**Iraqi Journal of Statistical Science (20) 2011**
**The Fourth Scientific Conference of the College of Computer Science & Mathematics**
pp [52-62]

# A study of the stability of an amplitude-dependent exponential autoregressive model with application

Dr.Abdul Ghafoor Jasim Salim* Dr.Kaies Esmaeel**
Haneen Talal Jasim***

## Abstract

In this paper we studied and found the stability conditions of the (AEXPAR) model by using a dynamical approach and by adopting the linear approximation technique.

دراسة استقرارية نموذج انحدار ذاتي أسي معتمد على السعة
مع التطبيق

## المستخلص

تم في هذا البحث دراسة استقرارية (AEXPAR) وإيجاد شروط استقراريته باستخدام نموذج أسلوب ديناميكي (حركي) وباستخدام طريقة التقريب الخطية .

## 1-Introduction:

Linear models, in time-series analysis as well as in many other areas, have been widely investigated and applied in handling real-life data. In the last four decades, many authors have questioned the omnipresence of linear approaches, and thus time-series analysis has moved towards the nonlinear area due to the complexity of most measured time series characterizing. Much attention has been given, in this respect, to bilinear models, threshold models, random coefficient autoregressive models.

Haggan and T. Ozaki studied in (1981) Modeling nonlinear random vibrations using an amplitude-dependent autoregressive time series model.[3]

Ozaki in(1982)proposed a dynamic approach in order to find the necessary and sufficient condition for stability of exponential

*Prof. \ College of Computers Sciences and Math.\ University of Mosul
**Asst. Prof. \ College of Education\ University of Mosul
***College of Computers Sciences and Math.\ University of Mosul

Received:1/10 /2011 ____________________Accepted: 21 /12 / 2011

autoregressive models. This approach is a local linearization technique used to find the approximated linear autoregressive model near the non-zero singular point of a nonlinear model.By using a variational equation near the nonzero singular point of an exponential autoregressive model. Ozaki finds the necessary and sufficient condition for existence and stability of a nonzero singular point of (EXPAR) model, furthermore he finds the stability condition of a limit cycle if it exists[1]

T.Ozaki,J.C.Jimenez,R.Biscay and P.Valdes studied in (1999) the Nonlinear time series models and Neural dynamical systems. Especially continuous time deterministic and stochastic dynamical system models and nonlinear autoregression model. (EXPAR) considered from the view point of prediction [7]

Kentaro Ishizuka, Hiroko Kato and Tomohiro Nakatani studied in(2005) a speech signal analysis approach that uses the exponential autoregressive (EXPAR) model. In real speech signal, the amplitude and frequency are fluctuating randomly.These fluctuations are non-Gaussian and have nonlinear dynamics. [4]

Jelloul Allal and Saı¨d El Melhaoui studied in(2005) the problem of detecting the eventual existence of an exponential component in an AR(1) model, that is, the problem of testing ordinary AR(1) dependence against the alternative of an exponential autoregression [EXPAR(1)] model,was considered. A local asymptotic normality property was established for EXPAR(1) models in the vicinity of AR(1) ones. Two problems arose in this context, which were quite typical in the study of nonlinear time-series models. The first was a problem of parameter identification in the EXPAR(1) model. A special parameterization was developed so as to overcome this technical problem. The second problem was related to the fact that the underlying innovation density had to be treated as a nuisance. The problem at hand, indeed, appeared to be nonadaptive. These problems were solved using semi-parametrically efficient pseudo-Gaussian methods (which did not require Gaussian observations).[2]

Mika Meitz and Pentti Saikkonen studied in(2006) the stability of nonlinear autoregressive models with conditionally heteroskedastic errors. They consider a nonlinear autoregression of order p (AR(p)) with the conditional variance specified as a nonlinear first order generalized autoregressive conditional heteroskedasticity (GARCH(1,1)) model.[5]

