{\phi(y)^T(2P\tilde{x}\tilde{x}^T P+\tilde{x}^T P\tilde{x}P)\phi(y)\}$$
$$+2b\,Tr\{\psi(\tilde{x})^T(2P\tilde{x}\tilde{x}^T P+\tilde{x}^T P\tilde{x}P)\psi(\tilde{x})\} \qquad (18)$$

Since we have

$$z_{i+1}=\hat{x}_{i+1}-\alpha_i\left(\bar{\hat{x}}_i, y\right)\Rightarrow \hat{x}_{i+1}=z_{i+1}+\alpha_i\left(\bar{\hat{x}}_i, y\right) \quad i=1,\dots,n \qquad (19)$$

thus(16)

$$\mathcal{L}V=\sum_{i=1}^{n}z_i^3 z_{i+1}+\sum_{i=1}^{n}z_i^3\alpha_i+\sum_{i=1}^{n}z_i^3 L_i\sum_{i=1}^{n}c_i\tilde{x}_i-\sum_{i=1}^{n}z_i^3\sum_{l=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{x}_l}$$
$$\hat{x}_{l+1}-\sum_{i=1}^{n}z_i^3\sum_{l=1}^{i-1}\frac{\partial \alpha_{i\ 1}}{d\hat{x}_l}L_l\sum_{i=1}^{n}c_i\tilde{x}_i-\sum_{i=1}^{n}z_i^3\frac{\partial \alpha_{i\ 1}}{\partial y}$$
$$\sum_{i=1}^{n}c_i\tilde{x}_{i+1}-\sum_{i=1}^{n}z_i^3\frac{\partial \alpha_{i-1}}{\partial y}\sum_{i=1}^{n}c_i\hat{x}_{i+1}-\sum_{i=1}^{n}z_i^3\frac{\partial \alpha_{i-1}}{\partial y}\sum_{i=1}^{n}c_i f_i(\tilde{x}_i)$$
$$-\frac{1}{2}\sum_{i=1}^{n}z_i^3\left(\frac{\partial^2 \alpha_{i-1}}{\partial y^2}\right)\left(\sum_{i=1}^{n}c_i\phi_i(y)\right)^T\left(\sum_{i=1}^{n}c_i\phi_i(y)\right)$$
$$-\frac{1}{2}\sum_{i=1}^{n}z_i^3\left(\frac{\partial^2 \alpha_{i-1}}{\partial y^2}\right)\left(\sum_{i=1}^{n}c_i\psi_i(\tilde{x}_i)\right)^T\left(\sum_{i=1}^{n}c_i\psi_i(\tilde{x}_i)\right)$$
$$+\frac{3}{2}\sum_{i=1}^{n}z_i^2\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2\left(\sum_{i=1}^{n}c_i\phi_i(y)\right)^T\left(\sum_{i=1}^{n}c_i\phi_i(y)\right)$$
$$+\frac{3}{2}\sum_{i=1}^{n}z_i^2\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2\left(\sum_{i=1}^{n}c_i\psi_i(\tilde{x}_i)\right)^T\left(\sum_{i=1}^{n}c_i\psi_i(\tilde{x}_i)\right)$$
$$+b(\tilde{x}^T P\tilde{x})[\tilde{x}^T P(A_0+f(\tilde{x}))+P\tilde{x}(A_0^T+f(\tilde{x})^T)]$$
$$+2b\,Tr\{\phi(y)(2P\tilde{x}\tilde{x}^T P+\tilde{x}^T P\tilde{x}P)\phi(y)^T\}$$
$$+2b\,Tr\{\psi(\tilde{x})(2P\tilde{x}\tilde{x}^T P+\tilde{x}^T P\tilde{x}P)\psi(\tilde{x})^T\} \qquad (20)$$

Now, by applying Young's inequality of equation (2) onto some terms of equation (20), the following simplifications are needed and as follows:

(Note that the values of p and q will be selected as: $p = \frac{4}{3}, q = 4$)

1. $\sum_{i=1}^{n} z_i^3 z_{i+1} \le \frac{3}{4}\sum_{i=1}^{n} z_i^4 \sigma_i^{4/3} + \frac{1}{4}\sum_{i=2}^{n} \frac{1}{\sigma_{i-1}^4} z_i^4$

2. $\sum_{i=1}^{n} z_i^3 \frac{\partial \alpha_{i-1}}{\partial y} \sum_{i=1}^{n} c_i \tilde{x}_{i+1} \le \frac{3}{4}\sum_{i=1}^{n} \eta_i^{4/3} \left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^{4/3} z_i^4 + \frac{1}{4}\sum_{i=1}^{n} \frac{1}{\eta_i^4} (c_i)^4 (\tilde{x}_{i+1})^4$

$$\le \frac{3}{4}\sum_{i=1}^{n} \eta_i^{4/3} \left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^{4/3} z_i^4 + \frac{1}{4}\sum_{i=1}^{n} \frac{1}{\eta_i^4} |\tilde{c}|^4 |\tilde{x}|^4$$

where

$$c_i \tilde{x}_{i+1} \le |c_i \tilde{x}_{i+1}| \le |c_i||\tilde{x}_{i+1}| \le |\tilde{c}|^4 |\tilde{x}|^4, p = \frac{4}{3}, q = 4$$

and $|\tilde{c}|$ is the largest value of c_i for i=1,2,..,n.

3. $\frac{3}{2}\sum_{i=1}^{n} z_i^2 \left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2 \left(\sum_{i=1}^{n} c_i \phi_i(y)\right)^T \left(\sum_{i=1}^{n} c_i \phi_i(y)\right)$

$$\le \frac{3}{4}\sum_{i=1}^{n} \frac{1}{\xi_i^2}\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^4 z_i^4 + \frac{3}{4}\sum_{i=1}^{n} \xi_i^2 |\tilde{c}|^4 \left(\phi_i(y)^T \phi_i(y)\right)^2$$

since we have by the imposed condition of equation (11) we get:

$$\le \frac{3}{4}\sum_{i=1}^{n} \frac{1}{\xi_i^2}\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^4 z_i^4 + \frac{3}{4}\sum_{i=1}^{n} \xi_i^2 |\tilde{c}|^4 \left(\lambda_{max}(Q_2)\right)^4 |\tilde{x}|^4$$

(where we select the values of p and q as $p = q = 2$)

