Iraqi Journal of Statistical Science (20) 2011
The Fourth Scientific Conference of the College of Computer Science & Mathematics
pp [83-92]

Rate of Convergence for a New Family of Summation-Integral Beta Operators

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ABSTRACT

In the present paper, we define a new family of summation-integral Beta operators to approximate a class of unbounded continuous functions of polynomial growth $O(t^r)$, for some $r > 0$ and then we estimate the rate of convergence for this family for functions have derivatives of bounded variation.

الخلاصة

عرفنا في هذا البحث، عائلة جديدة من مؤثرات بيتا ذات النمط مجموع تكامل لتقريب دوال مستمرة غير مقيدة وذات نمو متعددة حدود $O(t^r)$ لبعض $r > 0$ وبعد ذلك قدرنا نسبة التقارب لهذه العائلة ولدوال لها مشتقات محدودة التغاير.

Keywords: Linear positive operators, Bounded variation, Rate of convergence.

2000 Mathematics Subject Classification 41A25, 41A36

1. Introduction

In the last 30 years, the rate of convergence of certain operators acting on functions of bounded variation has been investigated. Important papers on this topic are appeared. Zeng and Chen [13, 2000] estimated the rate of convergence of Durrmeyer-type operators for functions of bounded variation on the interval [0,1], Vijay Gupta [6,2002] is one of the researchers who were interested in this topic, where he estimated the rate of convergence of a new sequence of linear positive operators $B_{n,\alpha}(f,x)$, which is the Bezier variant of the well-known Baskakov Beta operators, Gupta et al [11, 2003] estimate the rate of convergence of the recently introduced generalized sequence of linear positive operators $G_{n,c}(f,x)$ with derivatives of bounded variation, Niraj kumar [5, 2004] gives the rate of convergence for the linear combinations of the generalized Durrmeyer type operators which includes the well-known Szasz-Durrmeyer operators and Baskakov-Durrmeyer operators as special cases, Ulrich et al [9, 2005] study the approximation properties of beta operators of second kind, and they obtain the rate of convergence of these operators for

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Received:1/10 /2011 ____________________Accepted: 21 /12 / 2011

absolutely continuous functions having a derivative equivalent to a function of bounded variation, Jyoti Sinha and V. K. Singh [4, 2006] investigated this problem for a mixed sequence of summation integral type operators $S_n(f,x)$ for functions having derivatives of bounded variation. Recently, Vijay Gupta and Harun Karsli [7,2007] extended this problem for Bézier variant of Durrmeyer type Meyer–König and Zeller operators for functions with derivatives of bounded variation defined on [0,1], P. N. Agrawal and Vijay Gupta [8,2007] study a certain integral modification of the well-known Baskakov operators with the weight function of Beta basis function and establish the rate of convergence for these operators for functions having derivatives of bounded variation. Harun Karsli and Vijay Gupta [3,2008] study the behavior of nonlinear integral operators and estimated the rate of convergence at a point x , which has a discontinuity of the first kind as $\chi \to \chi_0$, Harun Karsli[2,2008] estimates the rate of pointwise convergence of the Chlodowsky operators C_n for functions, defined on the interval $[0,b_n]$ with derivatives of bounded variation, where $\lim_{n\to\infty} b_n = \infty$. In [12,2009] some direct local and global approximation theorems were given for the q-Bernstein-Durrmeyer operators. Ali Aral and Vijay Gupta [1,2010] deal with Durrmeyer type generalization of q-Baskakov type operators using the concept of q-integral, which introduces a new sequence of positive q-integral operators and estimates for the rate of convergence and weighted approximation properties are also obtained. Vijay Gupta and Taekyun Kimb [9,2011] investigated this problem for the q-analogue of the modified Beta operators.

