

Numerical Solution of Coupled-BBM Systems of Boussinesq type by Implicit Finite Difference Method

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Abstract

In this paper we study a numerical solution of coupled BBM systems of Boussinesq type, which describes approximately the two ways propagation of surface wave in a uniform horizontal channel of length l filled in its undisturbed depth h . This paper is devoted to drive the matrix algebraic equation for the one-dimensional nonlinear BBM system which is obtaining from using the implicit finite difference method. The convergence analysis of the solution is proved. Numerical experiments are presented with a variety of initial conditions describing the generation and evolution of such waves, and their interactions.

الحل العددي لنظام BBM من نوع Boussinesq بطريقة الفروقات المنتهية الضمنية

الخلاصة

في هذا البحث درسنا حلا عدديا لنظام BBM من نوع Boussinesq الذي يصف تقريبا انتشار الموجة السطحية في قناة أفقية منتظمة وغير مضطربة بطول L وعمق h . هذا البحث مخصص لاشتقاق معادلة المصفوفة ذات البعد الواحد لنظام BBM غير الخطي باستخدام طريقة الفروقات المنتهية الضمنية. كما تم تحليليا إثبات تقارب الحل. اجريت التجارب العددية على أنواع مختلفة من الشروط لوصف توليد هذه الموجات وتطورها وتصادفها.

Keywords: Boussinesq system; solitary waves; finite difference method.

Mathematics Subject Classification: 65M06, 76B25, 76B15

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1. Introduction

There are many models for studying weakly nonlinear dispersive water waves in a channel or in the near shore zone. For one-way waves, namely when the wave motion occurs in one-direction, the well known KdV (Korteweg–de Vries) and BBM (Benjamin–Bona–Mahoney) equations are the most studied. For two-way waves, a four parameter class of model equations (which are called Boussinesq-type systems)

$$\begin{aligned} v_t + u_x + (vu)_x + au_{xxx} - bv_{xxt} &= 0 \\ u_t + v_x + uu_x + cv_{xxx} - du_{xxt} &= 0 \end{aligned} \quad (1)$$

was put forward by Bona, Chen and Saut [7], to describe approximately the two-dimensional propagation of surface waves in a uniform horizontal channel of length L filled with an irrotational, incompressible, inviscid fluid, which in its undisturbed state has depth h . The non-dimensional variables $v(x, t)$ and $u(x, t)$ represent, respectively, the deviation of the water surface from its undisturbed position and the horizontal velocity. It is shown in [7, 8] that a physically relevant system in (1) is linearly well posed and locally nonlinearly well posed in certain natural Sobolev spaces if the constants a, b, c, d satisfy

$$b, d \geq 0, a, c \leq 0$$

or $b, d \geq 0, a = c > 0$.

It is also shown in [6, 11] that the above systems have the capacity to capture the main characteristics of the flow in an ideal fluid. But when the damping effect is comparable with the effects of nonlinearity and/or dispersion, as occurs in the real laboratory-scale experiments and in the fields (see [11, 5, 2, 10]), it should be considered in order for the model and its numerical results to correspond in detail with the experiments.

The authors in [15] show that the initial-boundary value problem for the form system (1) posed on a bounded smooth plane domain with homogenous Dirichlet or Neumann or reflective (mixed) boundary conditions is locally well-posed in H^1 . Also this system is a good candidate for modeling long waves of small to moderate amplitude, with homogenous Dirichlet boundary conditions and it is well-posed and with nice properties, such as the presence of the operator $1 - \partial_{xx}^2$, the existence of Hamiltonian and well-developed numerical schemes (see [6,8]). The existing of line solitary waves, line cnoidal waves symmetric and asymmetric periodic wave pattern are proved in [4, 12, 13]. We refer the reader to paper [4] to show the existence of periodic travelling-wave solutions $(v(x, t), u(x, t))$ of the form:

$$\begin{aligned} v(x, t) &= v(x - wt) = \sum_{n=-\infty}^{\infty} v_n e^{i(n\pi/l)(x-wt)} \\ u(x, t) &= u(x - wt) = \sum_{n=-\infty}^{\infty} u_n e^{i(n\pi/l)(x-wt)} \end{aligned}$$

where l and w connote the half-period and the phase speed, respectively, and it is proved that for any $|w|^2 > \max\left\{1, \frac{ac}{bd}\right\}$ or $|w|^2 < \min\left\{1, -\frac{a+c}{b+d}, \frac{ac}{bd}\right\}$, and for any large enough l , there exists an infinitely smooth non-trivial periodic traveling-wave solutions to system (1).

In [8], the authors proved that the solution of the BBM-BBM type system exists and unique by defining the solution of the system as integral equation. Chen in [11],

established the existence of solitary waves for several Boussinesq systems, including the BBM-BBM type. In [3] the authors studied numerically the solution of the system of Bona-smith family. In this paper we will study the Boussinesq system of the type BBM-BBM corresponds to $a = c = 0, b = d = \frac{1}{6}$, i.e.

$$v_t + u_x + (vu)_x - \frac{1}{6}v_{xxt} = 0 \quad (2a)$$

$$u_t + v_x + uu_x - \frac{1}{6}u_{xxt} = 0 \quad (2b)$$

Where $v = v(x, t)$, $u = u(x, t)$ are functions from $(0, L) \times (0, \infty) \subset \mathfrak{R} \times \mathfrak{R}$ to \mathfrak{R} with Dirichlet conditions: $v(0, t) = v(L, t) = u(0, t) = u(L, t) = 0$, and the initial conditions are $v(x, 0) = f(x), u(x, 0) = g(x)$.