4. $\frac{3}{2}\sum_{i=1}^{n} z_i^2 \left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2 \left(\sum_{i=1}^{n} c_i \psi_i(\tilde{x}_i)\right)^T \left(\sum_{i=1}^{n} c_i \psi_i(\tilde{x}_i)\right)$

$$\le \frac{3}{4}\sum_{i=1}^{n} \frac{1}{\delta_i^2}\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^4 z_i^4 + \frac{3}{4}\sum_{i=1}^{n} \delta_i^2 |\tilde{c}|^4 \left(\lambda_{max}(Q_3)\right)^4 |\tilde{x}|$$

(where the values of p and q as $p = q = 2$)

5. $2b\, Tr\{\phi(y)(2P\tilde{x}\tilde{x}^T P + \tilde{x}^T P \tilde{x} P)\phi(y)^T\}$, with reference to [14] we have:

$$\le 2b_n |\phi(y)(2P\tilde{x}\tilde{x}^T P + \tilde{x}^T P\tilde{x}P)\phi(y)^T|_\infty$$

$$\le 2b_n \sqrt{n} |\phi(y)(2P\tilde{x}\tilde{x}^T P + \tilde{x}^T P\tilde{x}P)\phi(y)^T|$$

$$\le 6b_n \sqrt{n}\, |\phi(y)|^2 |P|^2 |\tilde{x}|^2$$

$$\le 6b_n \sqrt{n}\ \lambda_{max}(P) |\phi(y)|^2 |\tilde{x}|^2,$$

finally applying Young's inequality with $= q = 2$, gives:

$$\le \frac{3b_n\sqrt{n}}{\varepsilon_1^2}\left(\lambda_{max}(Q_2)\right)^4 |\tilde{x}|^4 + 3b_n\sqrt{n}\,\varepsilon_1^2 \left(\lambda_{max}(P)\right)^4 |\tilde{x}|^4 \quad (25)$$

6.(21) $2b\, Tr\{\psi(\tilde{x})(2P\tilde{x}\tilde{x}^T P + \tilde{x}^T P\tilde{x}P)\psi(\tilde{x})^T\}$

$$\le 2b_n |\psi(\tilde{x})(2P\tilde{x}\tilde{x}^T P + \tilde{x}^T P\tilde{x}P)\psi(\tilde{x})^T|_\infty$$

$$\le 2b_n \sqrt{n} |\psi(\tilde{x})(2P\tilde{x}\tilde{x}^T P + \tilde{x}^T P\tilde{x}P)\psi(\tilde{x})^T|$$

(22) $\le 6b_n\sqrt{n}\, |\psi(\tilde{x})|^2 |P|^2 |\tilde{x}|^2$

with the help of Young's inequality, $p = q = 2$, we have:

$$\le \frac{3b_n\sqrt{n}}{\varepsilon_2^2}\left(\lambda_{max}(Q_3)\right)^4 |\tilde{x}|^4 + 3b_n\sqrt{n}\,\varepsilon_2^2 \left(\lambda_{max}(P)\right)^4 |\tilde{x}|^4 \quad (26)$$

7. $[\tilde{x}^T P(A_o + f(\tilde{x})) + (A_0^T + f(\tilde{x})^T)P\tilde{x}]$ is simplified as follows:

$$\tilde{x}^T(PA_o + A_o^T P)\tilde{x} \le -(\lambda_{min}(P))|\tilde{x}|^2 \quad (27)$$

where

$$f(\tilde{x})^T P\tilde{x} + \tilde{x}^T P\, f(\tilde{x}) \le |f(\tilde{x})^T P\tilde{x} + \tilde{x}^T P\, f(\tilde{x})|$$

$$\le |f(\tilde{x})^T P\tilde{x}| + |\tilde{x}^T P\, f(\tilde{x})| \quad (28)$$

$$|f(\tilde{x})^T P\tilde{x}| \le |f(\tilde{x})||P||\tilde{x}| \le \lambda_{max}(Q_1)|\tilde{x}|(\lambda_{max}(P))|\tilde{x}|$$

$$\le (\lambda_{max}(Q_1))(\lambda_{max}(P))|\tilde{x}|^2$$

thus, we get:

$$f(\tilde{x})^T P\tilde{x} + \tilde{x}^T P\, f(\tilde{x}) \le 2(\lambda_{max}(Q_1))(\lambda_{max}(P))|\tilde{x}|^2 \quad (29)$$

Now, by substituting the equations (21-29) into equation (20), we have:

$$\begin{aligned}
\mathcal{L}V \le -\Big[& b\,\lambda_{min}(P)(\lambda_{min}(P)-2(\lambda_{max}(Q_1))(\lambda_{max}(P))) - 3b_n\sqrt{n}\,\varepsilon_1^2(\lambda_{max}(P))^4 \\
& -\frac{3b_n\sqrt{n}}{\varepsilon_1^2}(\lambda_{max}(Q_2))^4 - 3b_n\sqrt{n}\,\varepsilon_2^2(\lambda_{max}(P))^4 - \frac{3b_n\sqrt{n}}{\varepsilon_2^2}(\lambda_{max}(Q_3))^4 \\
& -\frac{1}{4}\sum_{i=1}^{n}\frac{1}{\eta_i^4}|\tilde{c}|^4 - \frac{3}{4}\sum_{i=1}^{n}\xi_i^2|\tilde{c}|^4(\lambda_{max}(Q_2))^4 - \frac{3}{4}\sum_{i=1}^{n}\delta_i^2|\tilde{c}|^4(\lambda_{max}(Q_3))^4\Big]|\tilde{x}|^4 \\
& +\sum_{i=1}^{n-1} z_i^3\Big[\alpha_i + L_i\sum_{i=1}^{n-1}c_i\tilde{x}_i - \sum_{i=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{x}_i}\Big(\hat{x}_{i+1}+L_i\sum_{i=1}^{n-1}c_i\tilde{x}_i\Big) \\
& -\frac{\partial \alpha_{i-1}}{\partial y}\sum_{i=1}^{n-1}c_i\hat{x}_{i+1} - \frac{\partial \alpha_{i-1}}{\partial y}\sum_{i=1}^{n-1}c_i f_i(\tilde{x}_i) \\
& -\frac{1}{2}\Big(\frac{\partial^2 \alpha_{i-1}}{\partial y^2}\Big)\Big(\sum_{i=1}^{n-1}c_i\phi_i(y)\Big)^T\Big(\sum_{i=1}^{n-1}c_i\phi_i(y)\Big) \\
& -\frac{1}{2}\Big(\frac{\partial^2 \alpha_{i-1}}{\partial y^2}\Big)\Big(\sum_{i=1}^{n-1}c_i\psi_i(\tilde{x}_i)\Big)^T\Big(\sum_{i=1}^{n-1}c_i\psi_i(\tilde{x}_i)\Big) + \frac{3}{4}\sigma_i^{4/3}z_i + \frac{1}{4\sigma_{i-1}^4}z_i \\
& +\frac{3}{4}\eta_i^{4/3}\Big(\frac{\partial \alpha_{i-1}}{\partial y}\Big)^{4/3}z_i + \frac{3}{4\xi_i^2}\Big(\frac{\partial \alpha_{i-1}}{\partial y}\Big)^4 z_i + \frac{3}{4\delta_i^2}\Big(\frac{\partial \alpha_{i-1}}{\partial y}\Big)^4 z_i\Big] \\
& + z_n^3\Big[u + L_n c_n\tilde{x}_n - \sum_{i=1}^{n-1}\frac{\partial \alpha_{n-1}}{\partial \hat{x}_i}\hat{x}_{i+1} - \sum_{i=1}^{n-1}\frac{\partial \alpha_{n-1}}{\partial \hat{x}_i}L_i c_n\tilde{x}_n - \frac{\partial \alpha_{n-1}}{\partial y}\sum_{i=1}^{n-1}c_i\hat{x}_{i+1} \\
& -\frac{\partial \alpha_{n-1}}{\partial y}c_n f_n(\tilde{x}_n) - \frac{1}{2}\Big(\frac{\partial^2 \alpha_{n-1}}{\partial y^2}\Big)(c_n\phi_n(y))^T(c_n\phi_n(y)) \\
& -\frac{1}{2}\Big(\frac{\partial^2 \alpha_{n-1}}{\partial y^2}\Big)(c_n\psi_n(\tilde{x}_n))^T(c_n\psi_n(\tilde{x}_n)) + \frac{3}{4}\sigma_n^{4/3}z_n + \frac{1}{4\sigma_{n-1}^4}z_n \\
& +\frac{3}{4}\eta_n^{4/3}\Big(\frac{\partial \alpha_{n-1}}{\partial y}\Big)^{4/3}z_n + \frac{3}{4\xi_n^2}\Big(\frac{\partial \alpha_{n-1}}{\partial y}\Big)^4 z_n + \frac{3}{4\delta_n^2}\Big(\frac{\partial \alpha_{n-1}}{\partial y}\Big)^4 z_n\Big]
\end{aligned}$$