**Definition 1** :A singular point of $x_t = f\left(x_{t-1}, x_{t-2}, \ldots\ldots, x_{t-p}\right)$ is defined as a point, which every trajectory of $x_t = f\left(x_{t-1}, x_{t-2}, \ldots\ldots, x_{t-p}\right)$ beginning sufficiently near it approaches either for $t \to \infty$ or for $t \to -\infty$. If it approaches it for $t \to \infty$ we call it stable singular point and if it approaches it for $t \to -\infty$ we call it unstable singular point. Obviously a singular point $\zeta$ satisfies $\zeta = f(\zeta, \zeta, \zeta, \ldots\ldots\ldots, \zeta)$.[6]

**Definition 2**: A limit cycle of $x_t = f(x_{t-1}, x_{t-2}, \ldots\ldots, x_{t-p})$ is defined as an isolated and closed trajectory $x_{t+1}, x_{t+2}, \ldots\ldots, x_{t+q}$ where **q** is a positive integer. Closed means that if initial values $(x_1, \ldots\ldots, x_p)$ belong to the limit cycle then $(x_{1+kq}, x_{2+kq}, \ldots\ldots, x_{p+kq}) = (x_1, \ldots\ldots, x_p)$ for any integer k. Isolated means that every trajectory beginning sufficiently near the limit cycle approaches either for $t \to \infty$ or for $t \to -\infty$. If it approaches it for $t \to \infty$ we call it stable limit cycle and if it approaches it for $t \to -\infty$ we call it unstable limit cycle.[6]

## Theorem 1:

Let $\{x_t\}$ be expressed by the exponential model $x_t = \left(\phi_1 + \pi_1 e^{-x_{t-1}^2}\right)x_{t-1} + \varepsilon_t$

A limit cycle of period q $x_t, x_{t+1}, x_{t+2}, \ldots\ldots, x_{t+q}$ of the model is orbitally stable if

$$\left|\frac{\zeta_{t+q}}{\zeta_t}\right| < 1$$

For the prove see [6]

## Amplitude-dependent exponential autoregressive[AEXPAR] models:

This model is a class of nonlinear autoregressive models. These models were independently introduced by Jones(1976) and [Ozaki and Oda (1978)]. The basic form of an AEXPAR model of order (k) is

$$x_t = \sum_{j=1}^{k}\left[\alpha_j + \beta_j\, e^{\left(-\delta x_{t-1}^2\right)}\right]x_{t-j} + \varepsilon_t \qquad \ldots (1)$$

Where $\delta > 0, \{\varepsilon_t\}$ is white noise.[8]

## The proposed model:

Let the model in equation (1) be given, for k=1 and by replacing $\beta_j$ by $f_j(x_{t-1})$ where

$$f_j(x_{t-1}) = \sum_{i=0}^{p}\alpha_{i+2}\, x_{t-j}^{i} \quad ; \quad j = 1,2,3,\ldots.,k$$

Then the proposed model will be as follows:

$$x_t = \left[\alpha_1 + \left(\sum_{i=0}^{p}\alpha_{i+2}\, x_{t-j}^{i}\right)e^{\left(-\delta\, x_{t-1}^2\right)}\right]x_{t-1} + \varepsilon_t$$

$$\Rightarrow \quad x_t = \Big[\alpha_1 + \alpha_2 e^{\left(-\delta\, x_{t-1}^2\right)} + \alpha_3\, x_{t-1}\, e^{\left(-\delta\, x_{t-1}^2\right)} + \ldots\ldots\ldots\ldots\ldots + \alpha_p\, x_{t-1}^{p-2}\, e^{\left(-\delta\, x_{t-1}^2\right)}\Big] x_{t-1} + \varepsilon_t$$

$$x_t = \alpha_1 x_{t-1} + \alpha_2\, x_{t-1}\, e^{\left(-\delta\, x_{t-1}^2\right)} + \alpha_3\, x_{t-1}^2\, e^{\left(-\delta\, x_{t-1}^2\right)} + \ldots\ldots\ldots\ldots\ldots + \alpha_p\, x_{t-1}^{p-1}\, e^{\left(-\delta\, x_{t-1}^2\right)} + \varepsilon_t \qquad \ldots\ (2)$$

Where $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_p$ are arbitrary constants.