In section two of this paper we derive the matrix equation for the system (2) using implicit finite difference method. Also we proposed a theorem that the solution of system (2) can exist by finite difference method. In the last section we present numerically the generation, evolution and the interaction between two solitary waves of the system.

2. Derivation of the matrix equation using the finite Difference Method

The solutions are approximated numerically on the domain $0 \leq x \leq L$, with uniform mesh size taken:

$$x_i = i\Delta x \quad i = 0, 1, 2, \dots, N$$

$$t_j = j\Delta t \quad j = 0, 1, 2, \dots, M$$

The discretized solution of the equation (2a) is

$$\begin{aligned} & \frac{1}{k}(v^j(x_i) - v^{j-1}(x_i)) + \frac{1}{2h}(u^{j-1}(x_{i+1}) - u^{j-1}(x_{i-1})) + \frac{u^j(x_i)}{2h}(v^j(x_{i+1}) - v^j(x_{i-1})) + \\ & \frac{v^j(x_i)}{2h}(u^j(x_{i+1}) - u^j(x_{i-1})) - \frac{1}{6h^2k}(v^j(x_{i+1}) - 2v^j(x_i) + v^j(x_{i-1}) - v^{j-1}(x_{i+1}) + \\ & 2v^{j-1}(x_i) - v^{j-1}(x_{i-1})) = 0 \end{aligned} \quad (3)$$

Divide the interval $(0, L)$ into n subintervals (x_{i-1}, x_i) with length $\frac{1}{n}, i = 1, 2, \dots, n, 0 = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = L$, and multiply equation (3) by k , we get the following equation

$$\begin{aligned} & (v^j(x_i) - v^{j-1}(x_i)) + \frac{k}{2h}(u^{j-1}(x_{i+1}) - u^{j-1}(x_{i-1})) + \frac{ku^j(x_i)}{2h}(v^j(x_{i+1}) - v^j(x_{i-1})) + \\ & \frac{kv^j(x_i)}{2h}(u^j(x_{i+1}) - u^j(x_{i-1})) - \frac{1}{6h^2}(v^j(x_{i+1}) - 2v^j(x_i) + v^j(x_{i-1}) - v^{j-1}(x_{i+1}) + 2v^{j-1}(x_i) - v^{j-1}(x_{i-1})) = 0 \end{aligned}$$

After arranging the above equation in terms of $v^j(x_{i-1}), v^j(x_i)$ and $v^j(x_{i+1})$, we get

$$A_1(u^j(x_i))v^j(x_{i-1}) + B_1(u^j(x_i))v^j(x_i) + C_1(u^j(x_i))v^j(x_{i+1}) = F_1(u^{j-1}, v^{j-1}) \quad (4)$$

$$i = 1, 2, \dots, n-1$$

where

$$A_1(u^j(x_i)) = \left(-\frac{k}{2h}u^j(x_i) - \frac{1}{6h^2}\right)$$

$$B_1(u^j(x_i)) = \left(1 + \frac{k}{2h}(u^j(x_{i+1}) - u^j(x_{i-1})) + \frac{1}{3h^2}\right)$$

$$C_1(u^j(x_i)) = \left(\frac{k}{2h}u^j(x_i) - \frac{1}{6h^2}\right)$$

$$F_1(u^{j-1}, v^{j-1}) = v^{j-1}(x_i) + \frac{k}{2h}(u^{j-1}(x_{i+1}) - u^{j-1}(x_{i-1})) - \frac{1}{6h^2}v^{j-1}(x_{i+1}) + \frac{1}{3h^2}v^{j-1}(x_i) - \frac{1}{6h^2}v^{j-1}(x_{i-1})$$

Note that the definition of $A_1(u^j(x_i)), C_1(u^j(x_i))$ depends on $u^j(x_i)$, $B_1(u^j(x_i))$ depends on $u^j(x_{i+1}), u^j(x_{i-1})$ and $F(u^{j-1}, v^{j-1})$ depends on $v^{j-1}(x_i), u^{j-1}(x_{i+1}), u^{j-1}(x_{i-1}), v^{j-1}(x_{i+1}), v^{j-1}(x_i)$, and $v^{j-1}(x_{i-1})$. Equation (4) now becomes in the following algebraic matrix equation $E_1(u^j(x_i))\{v^j\} = \{F_1(u^{j-1}, v^{j-1})\}$ Where $E_1(u^j) = \{c_{l,s}(u^j(x_i))\}_{(n-1) \times (n-1)}$ is the tridiagonal matrix

$$\begin{bmatrix} B_1(u^j(x_1)) & C_1(u^j(x_1)) & \dots & 0 & \dots & 0 & 0 \\ A_1(u^j(x_2)) & B_1(u^j(x_2)) & & C_1(u^j(x_2)) & & 0 & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & & A_1(u^j(x_{n-2})) & & B_1(u^j(x_{n-2})) & C_1(u^j(x_{n-2})) \\ 0 & 0 & & 0 & & A_1(u^j(x_{n-1})) & B_1(u^j(x_{n-1})) \end{bmatrix}$$