Now, we study new family of summation-integral Beta operators (1.1) and estimate the rate of convergence for functions having derivatives of bounded variation. For $p \in \mathrm{N}^0 = \{0,1,2,3,....\}$, we defined a family of summation-integral type Beta operators as:

$$B_n(f;x) = \frac{1}{n+1}\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} b_{n,k+p}(t)f(t)dt \qquad (1.1)$$

$$= \int_0^{\infty} W_n(x,t)f(t)dt$$

Where $b_{n,k}(x) = \frac{(n+k)!}{k!(n-1)!}x^k(1+x)^{-(n+k+1)}$

and $W_n(x,t) = \sum_{k=0}^{\infty} b_{n,k}(x)b_{n,k+p}(t) + (1+x)^{-n}\delta(t)$

$\delta(t)$ being the Dirac delta function.

We denote $\beta_n(x;t) = \int_0^t W_n(x,s)ds$ then, we have: $\beta_n(x;\infty) = \int_0^\infty W_n(x,s)\,ds = 1$

In our work we suppose that $DB_r(0,\infty)\,; r \ge 0$ means the class of absolutely continuous functions f defined on the interval $(0,\infty)$ such that:

i) $f(t) = O(t^r), t \to \infty$
ii) having a derivative f' on the interval $(0,\infty)$ coinciding a.e. with a function which is of bounded variation on every finite subinterval of $(0,\infty)$. It can be observed that all functions $f \in DB_r(0,\infty)$ possess for each real value $c > 0$ the equation:

$$f(x) = f(c) + \int_c^x \psi(t)dt \;\; , x \ge c$$

Our main result is stated as follows:

Theorem: Let $f \in DB_r(0,\infty)$, $r \in \mathrm{N}$ and $x \in (0,\infty)$. Then for n sufficiently large, we have

$$\left| B_n(f,x) - f(x) \right| \le M 2^\gamma O(n^{-\frac{\gamma}{2}}) \sqrt{\frac{(2n^2+8n)x^2+(2n^2+8n+6p)x+(p^2+3p+2)n}{(n+1)(n-1)(n-2)}}$$

$$+ \left| f(x) \right| \frac{(2n^2+8n)x^2+(2n^2+8n+6p)x+(p^2+3p+2)n}{(n+1)(n-1)(n-2)x} +$$

$$\left| f'(x^+) \right| \sqrt{\frac{(2n^2+8n)x^2+(2n^2+8n+6p)x+(p^2+3p+2)n}{(n+1)(n-1)(n-2)}} +$$

$$\frac{(2n^2+8n)x^2+(2n^2+8n+6p)x+(p^2+3p+2)n}{(n+1)(n-1)(n-2)x} \left| f(2x) - f(x) - x f'(x^+) \right| +$$

$$\frac{(2n^2+8n)x^2+(2n^2+8n+6p)x+(p^2+3p+2)n}{(n+1)(n-1)(n-2)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x}^{x+\frac{x}{k}} (f_x) + \frac{x}{\sqrt{n}} \bigvee_{x}^{x+\frac{x}{\sqrt{n}}} (f_x)$$

$$+ \frac{(2n^2+8n)x^2+(2n^2+8n+6p)x+(p^2+3p+2)n}{(n+1)(n-1)(n-2)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{u}}^{x} (f_x) + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} (f_x)$$

where the auxiliary function f_x is given by

$$f_x(t)=\begin{cases} f(t)-f(x^-) & 0\le t<x \\ 0 & t=x \\ f(t)-f(x^+) & x<t<\infty \end{cases}$$

$\bigvee_a^b f(x)$ denotes the total variation of f_x on [*a, b*]

(If f is differentiable and its derivative is integrable, its total variation is the vertical component of the arc-length of its graph, that is to say: $\bigvee_a^b f(x)=\int_a^b |f'(x)|\,dx$).

2. Basic Results

We shall use the following lemmas to prove our main theorem.