## Statistical properties for the proposed model

### The singular point

To find the singular point of the model (2) we use the definition (1)of the singular point and suppose that the white noise is neglected then we have

$$\zeta = \alpha_1 \zeta + \alpha_2\, \zeta\, e^{\left(-\delta\, \zeta^2\right)} + \ldots\ldots\ldots + \alpha_p \zeta^{\,p-1} e^{\left(-\delta\, \zeta^2\right)} \qquad \ldots \quad (3)$$

Since $\zeta$ is small let $\zeta^q \to 0$ for $q \geq 3$; then we have:

$$\zeta = \alpha_1 \zeta + \alpha_2\, \zeta\, e^{\left(-\delta\, \zeta^2\right)} + \alpha_3\, \zeta^2\, e^{\left(-\delta\, \zeta^2\right)} \qquad \ldots \quad (4)$$

$$\zeta - \alpha_1 \zeta - \alpha_2 \zeta e^{-\delta \zeta^2} - \alpha_3 \zeta^2 e^{-\delta \zeta^2} = 0$$

$$\zeta \Big[1 - \alpha_1 - \alpha_2 e^{-\delta \zeta^2} - \alpha_3 \zeta e^{-\delta \zeta^2}\Big] = 0$$

Either $\zeta = 0$

Or $\Big[1 - \alpha_1 - \alpha_2 e^{-\delta \zeta^2} - \alpha_3 \zeta e^{-\delta \zeta^2}\Big] = 0$

$$1 - \alpha_1 - \alpha_2 e^{-\delta \zeta^2} - \alpha_3 \zeta e^{-\delta \zeta^2} = 0$$

$$1 - \alpha_1 - \alpha_2 \left[1 - \delta \zeta^2 + \frac{\delta^2}{2} \zeta^4 + \ldots.\right] - \alpha_3 \zeta \left[1 - \delta \zeta^2 + \frac{\delta^2}{2} \zeta^4 + \ldots.\right] = 0$$

$$1 - \alpha_1 - \alpha_2 + \alpha_2 \delta \zeta^2 - \frac{\alpha_2 \delta^2}{2} \zeta^4 + \ldots - \alpha_3 \zeta + \alpha_3 \delta \zeta^3 - \frac{\alpha_3 \delta^2}{2} \zeta^5 + \ldots. = 0$$

Since $\zeta$ is small let $\zeta^q \to 0$ for $q \geq 3$; then

$$1-\alpha_1-\alpha_2+\alpha_2\delta\zeta^2-\alpha_3\zeta=0$$

$$\zeta=\frac{\alpha_3\pm\sqrt{\alpha_3^2-4(\alpha_2\delta)(1-\alpha_1-\alpha_2)}}{2\alpha_2\delta}$$

This means that we get two singular points

$$\zeta_1=\frac{\alpha_3+\sqrt{\alpha_3^2-4(\alpha_2\delta)(1-\alpha_1-\alpha_2)}}{2\alpha_2\delta}$$

$$\zeta_2=\frac{\alpha_3-\sqrt{\alpha_3^2-4(\alpha_2\delta)(1-\alpha_1-\alpha_2)}}{2\alpha_2\delta}$$

## The stability condition for the singular point:

We will find the stability condition for the non-zero singular point as follows :

put $x_s=\zeta+\zeta_s$ ,s=t,t-1 in equation (4) then we have:

$$\zeta+\zeta_t=\alpha_1(\zeta+\zeta_{t-1})+\alpha_2(\zeta+\zeta_{t-1})e^{-\delta(\zeta+\zeta_{t-1})^2}+\alpha_3(\zeta+\zeta_{t-1})^2e^{-\delta(\zeta+\zeta_{t-1})^2}$$

$$\begin{aligned}\zeta+\zeta_t=&\left[\ \alpha_1+\alpha_2e^{-\delta(\zeta+\zeta_{t-1})^2}+\alpha_3(\zeta+\zeta_{t-1})e^{-\delta(\zeta+\zeta_{t-1})^2}\ \right]\zeta\\&+\left[\alpha_1+\alpha_2e^{-\delta(\zeta+\zeta_{t-1})^2}+\alpha_3(\zeta+\zeta_{t-1})e^{-\delta(\zeta+\zeta_{t-1})^2}\right]\zeta_{t-1}\end{aligned}$$

