Hyperelastic Constitutive Modeling of Rubber and Rubber-Like Materials under Finite Strain

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Abstract

This paper is concerned with determining material parameters for incompressible isotropic hyperelastic strain-energy functions. A systematic procedure analysis is implemented based on the use of least squares optimization method for fitting incompressible isotropic hyperelastic constitutive laws to experimental data from the classical experiments of Treloar [3] on natural rubber. Two phenomenological constitutive models are used to fit the experimental data of natural rubber, these are Mooney-Rivlin and Ogden models. The material parameters using Mooney-Rivlin are obtained using the linear least squares method, while for Ogden model the material coefficients are nonlinear, consequently the nonlinear least squares approach has been used. In this work the nonlinear least squares method with trusted region TD have been used using MATLAB Ver. 7 to find these coefficients. The comparison shows that the present mathematical formulations are correct and valid for modeling rubbery materials. Also it was found that Mooney-Rivlin model is suitable when the deformation is not to exceed 100%, while Ogden model is more appropriate when deformation exceed 100%. In addition, as the degree of non-linearity in material behaviour increases more material coefficients are required.

Keywords: Constitutive laws, Finite deformation, Incompressible materials, Rubber.

التمثيل بالأنموذج المفرط الاستطالـة للمواد المطاطيـــة وشبه المطاطيـة عند الانفعالات الكـبيرة

الخلاصة

يهتم هذا البحث بايجاد معاملات المادة لدوال **الانفعال- الطاقة** للمواد المفرطة الاستطالة اللاانضغاطية. اعتمد التحليل على الاستخدام الامثل لطريقة المربعات الصغرى لاحكام معادلات التصرف المفروضة رياضيا مع تلك المقابلة لها عمليا، حيث ان البيانات العملية اعتمدت بيانات **تريلير** الخاصة بالمطاط الطبيعي [3]. تم في هذا البحث استخدام طريقتيين: طريقة **موني – رفلن** و طريقة **اوجدن**. عند استخدام طريقة **موني – رفلن** تم ايجاد المعاملات الخاصة بالمواد المطاطية اعتمادا على استخدام طريقة المربعات الصغرى الخطية. بينما اعتمدت طريقة المربعات الصغرى اللاخطية عند استخدام طريقة **اوجدن،** حيث تم ايجاد المعملات بهذة الطريقة باستخدام برنامج الـــ MATLAB باعتماد منطقة وثوقية معينة. اظهرت النتائج ان الاشتقاق الرياضي المتبع صحيح وان طريقة **موني – رفلن** مناسبة عندما تكون الاستطالات اقل او تساوي 100%، بينما يفضل استخدام طريقة اوجدن في حالة الاستطالات الاكبر. كذلك تكون الحاجة اكبر لمعاملات اكثر عند ازدياد درجة اللاخطية لتصرف المادة.

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Nomenclature

$e_r, e_q\ e_z$	Strains in *r*, *q*, *z* direction	m/m
$\boldsymbol{\varepsilon}_G, [e_G]$	Green's or Lagrangian strain tensor	m/m
l	Stretch or stretch ratio	m/m
l_1, l_2, l_3	Principal Stretch's	m/m
υ	Poisson's ratio	
σ_o, σ_n	Engineering stress	N/mm^2
σ	True or Cauchy stress	N/mm^2
B	Left Cauchy-Green strain tensor	N/mm^2
C	Right Cauchy-Green strain tensor	N/mm^2
F	Deformation gradient matrix	
f	Displacement derivative matrix	
G	Shear modulus	N/mm^2
I_1, I_2, I_3	Stretch or strain invariants	
J	Determinant of deformation gradient	
J_D	Jacobain matrix	
p	Hydrostatic pressure	N/mm^2
R	Orthogonal rotation tensor	
S_I	1st Piola-Kirchhof stress	N/mm^2
S_{II}	2nd Piola-Kirchhof stress	N/mm^2
u	Displacements components	m
U	Right stretch tensor	m/m
W	Strain energy function	

1. Introduction

Rubber or rubber-like materials, have many engineering applications due to their wide availability and low cost. They are also used because of their excellent damping and energy absorption characteristics, flexibility, resiliency, long service life, ability to seal against moisture, heat, and pressure, and non-toxic. It can be easily molded into almost any shape.

deformation at a certain frequency or over a range of frequencies. Typical examples of this include tires and engine mounts. In this type of

Applications of rubbers include solid propellant, biomechanics, medical/dental, tires, gaskets, and engine mounts.

