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Idempotent Reflexive rings whose every simple singular right module are YJ-injective

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ABSTRACT

In this paper we study rightidempotent reflexive ring whose simple singular right R-module isYJ-injective, we prove that this type of ring is right weakly $\pi-$ regular ring, we show that if R is N duo or NCI ring and R is right idempotent reflexive ring whose every simple singular right R-module is YJ-injective then R is reduced weakly regular ring

الحلقات المنعكسة المتحايدة التي فيها كل مقاس بسيط منفرد غامر من النمط YJ

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الملخص

في هذا البحث درست الحلقات المنعكسة المتحايدة اليمنى التي كل مقاس بسيط منفرد ايمن عليها غامر من النمط YJ ، لقد تم برهان ان هذا النوع من الحلقات تكون منتظمة ضعيفة يمنى من النمط π ، كذلك تم برهان اذا كانت R هي حلقة N duo او حلقة NCI ومنعكسة متحايدة يمنى التي كل مقاس بسيط ايمن عليها هو غامر من النمط -YJفأن R هي حلقة مختزلة ومنتظمة ضعيفة.

1- *Introduction*

Throughout in this paper R is associative ring with identity and all modules are unitary. For a subset X of R, the left(right) annihilator of X in R is denoted by $l(X)(r(X))$.*If* $X=\{a\}$, we usually abbreviate it to $l(a)(r(a))$. We write $J(R)$, $N(R)$, $N^*(R)$, $P(R)$, $Z(R)$ for the Jacobson radical, the set of nilpotent elements,the nil radical (that means the sum of all nil ideals), prime radical (that means the intersection of all prime ideals) and left singular ideal respectively.A ring R is called NI if $N^*(R)=N(R)$ [6]. A ring R is called 2-primal if $N(R)=P(R)$**[1]**. Let I be a right (left) ideal of R,then R/I is a right (left) N-flat if and only if for each $a\in I$, there exists $b\in I$ and positive integer n such that $a^n\neq 0$ and $a^n=ba^n(a^n=a^nb)$[8]. A ring R is said to be right **weakl□** $\pi-$

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regular ring if for each $a \in R$, there exists $b, c \in R$ and positive integer n such that

A ring R is said to be right weakly regular ring for each $a \in R$, there exists

$b, c \in R$ such that Call a ring R, S-weakly regular ring if

$a \in aRa^2R,$ for all Call a ring R NCI if N(R) is contain a non-zero ideal of R whenever $N(R) \neq 0.$ Clearly, NI ring is NCI [3]. A ring R is said to be N duo if aR=Ra, for all $a \in N(R)$[12].

A ring R is said to be reflexive if $aRb = 0$ implies $bRa = 0$ for A ring R is said to be right idempotent reflexive if $eRa = 0$ implies $aRe = 0,$ for

A right R-module M is called YJ-injective if for any $0 \neq a \in R,$ there exists a positive integer n such that $a^n \neq 0$ and every right R-homomorphism of a^nR into M extends to one of R into M [11].YJ-injectivity is also called GP-injectivity, by several authors [5].

2. *Some properties of idempotent reflexive ring whose simple singular module is YJ-injective.*

In this section we give some properties of right idempotent reflexive ring whose every simple singular right R-module is YJ-injective then R is semiprime, J(R)=0, and right weakly $\pi -$ regular ring.

Every semiprime ring is reflexive, and every reflexive ring is right idempotent reflexive, but Kim in [4], gives example of right and left idempotent reflexive ring but notsemiprimenor reflexive ring. The next theorem gives condition makes the idempotent reflexive ring implies to reflexive and semiprime ring.

Theorem 2.1

Let R be a ring whose every simple singular rightR-module is YJ-injective.Then the following conditions are equivalent:

1- *R is semiprime.*
2- *R is reflexive ring.*

3- R is right idempotent reflexive.

Proof:

$1 \to 2 \to 3,$ it is clear.