And

$$\{v^j\} = \{v^j(x_1), v^j(x_2), \dots, v^j(x_{n-1})\}^T$$

$$\{F_1(u^{j-1}, v^{j-1})\} = \{F_1(u^{j-1}(x_1), v^{j-1}(x_1)), \dots, F_1(u^{j-1}(x_{n-1}), v^{j-1}(x_{n-1}))\}$$

Note that the matrix $E_1(u^j)$ is nonsymmetrical. By using the same approach for the equation (2b), we get

$$A_2(u^j(x_i))u^j(x_{i-1}) + B_2(u^j(x_i))u^j(x_i) + A_2(u^j(x_i))u^j(x_{i+1}) = F_2(u^{j-1}, v^{j-1}) \quad (5)$$

$$i = 1, 2, \dots, n-1$$

Where

$$A_2(u^j(x_i)) = -\frac{1}{6h^2}$$

$$B_2(u^j(x_i)) = (1 + \frac{k}{2h}(u^j(x_{i+1}) - u^j(x_{i-1}))) + \frac{1}{3h^2}$$

$$F_2(u^{j-1}, v^{j-1}) = u^{j-1}(x_i) - \frac{k}{2h}(v^{j-1}(x_{i+1}) - v^{j-1}(x_{i-1})) - \frac{1}{6h^2}u^{j-1}(x_{i+1}) + \frac{1}{3h^2}u^{j-1}(x_i) - \frac{1}{6h^2}u^{j-1}(x_{i-1})$$

Equation (3) now becomes in the following matrix equation

$$E_2(u^j(x_i))\{u^j\} = \{F_2(u^{j-1}, v^{j-1})\}$$

Where $E_2(u^j)$ is tridiagonal matrix

$$\begin{bmatrix} B_2(u^j(x_1)) & A_2(u^j(x_1)) & \dots & 0 & \dots & 0 & 0 \\ A_2(u^j(x_2)) & B_2(u^j(x_2)) & & A_2(u^j(x_2)) & & 0 & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & & A_2(u^j(x_{n-2})) & & B_2(u^j(x_{n-2})) & A_2(u^j(x_{n-2})) \\ 0 & 0 & & 0 & & A_2(u^j(x_{n-1})) & B_2(u^j(x_{n-1})) \end{bmatrix}$$

And

$$\{u^j\} = \{u^j(x_1), u^j(x_2), \dots, u^j(x_{n-1})\}^T$$

$$\{F_2(u^{j-1}, v^{j-1})\} = \{F_2(u^{j-1}(x_1), v^{j-1}(x_1)), \dots, F_2(u^{j-1}(x_{n-1}), v^{j-1}(x_{n-1}))\}$$

Since $A_2(u^j(x_i))$ is constant then the matrix E_2 is symmetric.

The matrices $E_1(u^j), E_2(u^j)$ involves the nonlinear terms u^j . Thus in each time step $j \in N$, we use an iterative solution method to compute v^j and u^j . We stop when the norm of the difference of u_i^j, u_{i-1}^j , and v_i^j, v_{i-1}^j are sufficiently small.

Now we prove the existence of the solution of matrices equation

Theorem 2.1. The solutions of the matrices equation

$$(i) E_1(u^j(x_i))\{v^j\} = \{F_1(u^{j-1}, v^{j-1})\}$$

$$(ii) E_2(u^j(x_i))\{u^j\} = \{F_2(u^{j-1}, v^{j-1})\}$$

exist

Proof (i): let $j \in N$ be fixed. Consider the following iteration

$$E_1(u_{i-1}^j)\{v_i^j\} = \{F_1(u^{j-1}, v^{j-1})\}, \forall i \in N \quad (6)$$

Where $u_0^j = u^{j-1}$. By subtracting the equation (6) from

$$E_1(u_i^j)\{v_{i+1}^j\} = \{F_1(u^{j-1}, v^{j-1})\}, \text{ we have}$$

$$E_1(u_i^j)\{v_{i+1}^j\} - E_1(u_{i-1}^j)\{v_i^j\} = 0$$

$$\Rightarrow E_1(u_i^j)\{v_{i+1}^j - v_i^j\} = (E_1(u_{i-1}^j) - E_1(u_i^j))\{v_i^j\} \quad (7)$$

The k-th element of the right hand side of the equation (7) is

$$\sum_{s=1}^{n-1} (c_{k,s}(u_{i-1}^j(x_k) - c_{k,s}(u_i^j(x_k)))\{v_i^j(x_s)\})$$

By the mean value theorem, it becomes

$$\sum_{s=1}^{n-1} \sum_{l=1}^{n-1} (c_{k,s}(u_{i^*}^j(x_k)))_{x_l} (u_{i-1}^j(x_l) - u_i^j(x_l))\{v_i^j(x_s)\}$$

Where the value of $u_{i^*}^j(x_k)$ is between $u_{i-1}^j(x_k)$ and $u_i^j(x_k)$, and $(c_{k,s}(u_{i^*}^j(x_k)))_{x_l}$ represents the partial derivatives of $(c_{k,s}(u_{i^*}^j(x_k)))$ with respect to $u_{i^*}^j(x_l)$.