Lemma: 2.1

Let the function $T_{n,m}(x)$, $m\in N\cup\{0\}$ be defined as:

$$T_{n,m}(x)=B_n((t-x)^m;x)=\frac{1}{n+1}\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} b_{n,k+p}(t)(t-x)^m\,dt$$

Then $T_{n,0}(x)=\dfrac{n}{n+1}$, $T_{n,1}(x)=\dfrac{2nx+n(p+1)}{(n+1)(n-1)}$

$$T_{n,2}(x)=\frac{(2n^2+8n)x^2+(2n^2+8n+6p)x+(p^2+3p+2)n}{(n+1)(n-1)(n-2)}$$

$T_{n,m}(x)$ is polynomial in x of degree *m*, $T_{n,m}(x)$ is polynomial in n^{-1} of degree *m ; m>0* also $T_{n,0}(x)$, $T_{n,1}(x)$ are polynomial in n^{-1} of degree 1 and $T_{n,2}(x)$ are polynomial in n^{-1} of degree *2*, and there holds the recurrence relation

$$x(1+x)T'_{n,m}(x)+2mx(1+x)T_{n,m-1}(x)+(m-n+1)T_{n,m+1}(x)+((p+m+1)+2x(m+1))T_{n,m}(x)=0$$

Consequently, for each $x\in[0,\infty)$ we get $T_{n,m}(x)=O(n^{-[\frac{m+1}{2}]})$

Proof: By using the direct computation, the values of $T_{n,0}(x)$, $T_{n,1}(x)$ and $T_{n,2}(x)$ are omitted

Now we have:

$$T_{n,m}(x)=\frac{1}{n+1}\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} b_{n,k+p}(t)(t-x)^m\,dt$$

$$T'_{n,m}(x)=\frac{1}{n+1}[\sum_{k=0}^{\infty} b'_{n,k}(x)\int_0^{\infty} b_{n,k+p}(t)(t-x)^m\,dt+m\sum_{k=0}^{\infty} b'_{n,k}(x)\int_0^{\infty} b_{n,k+p}(t)(t-x)^{m-1}.-1\,dt$$

$$x(x+1)(T'_{n,m}(x)+T_{n,m-1}(x))=\frac{1}{n+1}\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty}(k+p-nt-t-p+nt+t-nx-x)(t-x)^m b_{n,k+p}(t)dt$$

Using integration by parts we get:

$$x(1+x)T'_{n,m}(x)+2mx(1+x)T_{n,m-1}(x)+(m-n+1)T_{n,m+1}(x)+((p+m+1)+2x(m+1))T_{n,m}(x)=0$$

From above our lemmas are omitted.

Remark 2.1. From Lemma 2.1, using Cauchy-Schwarz inequality, it follows that

$$B_n(|t-x|,x)\le[B_n(1;x)]^{1/2}[B_n((t-x)^2;x)]^{1/2}$$
$$\le\sqrt{\frac{(2n^3+8n^2)x^2+(2n^3+8n^2+6pn)x+(p^2+3p+2)n^2}{(n+1)^2(n-1)(n-2)}}$$

Lemma 2.2. Let $x\in(0,\infty)$ and $W_n(x,t)$ be the kernel of defined in (1.1). Then for n sufficiently large, we have

$$(i)\ \beta_n(x,y)=\int_0^y W_n(x,t)\,dt$$
$$\le\frac{(2n^2+8n)x^2+(2n^2+8n+6p)x+(p^2+3p+2)n}{(n+1)(n-1)(n-2)(x-y)^2}\quad ,0\le y<x$$

$$(ii)\ 1-\beta_n(x,z)=\int_z^{\infty} W_n(x,t)\,dt$$
$$\le\frac{(2n^2+8n)x^2+(2n^2+8n+6p)x+(p^2+3p+2)n}{(n+1)(n-1)(n-2)(z-x)^2}\quad ,x<z<\infty$$

Proof : First we prove (i), by using Lemma 2.1, we have

$$\int_0^y W_n(x,t)\,dt \le \int_0^y \frac{(x-t)^2}{(x-y)^2} W_n(x,t)dt \le (x-y)^{-2} T_{n,2}$$

$$\le \frac{(2n^2+8n)x^2+(2n^2+8n+6p)x+(p^2+3p+2)n}{(n+1)(n-1)(n-2)(x-y)^2} \quad ,0 \le y < x$$

The proof of (ii) is similar, we omit the details.