$$\begin{aligned}\zeta+\zeta_t=&\alpha_1\zeta+\alpha_2\zeta e^{-\delta(\zeta+\zeta_{t-1})^2}+\alpha_3(\zeta+\zeta_{t-1})\zeta e^{-\delta(\zeta+\zeta_{t-1})^2}+\alpha_1\zeta_{t-1}+\\&\alpha_2\zeta_{t-1}e^{-\delta(\zeta+\zeta_{t-1})^2}+\alpha_3(\zeta+\zeta_{t-1})\zeta_{t-1}e^{-\delta(\zeta+\zeta_{t-1})^2}\end{aligned}$$

Taylor expansion of $e^{-\delta(\zeta+\zeta_{t-1})^2}$ can be calculated with respect to $\zeta_{t-1}$ as follows:

$$e^{-\delta(\zeta+\zeta_{t-1})^2}=e^{-\delta\zeta^2}-2\delta e^{-\delta\zeta^2}\zeta\zeta_{t-1}+\cdots$$

$$\begin{aligned}\zeta+\zeta_t=&\alpha_1\zeta+\alpha_2\zeta\left[\ e^{-\delta\zeta^2}-2\delta e^{-\delta\zeta^2}\zeta\zeta_{t-1}\ \right]\\&+\alpha_3(\zeta^2+\zeta\zeta_{t-1})\left[e^{-\delta\zeta^2}-2\delta e^{-\delta\zeta^2}\zeta\zeta_{t-1}\right]\\&+\alpha_1\zeta_{t-1}+\alpha_2\zeta_{t-1}\left[e^{-\delta\zeta^2}-2\delta e^{-\delta\zeta^2}\zeta\zeta_{t-1}\right]\\&+\alpha_3(\zeta\zeta_{t-1}+\zeta_{t-1}^2)\left[e^{-\delta\zeta^2}-2\delta e^{-\delta\zeta^2}\zeta\zeta_{t-1}\right]\end{aligned}$$

$$\zeta + \zeta_t = \alpha_1\zeta + \alpha_2\zeta + \alpha_3\zeta^2 + \alpha_3\zeta\zeta_{t-1} + \alpha_1\zeta_{t-1} + \alpha_2\zeta_{t-1} - \alpha_2\delta\zeta^2\zeta_{t-1}$$
$$- 2\alpha_2\delta\zeta\zeta^2_{t-1} - 2\alpha_2\delta\zeta^2\zeta^2_{t-1} - 2\alpha_3\delta\zeta^2\zeta^2_{t-1} + \alpha_3\zeta\zeta_{t-1} + 2\alpha_3\delta\zeta^2\zeta^2_{t-1}$$

since $\zeta, \zeta_{t-1}$ are very small and $\zeta\zeta_{t-1} \to 0$ let $\zeta^p\zeta_{t-1}{}^q \to 0; \forall p, q \geq 1$

$$\zeta_t = (\alpha_1 + \alpha_2 - 1 + \alpha_3\zeta)\zeta + (\alpha_1 + \alpha_2)\zeta_{t-1} \qquad \dots \quad (5)$$

Then

$$\zeta_t = A + B\zeta_{t-1} \qquad \dots \quad (6)$$

Where A and B are constant defined as follows:

$A = (\alpha_1 + \alpha_2 - 1 + \alpha_3\zeta)\zeta$

$B = \alpha_1 + \alpha_2$

Equation (6) is a first order linear autoregressive model which is stable if all roots of the characteristic equation

$\lambda - B + A = 0$

Lies inside the unit cycle

## The stability condition of the limit cycle:

Let the limit cycle of period q of the proppposed model(2) has the form $x_t, x_{t+1}, x_{t+2}, \dots\dots, x_{t+q} = x_t$ .The point $x_t, x_{t-1}$ on a trajectory near the limit cycle are replaced by $x_t + \zeta_t$ and $x_{t-1} + \zeta_{t-1}$ respectively we find the following difference equation

$$\zeta_t = T(x_{t-1})\zeta_{t-1} \qquad \dots \quad (7)$$

Where $T(x_{t-1}) = \alpha_1 + \alpha_2 e^{\left(-\delta x^2_{t-1}\right)} + 2\alpha_3 x_{t-1} e^{\left(-\delta x^2_{t-1}\right)} + 2\alpha_2 x^2_{t-1} e^{\left(-\delta x^2_{t-1}\right)}$

It is not easy to solve difference equation with periodic coefficients. What is required to know whether $\zeta_t$ of (7) converges to zero or not , and this can be checked by seeing whether $\left|\frac{\zeta_{t+q}}{\zeta_t}\right|$ less than one or not.