At this point, we can see that all the terms can be cancelled by u and α_i .

If we choose ϵ_1, ϵ_2, η_i, ξ_i, δ_i to satisfy:

$$\begin{aligned}
& b\,\lambda_{min}(P)(\lambda_{min}(P)-2(\lambda_{max}(Q_1))(\lambda_{max}(P))) - 3b_n\sqrt{n}\,\varepsilon_1^2(\lambda_{max}(P))^4 - \frac{3b_n\sqrt{n}}{\varepsilon_1^2}(\lambda_{max}(Q_2))^4 \\
& - 3b_n\sqrt{n}\,\varepsilon_2^2(\lambda_{max}(P))^4 - \frac{3b_n\sqrt{n}}{\varepsilon_2^2}(\lambda_{max}(Q_3))^4 \\
& -\frac{1}{4}\sum_{i=1}^{n}\frac{1}{\eta_i^4}|\tilde{c}|^4 - \frac{3}{4}\sum_{i=1}^{n}\xi_i^2|\tilde{c}|^4(\lambda_{max}(Q_2))^4 \\
& -\frac{3}{4}\sum_{i=1}^{n}\delta_i^2|\tilde{c}|^4(\lambda_{max}(Q_3))^4 \equiv \rho > 0 \qquad (31)
\end{aligned}$$

and α_i and u as:

$$\begin{aligned}
\alpha_i = & -S_i z_i - L_i\sum_{i=1}^{n-1}c_i\tilde{x}_i + \sum_{i=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{x}_i}\Big(\hat{x}_{i+1}+L_i\sum_{i=1}^{n-1}c_i\tilde{x}_i\Big) + \frac{\partial \alpha_{i-1}}{\partial y}\sum_{i=1}^{n-1}c_i\hat{x}_{i+1} + \frac{\partial \alpha_{i-1}}{\partial y}\sum_{i=1}^{n-1}c_i f_i(\tilde{x}_i) \\
& +\frac{1}{2}\Big(\frac{\partial^2 \alpha_{i-1}}{\partial y^2}\Big)\Big(\sum_{i=1}^{n-1}c_i\emptyset_i(\tilde{x}_i)\Big)^T\Big(\sum_{i=1}^{n-1}c_i\emptyset_i(\tilde{x}_i)\Big) \\
& +\frac{1}{2}\Big(\frac{\partial^2 \alpha_{i-1}}{\partial y^2}\Big)\Big(\sum_{i=1}^{n-1}c_i\Psi_i(\tilde{x}_i)\Big)^T\Big(\sum_{i=1}^{n-1}c_i\Psi_i(\tilde{x}_i)\Big) - \frac{3}{4}\sigma_i^{4/3}z_i - \frac{1}{4\sigma_{i-1}^4}z_i - \frac{3}{4}\eta_i^{4/3}\Big(\frac{\partial \alpha_{i-1}}{\partial y}\Big)^{4/3} \\
& -\frac{3}{4\xi_i^2}\Big(\frac{\partial \alpha_{i-1}}{\partial y}\Big)^4 z_i - \frac{3}{4\delta_i^2}\Big(\frac{\partial \alpha_{i-1}}{\partial y}\Big)^4 z_i \qquad (32)
\end{aligned}$$

$$\begin{aligned}
u = \Big[& -S_n z_n - L_n c_n\tilde{x}_n + \sum_{i=1}^{n-1}\frac{\partial \alpha_{n-1}}{\partial \hat{x}_i}(\hat{x}_{i+1}+L_i C_n\tilde{x}_n) + \frac{\partial \alpha_{n-1}}{\partial y}\sum_{i=1}^{n-1}c_i\hat{x}_{i+1} + \frac{\partial \alpha_{n-1}}{\partial y}c_n f_n(\tilde{x}_n) \\
& +\frac{1}{2}\Big(\frac{\partial^2 \alpha_{n-1}}{\partial y^2}\Big)(c_n\emptyset_n(y))^T(c_n\emptyset_n(y)) + \frac{1}{2}\Big(\frac{\partial^2 \alpha_{n-1}}{\partial y^2}\Big)(c_n\Psi_n(\tilde{x}_n))^T(c_n\Psi_n(\tilde{x}_n)) \\
& -\frac{3}{4}\sigma_n^{4/3}z_n - \frac{1}{4\sigma_{n-1}^4}z_n - \frac{3}{4}\eta_n^{4/3}\Big(\frac{\partial \alpha_{n-1}}{\partial y}\Big)^{4/3}z_n - \frac{3}{4\xi_n^2}\Big(\frac{\partial \alpha_{n-1}}{\partial y}\Big)^4 z_n \\
& -\frac{3}{4\delta_n^2}\Big(\frac{\partial \alpha_{n-1}}{\partial y}\Big)^4 z_n\Big] \qquad (33)
\end{aligned}$$