3. Proof of Theorem

Proof: By the application of mean value theorem of integral, we have

$$B_n(f,x) - f(x) = \int_0^\infty W_n(x,t)(f(t)-f(x))dt = \int_0^\infty \int_x^t W_n(x,t)(f'(u)du)dt$$

(3.1)

Using the identity [8]

$$f'(u) = \frac{1}{2}[f'(x^+)+f'(x^-)] + f_x(u) + \frac{1}{2}[f'(x^+)-f'(x^-)]\operatorname{sgn}(u-x) + [f'(x) - \frac{1}{2}[f'(x^+)+ f'(x^-)]\chi_x(u)$$

It is easily verified that if we substitute the above value $f'(u)$ in (3.1), the last term of the identity vanishes. Also

$$\int_0^\infty (\int_x^t \frac{1}{2}[f'(x^+)-f'(x^-)]\operatorname{sgn}(u-x)du)W_n(x,t)dt = \frac{1}{2}[f'(x^+)-f'(x^-)]B_n(|t-x|,x)$$

and $$\int_0^\infty (\int_x^t \frac{1}{2}[f'(x^+)-f'(x^-)]du)W_n(x,t)dt = \frac{1}{2}[f'(x^+)-f'(x^-)]M_n((t-x),x)$$

Thus in view of the above values, Lemma (2.1) and Remark (2.1) equation (3.1)are reduced to

$$|B_n(f,x)-f(x)| \le \left| \int_x^{\infty} \left(\int_x^t f_x(u)\,du \right) W_n(x,t)dt - \int_0^x \left(\int_x^t f_x(u)\,du \right) W_n(x,t)dt \right| + \frac{1}{2}[f'(x^+)$$
$$- f'(x^-)] \sqrt{\frac{(2n^3+8n^2)x^2+(2n^3+8n^2+6pn)x+(p^2+3p+2)n^2}{(n+1)^2(n-1)(n-2)}} + \frac{1}{2}[f'(x^+)$$
$$+ f'(x^-)]\frac{2nx+n(p+1)}{(n+1)(n-1)})$$

$$\le | E_n(f,x)+F_n(f,x)| + \frac{1}{2}[f'(x^+)-f'(x^-)]$$
$$\sqrt{\frac{(2n^3+8n^2)x^2+(2n^3+8n^2+6pn)x+(p^2+3p+2)n^2}{(n+1)^2(n-1)(n-2)}} + \frac{1}{2}[f'(x^+)$$
$$+ f'(x^-)]\frac{2nx+n(p+1)}{(n+1)(n-1)})$$

(3.2)

In order to complete the proof of theorem, it is sufficient to estimate the terms $E_n(f,x)$ and $F_n(f,x)$. Appling integration by parts, using Lemma (2.2) and taking $y = x - x/\sqrt{n}$, we have:

$$|F_n(f,x)| = \left| \int_0^x \left(\int_x^t f_x(u)\,du \right) \beta_n(x,t)\,dt \right|,$$

$$\left| \int_0^x \beta_n(x,t) f_x(t)dt \right| \le (\int_0^y + \int_y^x)|f_x(t)||\beta_n(x,t)|dt$$

$$\le \frac{(2n^2+8n)x^2+(2n^2+8n+6p)x+(p^2+3p+2)n}{(n+1)(n-1)(n-2)} \int_0^y \bigvee_t^x (f_x)\frac{1}{(x-t)^2}dt + \int_y^x \bigvee_t^x (f_x)\,dt$$

$$\le \frac{(2n^2+8n)x^2+(2n^2+8n+6p)x+(p^2+3p+2)n}{(n+1)(n-1)(n-2)} \int_0^y \bigvee_t^x (f_x)\frac{1}{(x-t)^2}dt \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} (f_x)$$

Let $u=\dfrac{x}{x-t}$ .Then we have

$$\frac{(2n^2+8n)x^2+(2n^2+8n+6p)x+(p^2+3p+2)n}{(n+1)(n-1)(n-2)}\int_0^y \bigvee_t^x (f_x)\frac{1}{(x-t)^2}dt=$$

$$\frac{(2n^2+8n)x^2+(2n^2+8n+6p)x+(p^2+3p+2)n}{(n+1)(n-1)(n-2)}\int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^{x} (f_x)\,du$$

$$\leq\frac{(2n^2+8n)x^2+(2n^2+8n+6p)x+(p^2+3p+2)n}{(n+1)(n-1)(n-2)}\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{u}}^{x} (f_x)$$