Now, the right hand side of equation (7) becomes

$$\sum_{l=1}^{n-1} \sum_{s=1}^{n-1} v_i^j(x_s) (c_{k,s}(u_{i^*}^j(x_k)))_{x_l} (u_{i-1}^j(x_l) - u_i^j(x_l)) = \Omega(v_i^j, u_{i^*}^j) \{u_{i-1}^j - u_i^j\}$$

Where $\Omega(v_i^j, u_{i^*}^j)$ is the following matrix

$$\begin{bmatrix} \sum_{s=1}^{n-1} v_i^j(x_s) (c_{1,s}(u_{i^*}^j(x_1)))_{x_1} & \cdots & \sum_{s=1}^{n-1} v_i^j(x_s) (c_{1,s}(u_{i^*}^j(x_1)))_{x_{n-1}} \\ \sum_{s=1}^{n-1} v_i^j(x_s) (c_{2,s}(u_{i^*}^j(x_2)))_{x_1} & \cdots & \sum_{s=1}^{n-1} v_i^j(x_s) (c_{2,s}(u_{i^*}^j(x_2)))_{x_{n-1}} \\ \vdots & \vdots & \vdots \\ \sum_{s=1}^{n-1} v_i^j(x_s) (c_{n-2,s}(u_{i^*}^j(x_{n-2})))_{x_1} & \cdots & \sum_{s=1}^{n-1} v_i^j(x_s) (c_{n-2,s}(u_{i^*}^j(x_{n-2})))_{x_{n-1}} \\ \sum_{s=1}^{n-1} v_i^j(x_s) (c_{n-1,s}(u_{i^*}^j(x_{n-1})))_{x_1} & \cdots & \sum_{s=1}^{n-1} v_i^j(x_s) (c_{n-1,s}(u_{i^*}^j(x_{n-1})))_{x_{n-1}} \end{bmatrix}$$

Since $c_{1,s}(u_{i^*}^j(x_1)), s = 1, 2, \dots, n-1$ only contains $u_{i^*}^j(x_1)$ and $u_{i^*}^j(x_2)$, hence

$$\Omega_{1,l}(u_{i^*}^j) = \sum_{s=1}^{n-1} v_i^j(x_s) (c_{1,s}(u_{i^*}^j(x_1)))_{x_l} = 0 \quad \text{if } l \neq 1, 2$$

If $l = 1, 2$

$$\begin{aligned} \Omega_{1,1}(v_i^j, u_{i^*}^j) &= \sum_{s=1}^{n-1} v_i^j(x_s) (c_{1,s}(u_{i^*}^j(x_1)))_{x_1} \\ &= v_i^j(x_1) (c_{1,1}(u_{i^*}^j(x_1)))_{x_1} + v_i^j(x_2) (c_{1,2}(u_{i^*}^j(x_1)))_{x_1} \\ &= v_i^j(x_1) (0) + v_i^j(x_2) \left(\frac{k}{2h}\right) \\ &= \frac{k}{2h} v_i^j(x_2) \end{aligned} \tag{8}$$

$$\begin{aligned} \Omega_{1,2}(v_i^j, u_{i^*}^j) &= \sum_{s=1}^{n-1} v_i^j(x_s) (c_{1,s}(u_{i^*}^j(x_1)))_{x_2} \\ &= v_i^j(x_1) (c_{1,1}(u_{i^*}^j(x_1)))_{x_2} + v_i^j(x_2) (c_{1,2}(u_{i^*}^j(x_1)))_{x_2} \\ &= v_i^j(x_1) \left(\frac{k}{2h}\right) + v_i^j(x_2) (0) \\ &= \frac{k}{2h} v_i^j(x_1) \end{aligned} \tag{9}$$

Since k and h are both small and v^j is bounded (see [1]), then $\Omega_{1,1}(u_{i^*}^j)$ and $\Omega_{1,2}(u_{i^*}^j)$ in equation (8) and (9) is bounded by a small numbers.

Therefore the elements in the first row of the matrix Ω are zero except the first and second elements are sufficiently small.

Similarly,

$c_{n-1,s}(u_{i^*}^j(x_{n-1}))$ Only contains $u_{i^*}^j(x_{n-2})$ and $u_{i^*}^j(x_{n-1})$, which implies that

$$\Omega_{n-1,l}(v_i^j, u_{i^*}^j) = 0, \text{ if } l \neq n-2, n-1$$

If $l = n-2$

$$\begin{aligned} \Omega_{n-1,n-2}(v_i^j, u_{i^*}^j) &= v_i^j(x_{n-2})(c_{n-1,n-2}(u_{i^*}^j(x_{n-1})))_{x_{n-2}} + v_i^j(x_{n-1})(c_{n-1,n-1}(u_{i^*}^j(x_{n-1})))_{x_{n-2}} \\ &= v_i^j(x_{n-2})(0) + v_i^j(x_{n-1})\left(\frac{-k}{2h}\right) \\ &= \frac{k}{2h} |v_i^j(x_{n-1})| \leq \epsilon_1 \end{aligned}$$

If $l = n-1$

$$\begin{aligned} \Omega_{n-1,n-1}(v_i^j, u_{i^*}^j) &= v_i^j(x_{n-2})(c_{n-1,n-2}(u_{i^*}^j(x_{n-1})))_{x_{n-1}} + v_i^j(x_{n-1})(c_{n-1,n-1}(u_{i^*}^j(x_{n-1})))_{x_{n-1}} \\ &= v_i^j(x_{n-2})\left(\frac{-k}{2h}\right) + v_i^j(x_{n-1})(0) \\ &= \frac{k}{2h} |v_i^j(x_{n-2})| \leq \epsilon_2 \end{aligned}$$