where $S_i > 0$, then the infinitesimal generator of the closed-loop stochastic system (9), (16) and (33) is negative definite, that is:

$$\mathcal{L}V \le -\sum_{i=1}^{n} s_i z_i^4 - \rho|\tilde{x}|^4 \qquad (34)$$

with (34) and hence $\mathcal{L}V < 0$, and from theorem (1) the critical point of (4) is globally asymptotically stable in probability. That completes the proof.

4. Inverse Optimal Output-Feedback Stabilization

After considering the stabilization of feedback stochastic dynamical systems in the previous section we shall show how our backstepping design which achieves stability can be redesigned to also achieve inverse optimality.

Theorem (4.1) [4]

Consider the simple class of nonlinear stochastic dynamical system described by:

$$dx_i = x_{i+1}\,dt + \varphi_i(y)^T\,dw\;; \qquad i = 1,\dots,n-1$$

$$dx_n = u dt + \varphi_n(y)^T\,dw$$

$$y = x_1$$

such that $\mathcal{L}V < 0$, with the suggested Lyapunov function of the form

$$V(z,\tilde{x}) = \frac{1}{4}y_1^4 + \frac{1}{4}\sum_{i=2}^{n}z_i^4 + \frac{b}{2}(\tilde{x}^T P\tilde{x})^2$$

if there exist a continuous positive function $M(y,\tilde{\tilde{x}})$ such that the control law of the above dynamical system can be rewritten as

$$u = \alpha(y, \hat{x}) = -M(y, \tilde{\hat{x}}) z_n$$

then the control law

$$u^* = \beta\alpha(y, \hat{x}) \quad , \quad \beta > 1$$

solves the problem of inverse optimal stabilization in probability.

Theorem (4.2)

Consider the nonlinear stochastic dynamical system described by equation (4) assuming that the conditions of theorem (2.1) are satisfied, if there exist a continuous positive function $M(y, \tilde{\hat{x}})$ such that the control law of theorem (3.1) can be rewritten as:

$$u = \alpha(y, \hat{x}) = -M(y, \tilde{\hat{x}}) z_n$$

Such that $\mathcal{L}V < 0$, with the suggested Lyapunov function

$$V(z, \tilde{x}) = \frac{1}{4}\sum_{i=1}^{n} z_i^4 + \frac{b}{2}(\tilde{x}^T P \tilde{x})^2$$

then the control law

$$u^* = \alpha^*(y, \hat{x}) = \beta\alpha(y, \hat{x}) \qquad \beta \geq \frac{4}{3}$$

solves the problem of inverse optimal stabilization in probability.

Proof

If we consider carefully the last bracket of equation (30), every term except the second, third, fourth, fifth, sixth, seventh, and eighth, has z_n as a factor, with the help of Young's inequality, we have:

1- $$L_n C_n \tilde{x}_n z_n^3 \leq \frac{3}{4} \epsilon_3^{4/3} z_n^4 + \frac{1}{4\epsilon_3^4} L_n^4 C_n^4 \tilde{x}_n^4$$

$$\leq \frac{3}{4} \epsilon_3^{4/3} z_n^4 + \frac{1}{4\epsilon_3^4} L_n^4 |\tilde{c}|^4 |\tilde{x}|^4 \qquad (37)$$

2 - $$-\sum_{\ell=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}\ell} L_e C_n \tilde{x}_n z_n^3 < \frac{3}{4}\left(\epsilon_4 \sum_{\ell=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}\ell} \mathrm{Le}\right)^{4/3} z_n^4 + \frac{1}{4\epsilon_4^4} |\tilde{c}|^4 |\tilde{x}|^4 \qquad (38)$$

3

$$-\frac{\partial \alpha_{n-1}}{\partial y} c_n f_n(x_n) z_n^3 \leq \frac{3}{4}\left(\epsilon_5 \frac{\partial \alpha_{n-1}}{\partial y}\right)^{4/3} z_n^4 + \frac{1}{4\epsilon_5^4} |\tilde{c}|^4 |f|^4$$

$$\leq \frac{3}{4}\left(\epsilon_5 \frac{\partial \alpha_{n-1}}{\partial y}\right)^{4/3} z_n^4 + \frac{1}{4\epsilon_5^4} |\tilde{c}|^4 (\lambda_{max}(Q_1))^4 |\tilde{x}|^4 \qquad (39)$$

4 - $$\frac{1}{2}\left(\frac{\partial^2 \alpha_{n-1}}{\partial y^2}\right)(c_n \phi_n(y))(c_n \phi_n(y))^T z_n^3$$

$$\leq \frac{3}{8}\left(\epsilon_6 \frac{\partial^2 \alpha_{n-1}}{\partial y^2}\right)^{4/3} z_n^4 + \frac{1}{4\epsilon_6^4} |\tilde{c}|^4 (\lambda_{max}(Q_2))^4 |\tilde{x}|^4 \qquad (40)$$

5 - $$\frac{1}{2}\left(\frac{\partial^2 \alpha_{n-1}}{\partial y^2}\right)(c_n \Psi_n(\tilde{x}_n))(c_n \Psi_n(\tilde{x}_n))^T z_n^3$$

$$\leq \frac{3}{8}\left(\epsilon_7 \frac{\partial^2 \alpha_{n-1}}{\partial y^2}\right)^{3/4} z_n^4 + \frac{1}{4\epsilon_7^4} |\tilde{c}|^4 (\lambda_{max}(Q_3))^4 |\tilde{x}|^4 \qquad (41)$$