From the relation (7) we get:

$$
\begin{aligned}
\zeta_{t+q} &= T(x_{t+q-1})\zeta_{t+q-1} \\
&= T(x_{t+q-1})T(x_{t+q-2})\,\zeta_{t+q-2} \\
&= T(x_{t+q-1})T(x_{t+q-2})\,T(x_{t+q-3})\zeta_{t+q-3} \\
&\vdots \\
&= T(x_{t+q-1})T(x_{t+q-2})\,T(x_{t+q-3})\ldots\ldots\ldots\; T(x_t)\zeta_t
\end{aligned}
$$

$$\zeta_{t+q} = \prod_{i=1}^{q} T(x_{t+q-i})\zeta_t$$

$$\frac{\zeta_{t+q}}{\zeta_t} = \prod_{i=1}^{q} T(x_{t+q-i}) \qquad \ldots \qquad (8)$$

Then by using theorem(1) the model (2) is stable iff $\left|\prod_{i=1}^{q} T(x_{t+q-i})\right| < 1$

## Examples:

We shall give two examples to explain the statistical properties for the proposed model

**Example(1)**

Let the time series $\{x_t\}$ is given by the following model:

$$x_t = 0.61547824\mathrm{x}_{\mathrm{t-1}} + 0.21134580\,\mathrm{x}_{\mathrm{t-1}}e^{-x_{t-1}^2} + 2.3\;x_{t-1}^2 e^{-x_{t-1}^2} + \varepsilon_t \ldots(9)$$

We shall use the above relation which we proved earlier for this example

**1- The singular point**

By using the relation in equation (5) we get

$$\zeta_{1,2} = \frac{\alpha_3 \pm \sqrt{\alpha_3^2 - 4(\alpha_2\delta)(1-\alpha_1-\alpha_2)}}{2\alpha_2\delta}$$