Since only $u_{i^*}^j(x_{k-1}), u_{i^*}^j(x_k)$ and $u_{i^*}^j(x_{k+1})$ are in $c_{k,s}(u_{i^*}^j(x_k))$ $k \neq 1, n-1, s = 1, \dots, n-1$, then $\Omega_{k,l}(u_{i^*}^j) = 0$ if $l \neq k-1, k, k+1$

When $l = k-1$

$$\begin{aligned} \Omega_{k,k-1}(v_i^j, u_{i^*}^j) &= v_i^j(x_{k-1})(c_{k,k-1}(u_{i^*}^j(x_k)))_{x_{k-1}} + v_i^j(x_k)(c_{k,k}(u_{i^*}^j(x_k)))_{x_{k-1}} + v_i^j(x_{k+1})(c_{k,k+1}(u_{i^*}^j(x_k)))_{x_{k-1}} \\ &= v_i^j(x_k)\left(\frac{-k}{2h}\right) \\ &= \frac{k}{2h} |v_i^j(x_k)| \leq \epsilon_3 \end{aligned}$$

if $l = k$

$$\begin{aligned} \Omega_{k,k}(v_i^j, u_{i^*}^j) &= v_i^j(x_{k-1})\left(\frac{-k}{2h}\right) + v_i^j(x_k)(0) + v_i^j(x_{k+1})\left(\frac{k}{2h}\right) \\ &= v_i^j(x_{k-1})\left(\frac{-k}{2h}\right) + v_i^j(x_{k+1})\left(\frac{k}{2h}\right) \\ &= \frac{k}{2h} (|v_i^j(x_{k-1})| + |v_i^j(x_{k+1})|) \leq \epsilon_4 \end{aligned}$$

If $l = k+1$

$$\begin{aligned} \Omega_{k,k+1}(v_i^j, u_{i^*}^j) &= v_i^j(x_{k-1})(c_{k,k-1}(u_{i^*}^j(x_k)))_{x_{k+1}} + v_i^j(x_k)(c_{k,k}(u_{i^*}^j(x_k)))_{x_{k+1}} + v_i^j(x_{k+1})(c_{k,k+1}(u_{i^*}^j(x_k)))_{x_{k+1}} \\ &= v_i^j(x_k)\left(\frac{k}{2h}\right) \\ &= \frac{k}{2h} |v_i^j(x_k)| \leq \epsilon_5 \end{aligned}$$

Therefore, the matrix $\Omega(v_i^j, u_{i^*}^j)$ has the following form

$$\begin{bmatrix} \Omega_{1,1}(v_i^j, u_{i^*}^j) & \Omega_{1,2}(v_i^j, u_{i^*}^j) \dots\dots\dots & 0 \dots\dots\dots & 0 & 0 \\ \Omega_{2,1}(v_i^j, u_{i^*}^j) & \Omega_{2,2}(v_i^j, u_{i^*}^j) & \Omega_{2,3}(v_i^j, u_{i^*}^j) & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \Omega_{n-2,n-3}(v_i^j, u_{i^*}^j) & \Omega_{n-2,n-2}(v_i^j, u_{i^*}^j) & \Omega_{n-2,n-1}(v_i^j, u_{i^*}^j) \\ 0 & 0 & 0 & \Omega_{n-1,n-2}(v_i^j, u_{i^*}^j) & \Omega_{n-1,n-1}(v_i^j, u_{i^*}^j) \end{bmatrix}$$

Where the nonzero elements are sufficiently small. Thus, the norm of the matrix Ω , which is defined by $\sup_{x \in \mathbb{R}^{n-1}} \|\Omega(v_i^j, u_{i^*}^j)\|$ is small and bounded.

Now, it remains to show that $E_1(u_i^j)$ is invertible and bounded away from zero.

The matrix $E_1(u_i^j)$ is positive definite and bounded away from zero, since we can choose k and h , to such that the diagonal of the matrix is positive. This completes the proof of part (i).

Proof (ii): as in part (i), note the following iteration

$$E_2(u_i^j) \{u_{i+1}^j - u_i^j\} = (E_2(u_{i-1}^j) - E_2(u_i^j)) \{u_i^j\} \tag{10}$$

From the right hand side of equation (8), the k -th element is

$$\sum_{s=1}^{n-1} (g_{k,s}(u_{i-1}^j(x_k)) - g_{k,s}(u_i^j(x_k))) \{u_i^j(x_s)\}$$

By the same way which used in proof of part (i) after using the mean value theorem, the right hand side of equation (10), becomes:

$$\sum_{l=1}^{n-1} \sum_{s=1}^{n-1} u_i^j(x_s) (g_{k,s}(u_{i^*}^j(x_k)))_{x_l} (u_{i-1}^j(x_l) - u_i^j(x_l)) = \Psi(v_i^j, u_{i^*}^j) \{u_{i-1}^j - u_i^j\}$$