The ability to accurately predict the mechanical behaviour of rubbery materials is an important technological problem that is still far from being completely understood [1]. In many applications rubber components are subjected to cyclic

applications the mechanical properties are often strongly dependent on the loading conditions such as temperature, frequency, deformation state, and the environment. To properly design new components of rubber material it is therefore of importance to be able to model the material behaviour under different loading conditions.

The unique properties of rubbery materials are such that [2]:

- It can undergo *large deformations* under load.
- Its load-extension behaviour is *markedly nonlinear*.
- Because it is *viscoelastic*, it exhibits significant damping properties.
- It is *incompressible or nearly incompressible*.

The non-linear relationship between stress and strain of rubbers can be obtained from the partial derivative of strain energy functions with respect to strain or stretch. One of the major difficulties encountered by engineers consists in the choice of a well-adapted constitutive model which satisfactorily reproduces the large strain or hyperelastic response of rubbers. Indeed, rubbers exhibit a time-dependent behaviour (creep, relaxation, and hysteresis) and a particular stress-softening phenomenon in the first few cycles, this phenomenon is known as the Mullins effect, Mullins and Tobin [4]. It is the aim of the present work is to clarify the procedure of determining material parameters for incompressible isotropic hyperelastic strain–energy functions using two phenomenological constitutive models, namely Mooney-Rivlin and Ogden models. The mathematical complexities are simplified as possible for this purpose.

2. Hyperelastic Constitutive Modeling

The aim of the constitutive theories is to develop mathematical models for representing the real behaviour of matter. Historically, two approaches have been developed for obtaining the strain energy functions in rubbery materials, or generally, elastomers. The first approach is based on statistical thermodynamic, where the microscopic molecular structure of the material is taken into account. The second is a phenomenological one, which treats the material as a continuum [3].

Constitutive theories are mathematical models for representing the real behaviour of matter. Nonlinear constitutive theory is suitable to model finite strains or hyperelastic materials. Constitutive equations are used to describe the mechanical behaviour of ideal materials by specification of the dependence of stress on kinematical variables such as the deformation gradient, rate of deformation, temperature …etc.

There are several material groups such as elastomers, polymers, foams and biological tissues which can undergo large deformations without permanent set, and hence exhibit large nonlinear elastic behaviour. The nonlinear elastic behaviour under load or prescribed displacement can be modeled using either a physical description of the molecular interplay through theories such as the classical Gaussian theory, slip-link, and macromolecular network theories or by a phenomenological approach [3]. The strain energy expression formulated using a molecular approach is often complex and material specific [5]. In the phenomenological approach, material is treated as a continuum and a strain energy density function is postulated, usually in terms of the deformation invariants, generally strain or stretch invariants. Several material parameters are usually needed to reflect the nonlinearity in the load-stretch relationships. Typically, the load- stretch response for rubber-like materials will display S-shaped behaviour with stiffening at large stretches, as shown in Figure 1. The number of material parameters depends on the level of nonlinearity.

3. Constitutive Equations for Hyperelastic Material

The strain energy function of hyperelastic materials is a scalar-valued function of tensorial variables. For the homogeneous isotropic materials, the strain energy function depends upon only the deformation gradient **F,** $W = W(\mathbf{F})$. The stress tensors of hyperelastic materials in terms of 1st Piola-Kirchhoff stresses are derived from the given strain energy function as [6]:

$$S_I = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} \qquad .....(1)$$

and the Cauchy stress tensor as [7]:

$$[s] = J^{-1}\frac{\partial W}{\partial \mathbf{F}}\mathbf{F}^T \qquad .....(2)$$

where J is the determinant of deformation gradient (or volume ratio). The strain energy function can be written in terms of Cauchy-Green strain tensor **W** (**C**). And alternative expressions for the first and second Piola-Kirchhoff stress tensor may be written as [8]:

$$S_I = 2\mathbf{F}\frac{\partial W(\mathbf{C})}{\partial \mathbf{C}} \qquad ......(3)$$

$$S_{II} = 2\frac{\partial W(\mathbf{C})}{\partial \mathbf{C}} \qquad ......(4)$$