$3 \to 1$

We shall show there is no nilpotent ideal in R, if not, suppose there exists $0 \neq a \in R$ with $(aR)^2 = 0$, $aRaR = 0$, that is mean $RaR \subseteq r(a)$, there exists a maximal right ideal M of R containing $r(a)$. If M is not essential, then $M = r(e)$, where $0 \neq e = e^2 \in R$,since $RaR \subseteq r(a) \subseteq M = r(e)$, then $eRaR = 0$, so $eRa = 0,$ since R is right idempotent reflexive ring, $aRe = 0, ae = 0, e \in r(a) \subseteq M = r(e), e^2 = 0,$ which is a contradiction. Therefore M

is an essential right ideal of R,we get that . Hence, there exists appositive integer n=1 such that $a \neq 0$ and any R-homomorphism of a R into R/M extends to one of R into R/M, we define $f : aR \to \frac{R}{\square}$ such that $f(ar) = r + M$ where $r \in R$. It is clear that f is well define right R-homomorphism, since R/M is YJ-injective, there exists $b + M \in \frac{R}{\square}$ such that $1 + M = f(a) = (b + M)(a + M) = ba + M, 1 + M = ba + M, 1 - ba \in M,$ since

, we get that $1 \in M,$ which is also contradiction. Therefore $a = 0$. This is shows

that R is

Theorem 2.2

Let R be a right idempotent reflexive ring whose every simple singular rightR-module is YJ-injective. Then R is left nonsingular.

Proof:

Let $a \in Z(R),$ such that $a^2 = 0,$ then either or not, if not, there exists a maximal right ideal M of R containing $RaR + r(a).$ If M is not essential, then $M = r(e),$ where $0 \neq e = e^2 \in R$, since $RaR \subseteq M = r(e)$, then $eRaR = 0$, so $eRa = 0.$ By the same method as in the proof of Theorem 2.1, we get that $RaR + r(a) = R,$ in particular there exists $y, z \in R$ and $v \in r(a)$ such that $yaz + v = 1,$ $ayaz + av = a, ayaz = a,$ since $a \in Z(R),$ then $yaz \in Z(R),$ that is mean $l(yaz)$ is essential left ideal of R, so $l(yaz)\cap I \neq 0$ for all left ideal I of R, in

special case we take $I = Ra,$ then $l(yaz)\cap Ra \neq 0,$ so there exists for some $d \in R, 0 = xyaz = dayaz = da = x,$ since $ayaz = a$, we get that $x = 0,$ therefore $l(yaz)\cap Ra = 0,$ but $l(yaz)$ is essential left ideal of R, it follows that $Ra = 0,$ that is mean $a = 0.$ Therefore R is left non singular.

Theorem 2.3

Let*R be a right idempotent reflexive ring whose every simple singular rightR-*

module is YJ-injective. Then R/ J(R) is N-flat left R-module.

Proof:

We shall to show that R/ J(R) is N-flat left R-module, if not, suppose there

exists $0 \neq a \in J(R),$ either or not, if not, there exists a maximal right ideal M of R containing $Ra^nR + r(a^n),$ If M is not essential, then $M = r(e),$ where $0 \neq e = e^2 \in R$, since $Ra^nR \subseteq M = r(e)$,then $eRa^nR = 0$, so $eRa^n = 0,$ since R is right idempotent reflexive ring, $a^nRe = 0, a^ne = 0, e \in r(a^n) \subseteq M = r(e), e^2 = 0,$ which is a contradiction. Therefore M is an essential right ideal of R, we get that R/M is YJ-injective, there exists a positive integer n and $a^n \neq 0$ such that any R-homomorphism of a^nR into R/M extends to one of R into R/M,let $f: a^nR \to \frac{R}{\square}$ such that $f(a^nr) = r + M,$ where $r \in R$, f is well defineright R-homomorphism, since R/M is YJ-injective, there exists $b + M \in \frac{R}{\square}$ such that 1+M=f(a^n)=(b+M)(a^n+M)=ba^n+M, 1+M=a^nb+M, $1 - ba^n \in M,$ since $ba^n \in Ra^nR \subseteq M,$ we get that $1 \in M,$ which is a contradiction. That is mean $Ra^nR + r(a^n) = R,$ in particular there exists $y, z \in R$ and $v \in r(a^n)$ such that

$ya^nz + v = 1,$ $a^nya^nz + a^nv = a^n, a^nya^nz = a^n,$ set , since $a \in J(R)$, so , $a^n d = a^n$, with $a^n \neq 0$. Therefore J(R)is N-flat left R-module.

Lemma 2.4

Let R be a right idempotent reflexive ring whose every simple singular rightR-module is YJ-injective. Then J(R)=0.