Where $\Psi(v_i^j, u_{i^*}^j)$ is the following matrix

$$\begin{bmatrix} \sum_{s=1}^{n-1} u_i^j(x_s) (g_{1,s}(u_{i^*}^j(x_1)))_{x_1} & \dots & \sum_{s=1}^{n-1} u_i^j(x_s) (g_{1,s}(u_{i^*}^j(x_1)))_{x_{n-1}} \\ \sum_{s=1}^{n-1} u_i^j(x_s) (g_{2,s}(u_{i^*}^j(x_2)))_{x_1} & \dots & \sum_{s=1}^{n-1} u_i^j(x_s) (g_{2,s}(u_{i^*}^j(x_2)))_{x_{n-1}} \\ \vdots & \vdots & \vdots \\ \sum_{s=1}^{n-1} u_i^j(x_s) (g_{n-2,s}(u_{i^*}^j(x_{n-2})))_{x_1} & \dots & \sum_{s=1}^{n-1} u_i^j(x_s) (g_{n-2,s}(u_{i^*}^j(x_{n-2})))_{x_{n-1}} \\ \sum_{s=1}^{n-1} u_i^j(x_s) (g_{n-1,s}(u_{i^*}^j(x_{n-1})))_{x_1} & \dots & \sum_{s=1}^{n-1} u_i^j(x_s) (g_{n-1,s}(u_{i^*}^j(x_{n-1})))_{x_{n-1}} \end{bmatrix}$$

By the same way in part (i), the matrix $\Psi(v_i^j, u_{i^*}^j)$, becomes

$$\begin{bmatrix} 0 & \Psi_{1,2}(u_{i^*}^j) \dots\dots\dots & 0 \dots\dots\dots & 0 & 0 \\ \Psi_{2,1}(u_{i^*}^j) & 0 & \Psi_{2,3}(u_{i^*}^j) & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \Psi_{n-2,n-3}(u_{i^*}^j) & 0 & \Psi_{n-2,n-1}(u_{i^*}^j) \\ 0 & 0 & 0 & \Psi_{n-1,n-2}(u_{i^*}^j) & 0 \end{bmatrix}$$

Where the nonzero elements are sufficiently small, since

$$\begin{aligned} \Psi_{1,2}(u_{i^*}^j) &= \sum_{s=1}^{n-1} u_i^j(x_s) (g_{1,s}(u_{i^*}^j(x_1)))_{x_2} \\ &= u_i^j(x_1) (g_{1,1}(u_{i^*}^j(x_1)))_{x_2} + u_i^j(x_2) (g_{1,2}(u_{i^*}^j(x_1)))_{x_2} \\ &= u_i^j(x_1) \left(\frac{k}{2h}\right) + v_i^j(x_2)(0) \\ &= \frac{k}{2h} |u_i^j(x_1)| \leq \epsilon_6 \end{aligned} \tag{11}$$

$$\begin{aligned} \Psi_{n-1,n-2}(u_{i^*}^j) &= u_i^j(x_{n-2}) (g_{n-1,n-2}(u_{i^*}^j(x_{n-1})))_{x_{n-2}} + u_i^j(x_{n-1}) (g_{n-1,n-1}(u_{i^*}^j(x_{n-1})))_{x_{n-2}} \\ &= u_i^j(x_{n-1}) \left(\frac{-k}{2h}\right) \\ &= \frac{k}{2h} |u_i^j(x_{n-1})| \leq \epsilon_7 \end{aligned}$$

$$\begin{aligned} \Psi_{k,k-1}(u_{i^*}^j) &= u_i^j(x_k) (g_{k,k}(u_{i^*}^j(x_k)))_{x_{k-1}} \\ &= u_i^j(x_k) \left(\frac{-k}{2h}\right) \\ &= \frac{k}{2h} |u_i^j(x_k)| \leq \epsilon_8 \end{aligned}$$

$$\begin{aligned} \Psi_{k,k+1}(u_{i^*}^j) &= u_i^j(x_{k+1}) (g_{k,k+1}(u_{i^*}^j(x_k)))_{x_{k+1}} \\ &= u_i^j(x_k) \left(\frac{k}{2h}\right) \\ &= \frac{k}{2h} |u_i^j(x_k)| \leq \epsilon_9 \end{aligned}$$

Where the nonzero elements are sufficiently small. Thus, the norm of the matrix Ψ , which is defined by $\sup_{x \in \mathbb{R}^{n-1}} \|\Psi(v_i^j, u_{i^*}^j)\|$ is small and bounded.

As in part (i), we can show that $E_2(u_i^j)$ is invertible and bounded away from zero. This completes the proof of part (ii).

3. Numerical experiments

We consider the initial-value problem for eq.(1) with a given general initial profile $(v_0(x), u_0(x))$ with homogenous Dirichlet boundary condition and we solve nonlinear system in each step by using fixed point iteration. The termination criterion for the inner iteration was: $\max |u_j^{(k+1)} - u_j^{(k)}| \leq .01$ and $\max |v_j^{(k+1)} - v_j^{(k)}| \leq .01$ $k = 0,1,2,\dots$ and $j \in N$ is the time step, we observe that solitary wave like pulse is generated, possibly indicating that the main pulses are GSW solutions of (1). It is well-known [1], that if the initial values of the solution of boussinesq system are suitably chosen, then the solution that emerges moves in one direction.

In the first experiment, consider an 'M-shaped' solitary wave:

$$v_0(x) = \frac{15}{4}(-2 + \cosh(3\sqrt{2/5}(x-x_0))) \operatorname{sech}^4\left(\frac{3(x-x_0)}{\sqrt{10}}\right)$$

$$u_0(x) = \frac{15}{2} \operatorname{sech}^2\left(\frac{3}{\sqrt{10}}(x-x_0)\right)$$

As an initial is known in [6].