6- To simplify the third term we use the equation:

$$\hat{x}_{\ell+1} = z_{\ell+1} + \alpha_\ell$$

thus

$$\hat{x}_\ell = z_{\ell+1} = z_{e+1} + \sum_{K=1}^{\ell} z_K \alpha_{\ell K} \qquad (35) \qquad (42)$$

substitute it back in the third term to get:

$$-z_n^3 \sum_{\ell=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}\ell} \hat{x}_{\ell+1} = -z_n^3 \sum_{\ell=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}\ell} z_{\ell+1} - z_n^3 \sum_{\ell=1}^{n=1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}\ell} \sum_{k=1}^{\ell} z_K \alpha_{\ell K}$$

$$(36) \sum_{\ell=1}^{n-1}\left[\frac{3}{4}\left(\epsilon_8 \frac{\partial \alpha_{n-1}}{\partial \hat{x}e}\right)^{3/4} z_n^4 + \frac{1}{4\epsilon_8^4} z_{e+1}^4\right]$$

$$+ \sum_{K=1}^{n-1}\left[\frac{3}{4}\left(\epsilon_q \sum_{\ell=K}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}e} \alpha_{\ell K}\right)^{4/3} z_n^4 + \frac{1}{4\epsilon_9^4} z_K^4\right] \qquad (43)$$

Thus, $\mathcal{L}$v is given as:-

$$\mathcal{L}V \leq -\Big[b\lambda_{min}(P)\lambda_{max}(P) - 2(b\lambda_{max}(Q_1))\lambda_{min}(P) - 3bn\sqrt{n}\,\epsilon_1^2 (b\lambda_{max}(P))^4$$

$$-\frac{3bn\sqrt{n}}{\epsilon_1^2}(b\lambda_{max}(Q_2))^4 - \frac{1}{4}\sum_{i=1}^{n}\frac{1}{\eta_i^4}|\tilde{c}|^4(b\lambda_{max}(Q_1))^4 - \frac{3}{4}\sum_{i=1}^{n}\delta_i^2|\tilde{c}|^4(b\lambda_{max}(Q_3))^4$$

$$-\frac{1}{4\epsilon_4^3}L_n^4|\tilde{c}|^4 - \frac{1}{4\epsilon_4^4}|\tilde{c}|^4 - \frac{1}{4\epsilon_5^4}|\tilde{c}|^4(b\lambda_{max}(Q_1))^4 - \frac{1}{4\epsilon_6^4}|\tilde{c}|^4(b\lambda_{max}(Q_2))^4$$

$$-\frac{1}{4\epsilon_7^4}|\tilde{c}|^4(b\lambda_{max}(Q_3))^4\Big]|\tilde{x}|^4$$

$$+\sum_{i=1}^{n-1} zi^{3}\Bigg[\alpha_i + L_i\sum_{i=1}^{n-1} Ci\tilde{x}i - \sum_{\ell=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{x}e}\Big(\hat{x}_{\ell+1}+L_\ell\sum_{i=1}^{n-1}Ci\tilde{x}i\Big) - \frac{\partial\alpha_{i-1}}{\partial y}\sum_{i=1}^{n-1}Ci\tilde{x}_{i+1} - \frac{\partial\alpha_{i-1}}{\partial y}c_i f_i(x)$$
$$-\frac{1}{2}\left(\frac{\partial^2\alpha_{i=1}}{dy_2}\right)\left(\sum_{i=1}^{n-1}c_i\phi_i(y)\right)\left(\sum_{i=1}^{n-1}c_i\phi_i(y)\right)^T$$
$$-\frac{1}{2}\left(\frac{\partial^2\alpha_{i=1}}{dy_2}\right)\left(\sum_{i=1}^{n-1}c_i\Psi_i(\tilde{x}_i)\right)\left(\sum_{i=1}^{n-1}c_i\Psi_i(\tilde{x}_i)\right)^T + \frac{3}{4}\partial_i^{4/3}z_i + \frac{3}{4}\eta_i^{4/3}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^{4/3}z_i$$
$$+\frac{3}{4\delta_i^4}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^4 z_i + \frac{1}{4\epsilon_8^4}z_i + \frac{1}{4\epsilon_9^4}z_i\Bigg]$$
$$+z_n^3\Bigg[u+\frac{3}{4}\epsilon_3^{4/3}z_n + \frac{3}{4}\left(\epsilon_4\sum_{\ell=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{x}_\ell}Le\right)^{4/3}z_n + \frac{3}{4}\left(\epsilon_5\frac{\partial\alpha_{n-1}}{\partial y}\right)^{4/3}z_n$$
$$+\frac{3}{8}\left(\epsilon_6\frac{\partial^2\alpha_{n-1}}{\partial y^2}\right)^{4/3}z_n + \frac{3}{8}\left(\epsilon_7\frac{\partial^2\alpha_{n-1}}{\partial y^2}\right)^{4/3}z_n + \sum_{\ell=1}^{n-1}\frac{3}{4}\left(\epsilon_8\frac{\partial\alpha_{n-1}}{\partial\hat{x}_\ell}\right)^{4/3} + z_n$$
$$+\sum_{K=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{x}_\ell}\left(\epsilon_9\sum_{\ell=K}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{x}_\ell}\alpha_{\ell K}\right)^{4/3}z_n - \frac{\partial\alpha_{n-1}}{\partial y}c_n U + \frac{3}{4}c^{4/3}z_n$$
$$+\frac{3}{4\delta_n^2}\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^4 z_n\Bigg] \qquad (44)$$

if $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7$, η_i, δ_i, and δ_i are chosen to satisfy :

$$b\,\lambda_{min}(P)\big(\lambda_{min}(P) - 2(\lambda_{max}(Q_1))(\lambda_{max}(P))\big) - 3b_n\sqrt{n}\,\varepsilon_1^2(\lambda_{max}(P))^4 - \frac{3b_n\sqrt{n}}{\varepsilon_1^2}(\lambda_{max}(Q_2))^4$$
$$-3b_n\sqrt{n}\,\varepsilon_2^2(\lambda_{max}(P))^4 - \frac{3b_n\sqrt{n}}{\varepsilon_2^2}(\lambda_{max}(Q_3))^4 - \frac{1}{4}\sum_{i=1}^{n}\frac{1}{\eta_i^4}|\tilde{c}|^4$$
$$-\frac{3}{4}\sum_{i=1}^{n}\xi_i^2|\tilde{c}|^4(\lambda_{max}(Q_2))^4 - \frac{3}{4}\sum_{i=1}^{n}\delta_i^2|\tilde{c}|^4(\lambda_{max}(Q_3))^4 - \frac{1}{4\epsilon_3^4}L_n^4|\tilde{c}|^4 - \frac{1}{4\epsilon_4^4}L_n^4|\tilde{c}|^4$$
$$-\frac{1}{4\epsilon_5^4}L_n^4|\tilde{c}|^4(\lambda_{max}(Q_1))^4 - \frac{1}{4\epsilon_6^4}L_n^4|\tilde{c}|^4(\lambda_{max}(Q_2))^4 - \frac{1}{4\epsilon_7^4}L_n^4|\tilde{c}|^4(\lambda_{max}(Q_3))^4$$
$$\equiv \rho > 0$$