Proof: see Kim [4].

Corollary 2.5

Let R be a right idempotent reflexive ring whose every simple singular rightR-module is YJ-injective. Then $N^*(R) = 0$.

Proof:

Since $N^*(R)$ is the large nil ideal of R, It is clearly that J(R) contains every nil ideal, so $N^*(R) \subseteq J(R)$, but $J(R) = 0$, by Lemma 2.4. This is shows that $N^*(R) = 0$.

Theorem 2.6

Let R be a right idempotent reflexive ring whose every simple singular rightR-module is YJ-injective. Then R is right weakly $\pi - regular$ *ring.*

Proof:

Let $a \in R$, then either or not, if not, there exists a maximal right ideal M of R containing $Ra^nR + r(a^n)$. If M is not essential, then $M = r(e)$, where $0 \neq e = e^2 \in R$, since $Ra^nR \subseteq M = r(e)$, then $eRa^nR = 0$, so $eRa^n = 0$, since R is right idempotent reflexive ring, $a^nRe = 0, a^ne = 0, e \in r(a^n) \subseteq M = r(e), e^2 = 0$, which is a contradiction. Therefore M is an essential right ideal of R, we get that R/M is YJ-injective, there exists a positive integer n and $a^n \neq 0$ such that any R-homomorphism of a^nR into R/M extends to one of R into R/M, let $f: a^nR \to \frac{R}{\square}$ such that $f(a^nr) = r + M$, where $r \in R$, as we show in Theorem 2.2, f is well define right R-homomorphism, since R/M is YJ-injective, there exists $b + M \in \frac{R}{\square}$ such that 1+M=f(a^n)=(b+M)(a^n+M)=ba^n+M,
1+M=ba^n+M, $1 - ba^n \in M$, since $ba^n \in Ra^nR \subseteq M$, we get that $1 \in M$, which is a contradiction. That is mean $Ra^nR + r(a^n) = R$, in particular there exists $y, z \in R$ and $v \in r(a^n)$ such that $ya^nz + v = 1$, $a^nya^nz + a^nv = a^n, a^nya^nz = a^n$, for all $a \in R$. Therefore R is *weakly* $\pi - regular$ *ring.*

3.*Idempotent reflexive ring whose every simple singular module is YJ-injective and it relation with other rings.*

In this section we give different condition to the Idempotent reflexive ring whose every simple singular module is YJ-injective to get the reduced, weakly regular, S-weakly regular ring.

Remark (1)

N duo ring whose every simple singular right R-module is YJ-injective need not to be reduced ring as we show in this example.

Let $R = \begin{bmatrix} Z_2 & Z_2 \\ 0 & Z_2 \end{bmatrix}$, where Z_2 is the ring of integer modulo 2. Then the only nilpotent element in R is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Z_2 & Z_2 \\ 0 & Z_2 \end{bmatrix} = \begin{bmatrix} 0 & Z_2 \\ 0 & 0 \end{bmatrix}$$
$$R\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} Z_2 & Z_2 \\ 0 & Z_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & Z_2 \\ 0 & 0 \end{bmatrix}$$

That is mean R is N duo ring, and clearly every simple singular right R-module is YJ-injective. But R with this condition need not be reduced, therefore R need another condition to get reduced ring. The following theorem with condition that the ring it well be idempotent reflexive ring make the ring is reduced.

Theorem 3.1

Let R be N duo and right idempotent reflexive ring whose every simple singular rightR-module is YJ-injective. Then R is reduced ring.

Proof:

We shall to show that N(R) =0, if not, there exists $0 \neq a \in N(R)$ with $a^2 = 0$, if $aR + r(a) \neq R$, there exists a maximal right ideal M of R containing $aR + r(a)$, then by the same method as in the proof of Theorem 2.1, M must be

essential right ideal of R, we have . Hence,there exists appositive integer n=1 such that $a \neq 0$ and any R-homomorphism of a R into R/M

extends to one of R into R/M. Let $f: aR \rightarrow \frac{R}{\square}$ such that $f(ar) = r + M$ where , f is well define right R-homomorphism, since R/M is $YJ-injective$ there exists $b + M \in \frac{R}{\square}$ such that $1 + M = f(a) = (b + M)(a + M) = ba + M, 1 + M = b + M, 1 - ba \in M,$ since R is

N duo ring and $a \in N(R)$, we get aR=Ra, , so $1 \in M$, which is a contradiction. Therefore $aR + r(a) = R$. In particular there exists $z \in R$ and $v \in r(a)$ such that $az + v = 1, a^2z + av = 0 = a$, for all $a \in N(R)$, This is shows that R is reduced ring.