We present in figure (1) generation and the propagation of v in the positive x -direction:

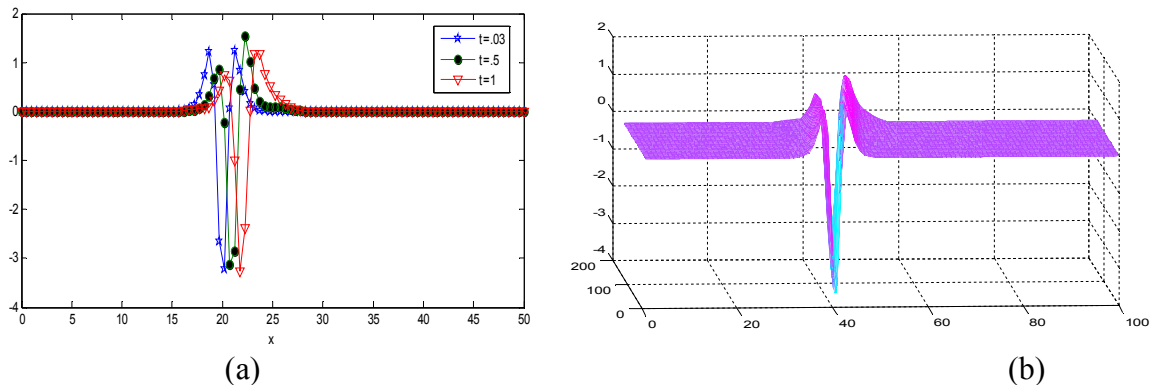


Figure 1:(a) A solution of system(1) with different time (b) surface profile of $v(x,t)$

In experiment (2) we took special initial data of the form :

$$v_0(x) = \emptyset(x), u_0(x) = \emptyset(x) - .25\emptyset^2(x), \text{ where } \emptyset(x) = \operatorname{sech}^2\left(\frac{(x-x_0)}{\sqrt{2}}\right)$$

Is a solitary wave of the BBM equation $w_t + w_x + ww_x - \frac{1}{6}w_{xxt} = 0$. [1]

The structure of the solution of system(1), when initiated with the BBM solitary wave appears to be solitary wave solution of system(1). The resulting evolution of the $v(x,t)$ of the solution of system(1) is shown in figure(2)

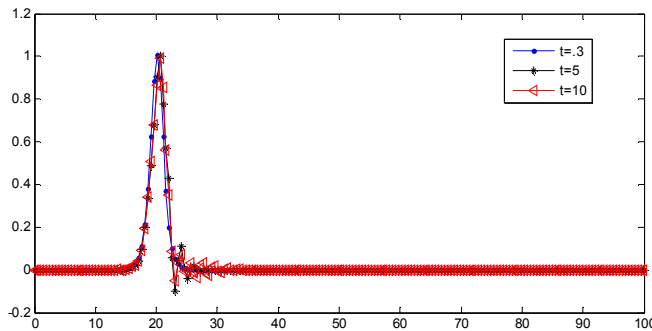


Figure (2): Solution of system(1) at t=0.3,5,10.

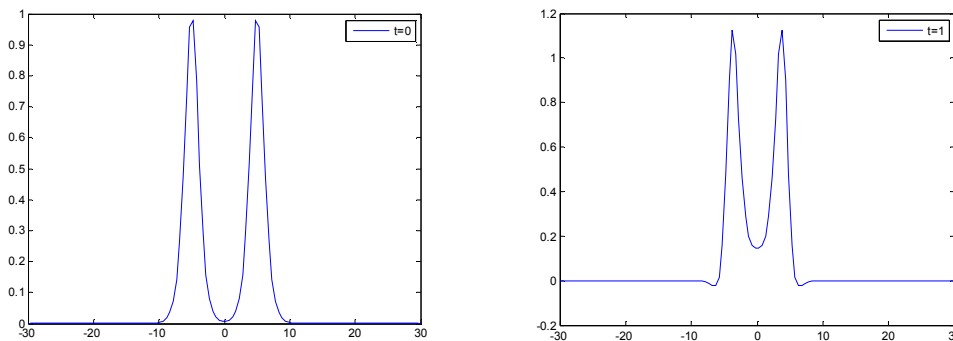
In the last experiment, we show the collision of two general solitary waves for the system (1) on $[-30,30]$, traveling in opposite directions with initial values of the form

$$v_0(x) = \phi(x + 10) + \phi(x - 10)$$

$$u_0(x) = \phi(x + 10) + \frac{1}{4}\phi^2(x + 10) - (\phi(x - 10) + \frac{1}{4}\phi^2(x - 10)) ,$$

where $\phi(x)$ as in experiment (2).