$$\frac{1}{4\epsilon_8^4} + \frac{1}{4\epsilon_9^4} = \frac{Si}{2}$$

where Si are those in (33), and

$$u = -M(y,\tilde{x})z_n$$

$$M(y,\tilde{x}) = S_n + \frac{3}{4}\epsilon_3^{4/3} + \frac{3}{4}\left(\epsilon_4\sum_{e=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{x}_e}Le\right)^{4/3} + \frac{3}{4}\left(\epsilon_5\frac{\partial\alpha_{n-1}}{\partial y}\right)^{4/3} + \frac{3}{8}\left(\epsilon_6\frac{\partial^2\alpha_{n-1}}{\partial y_2}\right)^{4/3}$$
$$+\frac{3}{8}\left(\epsilon_7\frac{\partial^2\alpha_{n-1}}{\partial y_2}\right)^{4/3} + \sum_{e=1}^{n-1}\frac{3}{4}\left(\epsilon_8\frac{\partial\alpha_{n-1}}{\partial y}\right)^{4/3} + \sum_{e=K}^{n-1}\frac{3}{4}\left(\epsilon_9\frac{\partial\alpha_{n-1}}{\partial\tilde{x}\ell}\alpha_{\ell K}\right)^{4/3}$$
$$-\frac{\partial\alpha_{n-1}}{\partial y}CnSn + \frac{3}{4}\sigma_n^{4/3} + \frac{1}{4\sigma_{n-1}^4} + \frac{3}{4}\eta_n^{4/3}\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^{4/3} + \frac{3}{4\delta_n^2}\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^4$$
$$+\frac{3}{4\delta_n^2}\left(\frac{\partial\alpha_{n-1}}{\partial y}\right)^4 \qquad (48)$$

Then

$$u = \beta\alpha(y,\hat{x}), \qquad \beta \ge \frac{4}{3}$$

Thus we get:

$$\mathcal{L}V \le -\frac{1}{2}\sum_{i=1}^{n}S_i z_i^4 - \rho|\tilde{x}|^4 < 0$$

Thus, according to theorem (4.2), we achieve not only global asymptotic stability in probability, but also inverse optimality, which completes our proof.

5. Algorithm

A robust controller stabilization in probability of the non-linear stochastic system presented in equation (4) with linear dynamic observer of equation (6), is found using the following steps.

Input: The dynamic control system described by

$$dx_i = x_{i+1}dt + f_i(\bar{x}_i)dt + \phi_i(y)^T dw + \Psi_i(\bar{x}_i)^T dw \qquad i = 1,2,\dots,n-1 \qquad (45)$$

$$dx_n = udt + f_n(x_n)dt + \phi_n(y)^T dw + \Psi_n(x_n)^T dw \qquad (46)$$

$$y(x) = \sum_{i=1}^{n}c_i x_i \qquad (47)$$

Output: Robust stabilizing control u in probability and the unknown design positive functions α_i, for backstepping procedure, $i = 1,\dots,n-1,$ as well as a suitable stabilized Lyapunov function $V(\tilde{x}, z)$.

Step 1: Check Lipschitz conditions for the functions f, ϕ, ψ otherwise, either approximate the function by another one that satisfies the Lipschitz condition or change the space into another one to ensure the condition is satisfied "the problem of extension", or go to the last step (12), for stopping the algorithm work.

Step 2: Define the following (suggested) dynamic observer:

$$d\hat{x}_i = \hat{x}_{i+1}\, dt + L_i \sum_{i=1}^{n-1} c_i\, \hat{x}_i\, dt \qquad i = 1, \dots, n-1$$

$$d\hat{x}_n = u\, dt + L_n\, C_n\, \hat{x}_n\, dt$$

Step 3: Set the error vector $e = \tilde{x}_i = x_i - \hat{x}_i$

Step 4: Compute $d\tilde{x}_i = dx_i - d\hat{x}_i$ using Itô formula such that:

$$d\tilde{x}_i = \tilde{x}_{i+1} dt - L_i \sum_{i=1}^{n} c_i\, \tilde{x}_i + f_i(\bar{x}_i) dt + \psi_i(\bar{x}_i)^T\, dw + \phi_i\,(y)^T\, dw$$

or

$$d\tilde{x}_i = A_0\, \tilde{x}\, dt + f_i(\bar{x}_i) dt + \psi_i(\bar{x}_i)^T\, dw + \phi_i\,(y)^T\, dw$$

$$A_0 = \begin{bmatrix} -L_1 C_1 & -L_1 C_2 + 1 & -L_1 C_3 & \dots & -L_1 C_1 & 0 \\ -L_2 C_1 & -L_2 C_2 & -L_2 C_3 + 1 & \dots & -L_2 C_n & 0 \\ \vdots & & \vdots & & & \vdots \\ \vdots & & & & & 0 \\ -L_n C_1 & \dots & \dots\dots & -L_n C_{n-1} & -L_n C_n & u \end{bmatrix}$$

Step 5: Compute $L_i,\ i = 1, \dots, n$ in order to make A_0 stable.

Step 6: Find the unique positive definite matrix P of the following linear algebraic Riccati equation:

$$A_0^T\, P + P\, A_0 = -I$$

Step 7: Suggest the Lyapunov function of the form:

$$V(z, \tilde{x}) = \frac{1}{4} \sum_{i=1}^{n} z_i^4 + \frac{b}{2}\, (\tilde{x}^T P \tilde{x})^2$$

where

$$z_i = \hat{x}_i - \alpha_{i-1} \qquad i = 1, \dots, n$$

Step 8: As discussed in theorem (4.1) above, select a suitable values for $\epsilon_1, \epsilon_2, \delta_i, \varepsilon_i$ and η_i to satisfy:

$$b\, \lambda_{min}(P)(\lambda_{min}(P) - 2(\lambda_{max}(Q_1))(\lambda_{max}(P))) - 3b_n\sqrt{n}\, \varepsilon_1^2 (\lambda_{max}(P))^4 - \frac{3b_n\sqrt{n}}{\varepsilon_1^2}(\lambda_{max}(Q_2))^4 - 3b_n\sqrt{n}\, \varepsilon_2^2 (\lambda_{max}(P))^4 - \frac{3b_n\sqrt{n}}{\varepsilon_2^2}(\lambda_{max}(Q_3))^4 - \frac{1}{4}\sum_{i=1}^{n} \frac{1}{\eta_i^4} |\tilde{c}|^4 - \frac{3}{4}\sum_{i=1}^{n} \xi_i^2 |\tilde{c}|^4 (\lambda_{max}(Q_2))^4 - \frac{3}{4}\sum_{i=1}^{n} \delta_i^2 |\tilde{c}|^4 (\lambda_{max}(Q_3))^4 = \rho > 0$$

where Q_1 Q_2, Q_3 are positive definite matrices, and $\lambda_{max}(Q_1)$ is the largest eigenvalue of Q_1, $\lambda_{max}(Q_2)$, $\lambda_{max}(Q_3)$ are the largest eigenvalues of Q_2 and Q_3 respectively and $|\tilde{c}|$ is the largest value of c_i for i=1,2,..,n.