$\zeta_1 = 0.4827 \quad , \quad \zeta_2 = 0.0034$

**2-The stability condition:**

From the relation

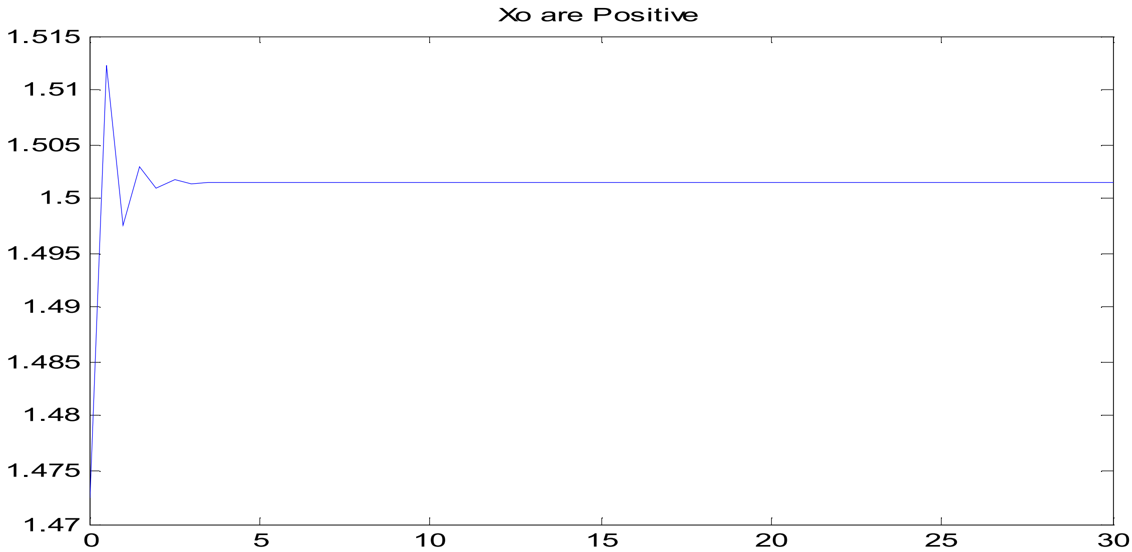
$$\zeta_t = A + B\zeta_{t-1}$$

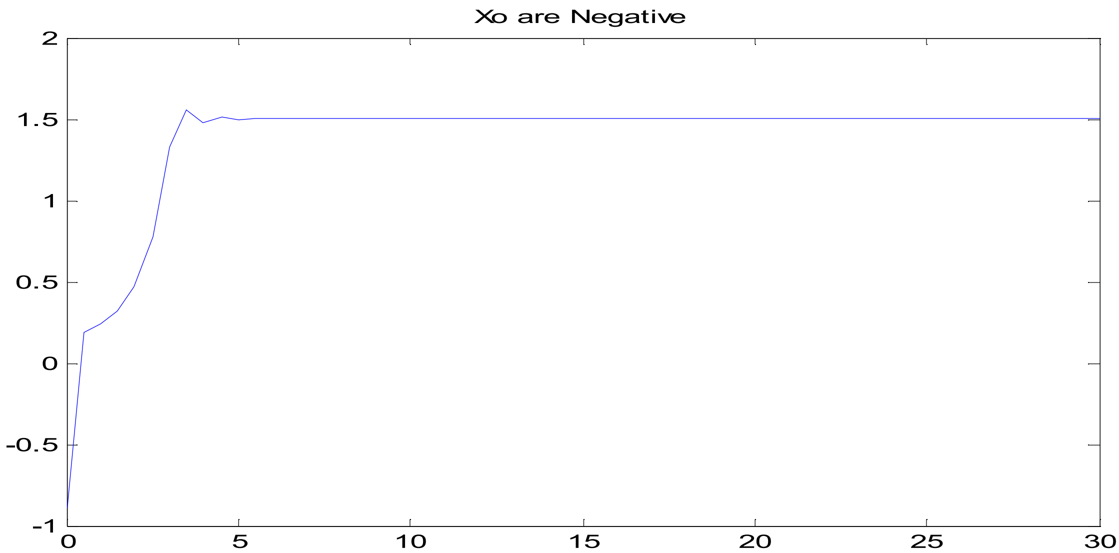
$$\zeta_t = 0.4523 + 0.8268\zeta_{\mathrm{t-1}} \qquad \ldots \qquad (10)$$

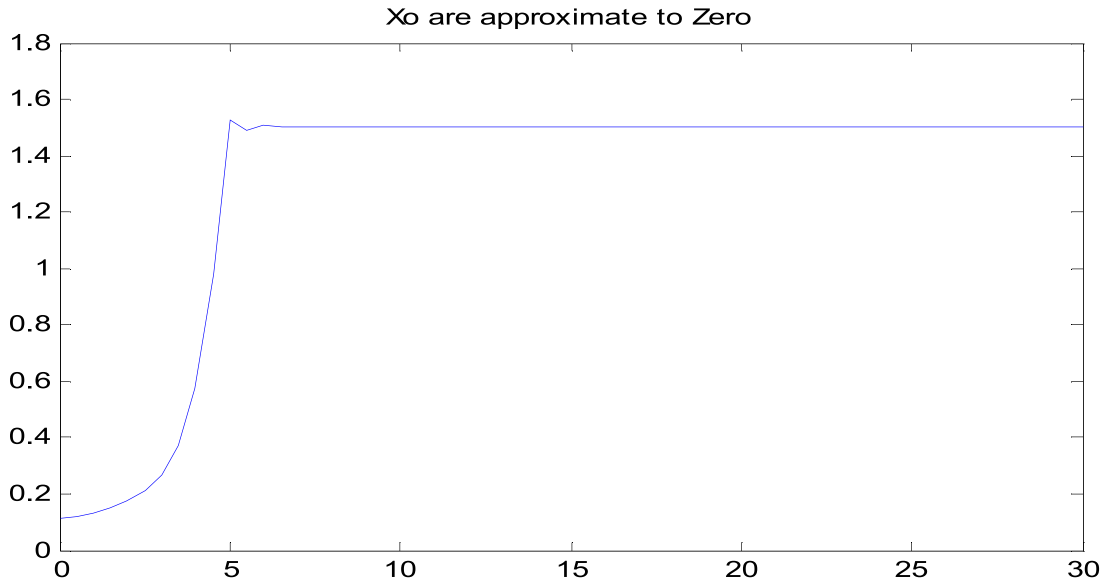
which is a first order linear autoregressive model.