When two solitary waves of same heights, specifically of initial v-amplitudes, were let to collide head-on, we observed that, during the head-on collision the amplitude is longer than the initial amplitude and after the collision, the two waves unreturned to the practically their initial amplitudes. This may be seen in the graph of the evolution of the total amplitude during the head-on collision in figure (3):



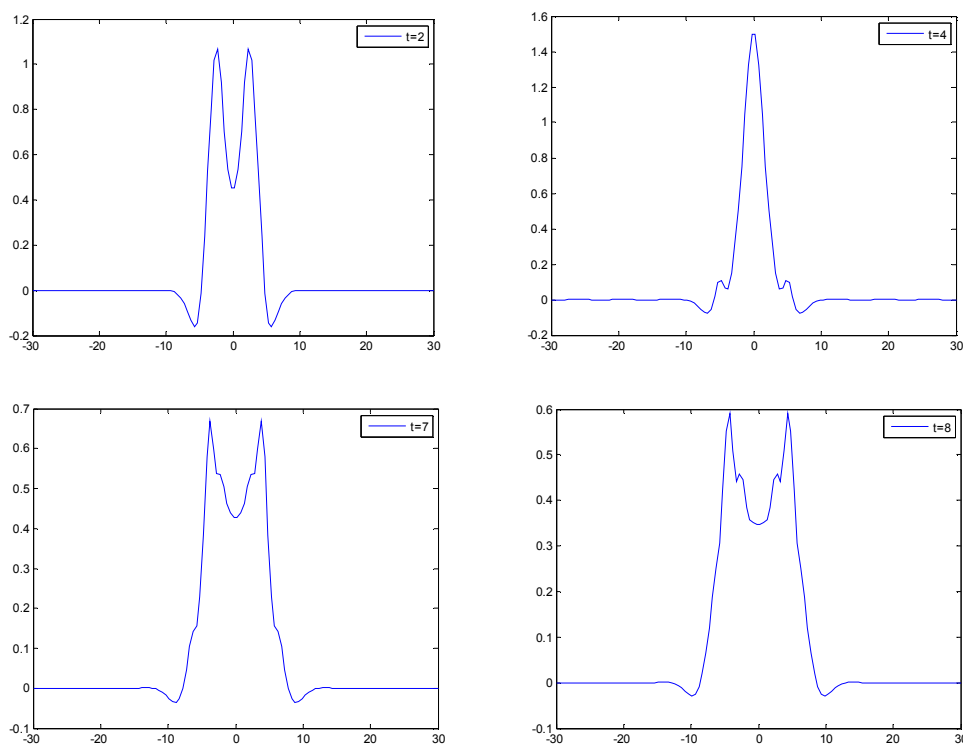


Figure (3) Head-on collision of two solitary wave of equal heights

4. Conclusion

The basic idea in this paper was to prove that the use of implicit finite difference method in solving the BBM-BBM system possible and, indeed, been proposed theory in this regard, due to the use of this method has emerged from a system of nonlinear algebraic equations and found the practical part of this paper that the method of fixed point iteration is to give acceptable results and can be used easily in solving nonlinear system for this types of systems

References

- [1] A.A.Alazman, J.P.Albert, J.L.Bona, M.Chen and J.Wu (2006), Comparison between the BBM equation and a Boussinesq systems, *adv. Differential equation*, 2, pp.121-166.
- [2] C.C.Mei and L.F.Liu (1973), The damping of surface gravity waves in a bounded liquid, *J. Fluid Mech.*, 59, pp. 239–256.
- [3] D.C.Antoropoulos, V.A.Dougalis and D.E.Mitsotakis (2010), Numerical Solution of Boussinesq systems of Bona-Smith family. *Applied Numerical Mathematics*, 30, pp.314-336.
- [4] H.Chen, M.Chen and N.V.Wguyen (2007), Cnoidal wave solutions of Boussinesq Systems, *Nonlinearity* 20(6) , pp.1443-1461.
- [5] J. J. Mahony and W. G. Pritchard, Wave reflexion from beaches, *J. Fluid Mech.*, 101(1980), 809–832.
- [6] J.L.Bona and M.Chen (1998) , A Boussinesq system for two-way propagation of Nonlinear dispersive waves, *physica D* 116, pp. 191-224.
- [7] J.L.Bona and M.Chen and J.C.Saut (2002), Boussinesq equations and other

- systems for small-amplitude longwaves in nonlinear dispersive media, I: Derivation and linear theory, *J. Nonlinear Sci.*, 12, pp. 283-318.
- [8] J.L. Bona and M. Chen and J.C. Saut (2004), Boussinesq equations and other systems for small-amplitude longwaves in nonlinear dispersive media, II: The nonlinear theory, *J. Nonlinear Sci.*, 17, pp. 925-952.
- [9] J. L. Bona, W. G. Pritchard, and L. R. Scott (1981), An evaluation of a model Equation for water waves, *Philos. Trans. Royal Soc. London, Ser. A*, 302, pp. 457-510.
- [10] J. W. Miles (1967), Surface-wave damping in a closed basin, *Proc. Royal Soc. London A*, 297, pp. 459-475.
- [11] M. Chen (2000), Solitary-Wave and multipulsed traveling wave solutions of Boussinesq systems, *Applicable Analysis* 75, pp. 213-240.
- [12] M. Chan and G. Iooss (2006), Periodic wave patterns of two-dimensional Boussinesq system, *European Journal of Mechanics B Fluids*, 25, pp. 393-405.
- [13] M. Chan and G. Iooss (2008), Asymmetric Periodic wave patterns of two-dimensional Boussinesq system, *Physica D*, 237, pp. 1539-1552.
- [14] V.A. Dougalis, D.E. Mitsotakis and J.C. Saut (2007), On some Boussinesq systems in two space dimensions theory and numerical analysis, *Siam. Numer. Anal.* 41, pp. 825-854.
- [15] V.A. Dougalis, D.E. Mitsotakis and J.C. Saut (2009), On initial boundary value problem for Boussinesq system of BBM-BBM type in plane domain, *Discrete Contin. Dyn. Syst.* 23, pp. 1191-1204.