Step 9: Compute α_i and u using equations (32) and (33) in the main theorem (4.2).

Step 10: On using the results of the previous steps, the infinitesimal generator $\mathcal{L}V$ will be negative, i.e.

$$\mathcal{L}V \le -\rho\, |\tilde{x}|^4 - \sum_{i=1}^{n} S_i\, z_i^4 < 0$$

where

$S_i > 0 \quad i = 1, \dots, n$

Step 11: Back substitution the values of step (10) into step (7) making the Lyapunov function of step (7) is completely defined.

Step 12: Stop "the algorithm work is completed".

6. EXAMPLE

Consider the following non-linear dynamical system

$$dx_1 = x_2 dt + x_1^3 dw + 3y_1 dw$$

$$dx_2 = x_3 dt + \cos x_1 \sin x_2 dw + y_2^2 dw$$

$$dx_3 = x_4 dt + \sin(x_2\, x_3) dw + \sin y_3\, dw$$

$$dx_4 = u\, dt + \cos(x_2\, x_3) dw + \cos y_4\, dw$$

$$y = c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4$$

Check Lipschitz Condition for f, ϕ, ψ

Check for f:

$$\|f(x_1,x_2,x_3) - f(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3)\| \le \left\|\frac{\partial f}{\partial x}\right\| \|x-\hat{x}\| = 3\|x-\hat{x}\|$$

1. Check the Lipschitz conditions for ψ : $\left\|\frac{\partial \psi}{\partial x}\right\| \le 9$ Thus $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ satisfies Lipschitz condition

$$\|\psi(x_1,x_2,x_3,x_4) - \psi(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tilde{x}_4)\| \le 9\|x - \tilde{x}\|$$

3. Check the Lipschitz condition for the function $\phi(y)$ satisfies:-

$$\|\phi(y) - \phi(\hat{y})\| \le 7$$

The observer system is:

$$d\hat{x}_i = \hat{x}_{i+1} dt + L_i \textstyle\sum_{i=1}^{n} c_i \tilde{x}_i\, dt$$

$$d\hat{x}_1 = \hat{x}_2 dt + L_1(c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4)dt$$

$$d\hat{x}_2 = \hat{x}_3 dt + L_2(c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4)dt$$

$$d\hat{x}_3 = \hat{x}_4 dt + L_3(c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4)dt$$

$$d\hat{x}_4 = udt + L_4(c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4)dt$$

The error is computed as follows:

$$d\tilde{x} = A_0\tilde{x}dt + (x_2 + x_3 + x_4)dt + x_1^3 + \cos x_1 \sin x_2 + \sin(x_2x_3) + \cos(x_3x_4)dw + (3y_1^2 + y_2^2 + \sin y_3 + \cos y_3 + \cos y_4)dw$$

we choose the values of the above uncounted matrix in order to satisfy the equation $A_0^T P + PA_0 = -I$ and computing P, after choosing the values of c_i, i=1,…, 4, and computing the values of L_i we have:

$$A_0 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ -6.25 & 0 & 1 & 0 \\ -7.5 & 0 & 0 & 1 \\ -9 & 0 & 0 & 0 \end{bmatrix} \quad \textit{so that:}$$

$$P = \begin{bmatrix} 481.9063 & -0.5 & -128.625 & 0.5 \\ -0.5 & 128.625 & -0.5 & -35.25 \\ -128.6250 & -0.5 & 35.25 & -0.5 \\ 0.5 & -35.25 & -0.5 & 10.4913 \end{bmatrix}$$

where the eigenvalues of P are $\lambda_1 = 0.3093, \lambda_2 = 1.3203, \lambda_3 = 138.345, \lambda_4 = 516.2997$

$$\frac{1}{4}\hat{x}_1^4 + \frac{1}{4}(\hat{x}_2 - \alpha_1)^4 + \frac{1}{4}(\hat{x}_3 - \alpha_2)^4 + \frac{1}{4}(\hat{x}_4 - \alpha_3)^4$$

$$+\frac{b}{2}\left([\tilde{x}_1\tilde{x}_2\tilde{x}_3\tilde{x}_4]^T \begin{pmatrix} 481.9063 & -0.5 & -128.625 & 0.5 \\ -0.5 & 128.625 & -0.5 & -35.25 \\ -128.6250 & -0.5 & 35.25 & -0.5 \\ 0.5 & -35.25 & -0.5 & 10.4913 \end{pmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix}\right)^2$$

$$= \frac{1}{4}\hat{x}_1^4 + \frac{1}{4}(\hat{x}_2 - a_1)^4 + \frac{1}{4}(\hat{x}_3 - a_2)^4 + \frac{1}{4}(\hat{x}_4 - a_3)^4 + \frac{b}{2}(481.9063x_1^2 + 128.625x_2^2 + 35.25x_3^2 + 10.4931x_4^2 - x_1x_2 - 257.25x_1x_3 + x_1x_4 - x_2x_3 - 70.5x_2x_4 - x_3x_4)^2$$

b is positive constant. Compute $\rho > 0$ such that

$$b\lambda_{min}(P)(\lambda_{min}(P) - 2(\lambda_{max}(Q_1))(\lambda_{max}(P))) - 3b_n\sqrt{n}\,\varepsilon_1^2(\lambda_{max}(P))^4 - \frac{3b_n\sqrt{n}}{\varepsilon_1^2}(\lambda_{max}(Q_2))^4$$
$$- 3b_n\sqrt{n}\,\varepsilon_2^2(\lambda_{max}(P))^4 - \frac{3b_n\sqrt{n}}{\varepsilon_2^2}(\lambda_{max}(Q_3))^4 - \frac{1}{4}\sum_{i=1}^{n}\frac{1}{\eta_i^4}|\tilde{c}|^4$$
$$-\frac{3}{4}\sum_{i=1}^{n}\xi_i^2|\tilde{c}|^4(\lambda_{max}(Q_2))^4$$
$$-\frac{3}{4}\sum_{i=1}^{n}\delta_i^2|\tilde{c}|^4(\lambda_{max}(Q_3))^4$$