From the characteristic equation of the model (10) we have the root $\lambda = 0.3745$.

Since $0< 0.3745<1$ this means that the model is stable .and the following figures show the stability of the model for different initial values






Fig(1)the series generated from model(9)

We can see from the above figures that it doesn't depend on the initial value, this implies that the proposed model has a limit cycle.

**Example(2)**

Let the time series $\{x_t\}$ is given by the following model:

$$x_t = 0.61547824\mathrm{x}_{\mathrm{t-1}} + 0.91244080\,\mathrm{x}_{\mathrm{t-1}} e^{-x_{t-1}^2} + 50x_{t-1}^2 e^{-x_{t-1}^2} + \varepsilon_t \ldots(11)$$

We shall use the above relation which we proved earlier for this example

**1- The singular point**

By using the relation in equation(5) we get

$$\zeta_{1,2} = \frac{\alpha_3 \pm \sqrt{\alpha_3^2 - 4(\alpha_2\delta)(1-\alpha_1-\alpha_2)}}{2\alpha_2\delta}$$

$\zeta_1 = 45.6308$ , $\zeta_2 = \text{-}0.0088$

**2-The stability condition:**

From the relation

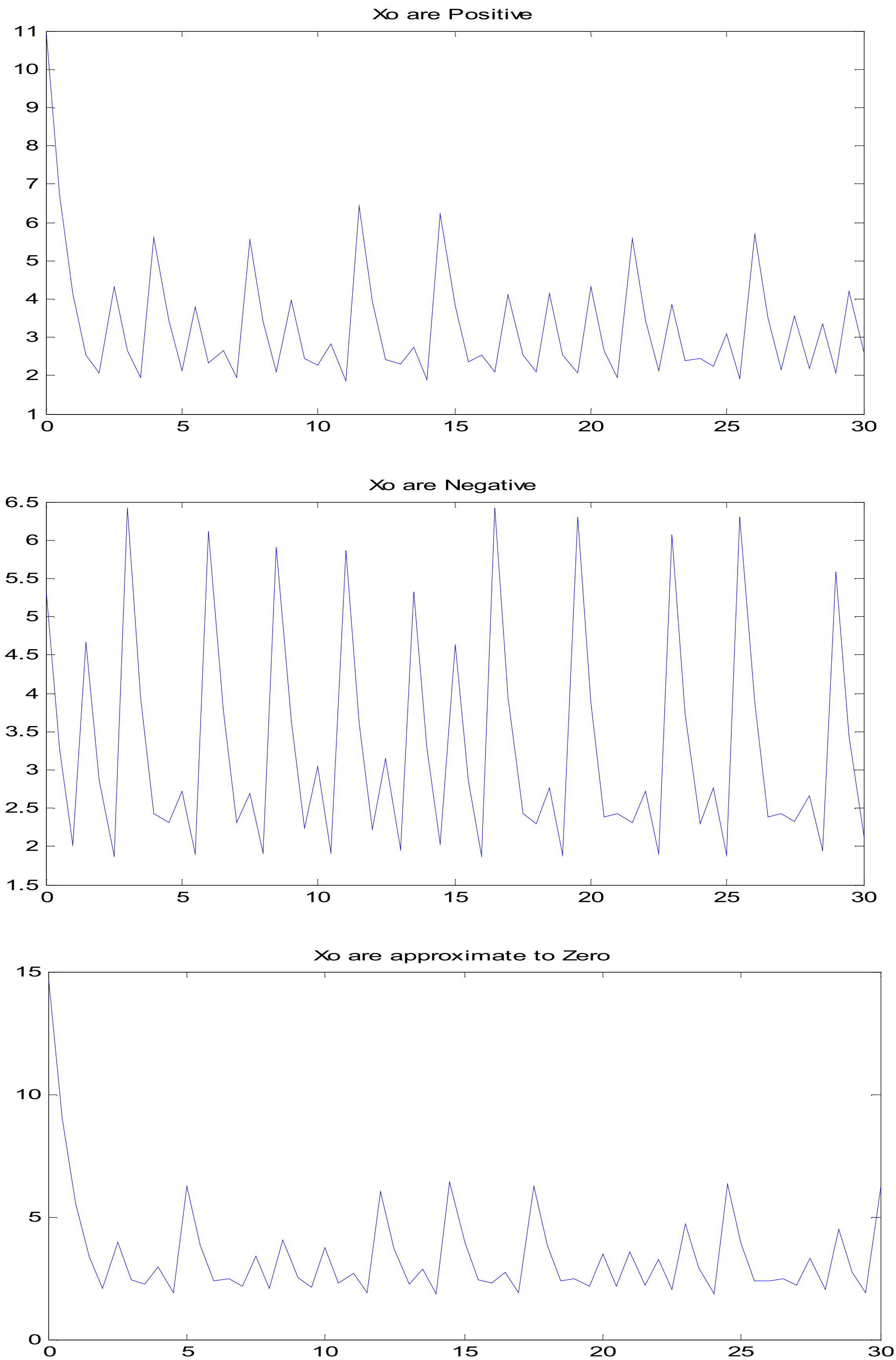
$$\zeta_t = A + B\zeta_{t-1}$$

$$\zeta_t = 104132 + 1.5279\zeta_{\mathrm{t-1}} \qquad \ldots \quad (12)$$

which is a first order linear autoregressive model. From the characteristic equation of the model (10) we have the root

$\lambda = \text{-}104131.$

Since $\lambda = \text{-}104130$ lies out the unit cycle this means that the model is unstable .The following figures show the instability of the model for different initial values




Fig(2)the series generated from model(11)