Remark (2)

NCI ring whose every simple singular right R-module is YJ-injective need not to be reduced ring, the ring in the example of Remark(1) every simple singular right R-module is YJ-injective, also the ring is NI ring, where $N(R) = \left\{\begin{bmatrix} 0 & Z_2 \\ 0 & 0 \end{bmatrix}\right\}$, every NI ring is NCI. The ring R is satisfies two condition and it is not reduced, the following theorem with the condition right idempotent reflexive ring make the ring reduced.

Theorem 3.2

Let R be NCI ring and right idempotent reflexive ring whose every simple singular rightR-module is YJ-injective. Then R is reduced ring.

Proof:

We shall to show that N(R) =0, if not, $0 \neq N(R)$, since R is NCI ring, so N(R) is contain a non-zero ideal I, but I is nil ideal, It is clearly that J(R) contain every nil ideal,since R is right idempotent reflexive, so form Proposition 2.4, we get that $J(R) = 0$, so $I \subseteq J(R) = 0$, I=0, that is mean must N(R) be in ideal, similarly $N(R) \subseteq J(R) = 0$, N(R)=0.This is shows that R is a reduced ring.

Theorem 3.3

Let R be a right idempotent reflexive ring whose every simple singular rightR-module is YJ-injective. Then the following conditions are equivalent:

1- R is reduced ring.
2- R is N duo ring.
3- R is 2-priaml ring.
4- R is NI ring.
5- R is NCI ring.

Proof:

$1 \rightarrow 2$, it is clear, $2 \rightarrow 1$, by Theorem 3.2

$1 \rightarrow 3 \rightarrow 4 \rightarrow 5$, it is clear, $5 \rightarrow 1$, by Theorem 3.1

Theorem3.4

Let R be a right idempotent reflexive ring whose every simple singular rightR-module is YJ-injective. Then R is weakly regular ring.If R satisfies one of the following condition.

1- R is reduced ring.
2- R is N duo ring.
3- R is 2-priaml ring.
4- R is NI ring.
5- R is NCI ring.

Proof:

We shall prove R is weakly regular when R is reduced, and prove of the other condition clearly from Theorem 3.3.

We shall to show that $RdR + r(d) = R,$ for all $d \in R,$ if not, there exists $0 \neq a \in R,$ such that $RaR + r(a) \neq R,$ there exists a maximal right ideal M of R containing $RaR + r(a),$ since R is reduced, then M is essential right ideal of R. we

have . Hence,there exists appositive integer n such that $a^n \neq 0$ and any R-homomorphism of a^n R into R/M extends to one of R into R/M. Let $f: a^n R \to \frac{R}{\square}$ such that $f(a^n r) = r + M$ where $r \in R.$ f is well define right R-homomorphism, since R/M is YJ-injective there exists $b + M \in \frac{R}{\square}$ such that $1 + M = f(a^n) = (b + M)(a^n + M) = ba^n + M, 1 - ba^n \in M,$ $ba^n \in RaR \subseteq M, 1 \in M,$

which is a contradiction. Therefore $RaR + r$ *(a)=R, for all a∈R.* In particular there

exists y,$z \in R$ and $v \in r(a)$ such that y$az + v = 1,$ $ayaz + av = ayaz = a.$ This is show that R is rightweakly regular ring. $a(1 - yaz) = 0, 1 - yaz \in r(a) = l(a),$ since R is reduced ring, $(1 - yaz)a = 0,$ $a = yaza,$ so R is also left weakly regular ring. So R is weakly regular ring.

Corollary 3.5

Let R be a right idempotent reflexive ring whose every simple singular rightR-module is YJ-injective. Then R is S-weakly regular ring.If R satisfies one of the following condition.

1- R is reduced ring.
2- R is N duo ring.
3- R is 2-priaml ring.
4- R is NI ring.
5- R is NCI ring.

Proof:

From Theorem 3.4, R is reduced ring if it is satisfies any above condition. Since R is reduced weakly regular ring, then R is S-weakly regular ring.

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