Thus

$$|F_n(f,x)|\leq\frac{(2n^2+8n)x^2+(2n^2+8n+6p)x+(p^2+3p+2)n}{(n+1)(n-1)(n-2)}\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{u}}^{x} (f_x)+\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} (f_x)$$

(3.3)

On the other hand, we have

$$|E_n(f,x)|=\left|\int_x^\infty\left(\int_x^t f_x(u)\,du\right)W_n(x,t)dt\right|$$

$$=\left|\int_{2x}^\infty\left(\int_x^t f_x(u)\,du\right)W_n(x,t)dt+\int_x^{2x}\left(\int_x^t f_x(u)\,du\right)(1-\beta_n(x,t))\,dt\right|$$

$$\leq\left|\int_{2x}^\infty (f(t)-f(x))W_n(x,t)dt\right|+|f'(x^+)|\left|\int_{2x}^\infty (t-x)W_n(x,t)dt\right|+\left|\int_x^{2x} f_x(u)du\right||1-\beta_n(x,2x)|$$
$$+\int_x^{2x}|f_x(t)||1-\beta_n(x,t)|dt$$

$$\le \frac{M}{x}\int_{2x}^{\infty} W_n(x,t)t^{\gamma}\,|t-x|\,dt + \frac{|f(x)|}{x^2}\int_{2x}^{\infty} W_n(x,t)(t-x)^2\,dt + \left|f'(x^+)\right|\left|\int_{2x}^{\infty} W_n(x,t)|t-x|\,dt\right| +$$

$$+\frac{(2n^2+8n)x^2+(2n^2+8n+6p)x+(p^2+3p+2)n}{(n+1)(n-1)(n-2)x}\left|f(2x)-f(x)-xf'(x^+)\right| + \qquad (3.4)$$

$$\frac{(2n^2+8n)x^2+(2n^2+8n+6p)x+(p^2+3p+2)n}{(n+1)(n-1)(n-2)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{x}{k}}(f_x)+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}(f_x)$$

For estimation of the first two terms in the right hand side of (3.4), we proceed as follows:

Applying Holder's inequality, Remark 2.1 and Lemma 2.1

$$\frac{M}{x}\int_{2x}^{\infty} W_n(x,t)t^{\gamma}\,|t-x|\,dt + \frac{|f(x)|}{x^2}\int_{2x}^{\infty} W_n(x,t)(t-x)^2\,dt \le$$

$$\frac{M}{x}\left(\int_{2x}^{\infty} W_n(x,t)t^{2\gamma}\,dt\right)^{\frac{1}{2}} + \left(\int_{0}^{\infty} W_n(x,t)(t-x)^2\,dt\right)^{\frac{1}{2}} + \frac{|f(x)|}{x^2}\int_{2x}^{\infty} W_n(x,t)(t-x)^2\,dt$$

$$\le M\,2^{\gamma}\,O(n^{-\frac{\gamma}{2}})\sqrt{\frac{(2n^2+8n)x^2+(2n^2+8n+6p)x+(p^2+3p+2)n}{(n+1)(n-1)(n-2)}}$$

$$+\left|f(x)\right|\frac{(2n^2+8n)x^2+(2n^2+8n+6p)x+(p^2+3p+2)n}{(n+1)(n-1)(n-2)x} \qquad (3.5)$$

Also, by Remark 2.1 the third term of the right side of (3.4) is given by

$$\left|f'(x^+)\right|\left|\int_{2x}^{\infty} W_n(x,t)|t-x|\,dt\right| \le \left|f'(x^+)\right|\left|\int_{0}^{\infty} W_n(x,t)|t-x|\,dt\right.$$

$$\le \left|f'(x^+)\right|\sqrt{\frac{(2n^2+8n)x^2+(2n^2+8n+6p)x+(p^2+3p+2)n}{(n+1)(n-1)(n-2)}}$$

Combining the estimates (3.2)-(3.4), we get the desired result. This completes the proof of the theorem.

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