## Conclusions:

In this paper we studied and found the stability conditions of the (AEXPAR) model by using a dynamical approach and by adopting the linear approximation technique.

We find that the proposed model has a stable non-zero singular point if it belongs to the interval [0,1](i.e. $0 < \zeta < 1$) and otherwise it will be unstable, so that the proposed model is stable and it has a limit cycle if all the roots of the characteristic equation lie inside the unit cycle i.e. $-1 < \lambda < 1$.

## Reference

[1] Mohammad A.A., and, Salim A.J.,(2007) *Stability Of Logistic Autoregressive Model.* Qatar university,Sci. J(2007),27,PP(17-28)

[2] Allal,J. and El Melhaoui.S.,(2006).*Optimal Detection Of Exponential Component.* Journal Of Time Series Analysis Vol. 27, No. 6, PP(793-810).

[3] Haggan, V .,and Ozaki,T.,(1981), *Modeling nonlinear random vibrations using an amplitude-dependent autoregressive time series model.* Biometrik Vol.68,No.1,PP(96-189).

[4] Ishizuka,K., Kato,H., and Nakatani,T.,(2005), *Speech Signal Analysis With Exponential Autoregressive Model*, 0-7803-8874-7/05 ©2005 IEEE,ICASSP 2005,PP(225-228).

[5] Meitz M., Saikkonen,P.,(2006). *Stability of nonlinear AR–GARCH models*. SSE/EFI Working Paper Series in Economics and Finance No. 632.

[6] Ozaki,T.,(1982),*The statistical analysis of Perturbed Limit Cycle Processes Using Nonlinear Time Series Models*, Journal of Time Series Analysis,Vol.3,No.1,PP(29-41).

[7] Ozaki,T.,Jimenez,J.C.,Biscay,R.,and Valdes,P.,(1999) *Nonlinear Time series Models And Neural Dynamical System*.

[8] Tong, H.,(1990),*Nonlinear Time series .A Dynamical System Approach*, Oxford University press, New York