$\lambda_{max}(P) = 516.2997, b = 1000, \epsilon_1 = 0.01, \epsilon_2 = 0.02$

$\eta_1 = \eta_2 = \eta_3 = \eta_4 = 1$, $\xi_i = 0.3$, $|\tilde{c}| = 1$, $\delta_i = 0.4$

=10000(516.2997) +2(516.2997)- (0.0003)$\sqrt{2}$(0.01)^2 (516.2997)^4

$$+\frac{0.0003(\sqrt{2})}{(0.01)^2} - (0.0003)\sqrt{2}\,(0.02)(516.2997)^4 + \frac{0.0003\sqrt{2}}{(0.02)^2} - \frac{1}{16}+\frac{3}{4}(0.36) + \frac{3}{4}(0.64)$$

$$=4558081.389 = \rho > 0$$

Find αi and u:-

$$\alpha_1 = -S_1 z_1 - L_1 (c_1x_1+c_2x_2+c_3x_3) - \frac{3}{4}\sigma_1^{4/3} z_1 - \frac{1}{4\sigma_0^4} z_1$$

$$\alpha_2 = -S_2 z_2 - L_2 (c_1x_1+c_2x_2+c_3x_3) + \frac{\partial\alpha_1}{\partial\hat{x}_1}\left((c_1f(x_1)+c_2f(x_2)+c_3f(x_3)\right) + \frac{1}{2}\left(\frac{\partial 2_{\alpha_1}}{\partial y^2}\right)(c1\emptyset1+c2\emptyset2+c3\emptyset3)(c1\emptyset_1^T+c2\emptyset_2^T+C3\emptyset_3^T) + \frac{1}{2}\frac{\partial 2_{\alpha^1}}{\partial y^2}(c_1\Psi_1+c_2\Psi_2+c_3\Psi_3)(c_1\Psi_1+c_2\Psi_2+c_3\Psi_3)^T - \frac{3}{4}\sigma^{4/3} z_2 - \frac{1}{4\sigma_1^4} - \frac{3}{4}\eta_2^{4/3}\left(\frac{\partial\alpha_1}{\alpha_y}\right)^{4/3} z_2 - \frac{3}{4\xi_2^2}\left(\frac{\partial\alpha_1}{\partial y}\right)^4 z_2 - \frac{3}{4\delta_2^2}\left(\frac{\partial\alpha_1}{\partial y}\right)^4 z$$

$$\alpha_3 = -s_3z_3 - L_3(c_1x_1+c_2x_2+c_3x_3) + \sum_{e=1}^{2}\frac{\partial\alpha_2}{\partial \hat{x}_e}(\hat{x}_{e+1} + L_e(c_1x_1+c_2x_2+c_3x_3)) + \frac{\partial\alpha_2}{\partial y} c_ix_i + 1\frac{\partial\alpha_2}{\partial y}\sum_{i=1}^{3} c_if_i(x_i) + \frac{1}{2}\left(\frac{\partial 3_{\alpha_2}}{\partial y_2}\right)\left(\sum_{i=1}^{3} c\, i\emptyset i\right)\left(\sum_{i=1}^{3} ci\emptyset i\right)^T + \frac{1}{2}\left(\frac{d2_{\alpha_2}}{dy_2}\right)\left(\sum_{i=1}^{3} ci\Psi_i\right)\left(\sum_{i=1}^{3} ci\Psi_i\right)^T - \frac{3}{4}\sigma_3^{4/3} z_3 - \frac{1}{4\sigma_2^4} - \frac{3}{4}\eta_3^{4/3}\left(\frac{\partial 2_{\alpha_2}}{\partial y}\right)^{4/3} z_3 - \frac{3}{4\xi_3^2}\left(\frac{\partial\alpha_2}{\partial y}\right)^4 z_3 - \frac{3}{4\delta_3^2}\left(\frac{\partial\alpha_2}{\partial y}\right)^4 z_3$$

$$u = [-s_4 z_4 - L_4C_4\tilde{X}_4 + \sum_{e=1}^{3}\frac{\partial\alpha_3}{\partial\tilde{x}_e}\hat{x}_{e+1} + \sum_{e=1}^{3}\frac{\partial\alpha_3}{\partial \hat{x}_e} L_e c_4 \tilde{x}_4 + \frac{\partial\alpha_3}{\partial y} c_4 f_4(\tilde{x}_4) + \frac{1}{2}\left(\frac{\partial 2_{\alpha_3}}{\partial y_2}\right)(c_4\emptyset_4(y))(c_4\emptyset_4(y))^T + \frac{1}{2}\left(\frac{\partial 2_{\alpha_3}}{\partial y_2}\right)(c_4\Psi_4(x_4))(c_4\Psi_4(x_4))^T - \frac{3}{4}\sigma_4^{4/3} z_4 - \frac{1}{4\sigma_3^4} z_4 - \frac{3}{4}\eta_4^{4/3}\left(\frac{\partial\alpha_3}{\partial y}\right)^{4/3} z_4 - \frac{3}{4\xi_4^2}\left(\frac{\partial\alpha_3}{\partial y}\right)^4 z_4 - \frac{3}{4\delta_4^2}\left(\frac{\partial\alpha_3}{\partial y}\right)^4 z_4\,]$$

7. Conclusions

1. A robust and optimal control law which guarantees global asymptotic stability in probability has been designed. The output feedback (observer-based) backstepping control law which guarantees global asymptotic stability in probability has been also discussed and proved supported by some theoretical justifications and an illustration.
2. A large class of nonlinear stochastic dynamic control systems in the presence of Brownian motion have been discussed and its controllability and hence stablizability are also been proved depending on the presented theorems.
3. The relation between inverse optimality and optimality as well as robust control is discussed supported by some theoretical results.
4. On depending on this work, the computational algorithm is easier and hence makes this work applicable and can be used to design some real life systems later on.
5. The given example are added to the research to be easy to follow the direction of theorems and how it can be applied to more complex dynamic systems in future.
6. We have been faced by a large of difficulties to follow this direction, like backstepping of stochastic dynamic systems, inverse optimality,

control Lyapunov functions, stochastic nonlinear output-feedback systems, stochastic stabilization, etc. So we recommend that any person who is interested in this direction should be familiar with these facts.

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