And in terms of Cauchy stress tensor as:

$$[s] = 2J^{-1}\mathbf{F}\left(\frac{\partial W(\mathbf{C})}{\partial \mathbf{C}}\right)^T \mathbf{F}^T \quad ..(5)$$

4. Constitutive Equations in Terms of Principals Invariants

The strain invariants are independent of the chosen coordinate system and can be expressed as functions of the principal stretches λ_1, λ_2, λ_3 [7] it can further be deduced that it is possible to have the chosen coordinate system axes aligned with the principal axes with two of the axes parallel but opposite in direction to two of the principal axes. Therefore in order to always obtain positive strain energy value the strain energy function should be based on the square of the principal stretches, l_1^2, l_2^2, l_3^2 .

The square of the principal stretches l_1^2, l_2^2, l_3^2 are the eigenvalues of the left and right Cauchy-Green tensors **B** and **C** respectively. The non-trivial solutions (eigenvalues) are obtained from the following equation:

$$\left|\mathbf{C} - l_p^2\mathbf{I}\right| = \left|\mathbf{B} - l_p^2\mathbf{I}\right| = 0 \quad .... (6)$$

This leads to the following cubic equation:

$$l_p^6 - I_1 l_p^4 + I_2 l_p^2 - I_3 = 0$$

....(7)

The coefficients of equation (7) are the strain invariants, and are expressed as follow:

$$I_1 = l_1^2 + l_2^2 + l_3^2 = \mathrm{tr}(\mathbf{C})$$

$$I_2 = l_1^2 l_2^2 + l_2^2 l_3^2 + l_3^2 l_1^2 = \frac{1}{2}\left((\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}\mathbf{C}^2\right) \quad \text{....(8)}$$

$$I_3 = l_1^2 l_2^2 l_3^2 = |\mathbf{C}| = J^2$$

The constitutive equation for the isotropic hyperelastic materials may be expressed in terms of stretch invariants as $W = W(I_1, I_1, I_1)$. Hence, equation (4), using chain rule, may be rewritten as:

$$S_{II} = 2\frac{\partial W(\mathbf{C})}{\partial \mathbf{C}} = 2\left(\frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial \mathbf{C}} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial \mathbf{C}} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial \mathbf{C}}\right) \quad \text{....(9)}$$

The derivatives of the stretch invariant with respect to Cauchy-Green strain tensor **C** are as follows:

$$\frac{\partial I_1}{\partial \mathbf{C}} = \frac{\partial \operatorname{tr}(\mathbf{C})}{\partial \mathbf{C}} = \mathbf{I}$$
$$\frac{\partial I_2}{\partial \mathbf{C}} = \operatorname{tr}(\mathbf{CI}) - \frac{1}{2}\frac{\partial \operatorname{tr}(\mathbf{C}^2)}{\partial \mathbf{C}} = I_1\mathbf{I} - \mathbf{C} \quad \text{....(10)}$$
$$\frac{\partial I_3}{\partial \mathbf{C}} = I_3\mathbf{C}^{-1}$$

Substituting equations (10) into equation (9), the second Piola-Kirchhoff stress tensor can be written as:

$$S_{II} = 2\left(\left(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right)\mathbf{I} - \frac{\partial W}{\partial I_2}\mathbf{C} + I_3\frac{\partial W}{\partial I_3}\mathbf{C}^{-1}\right) \quad \text{.....(11)}$$

In similar approach, the Cauchy stress tensor can be expressed as:

$$[s] = 2J^{-1}\left(I_3\frac{\partial W}{\partial I_3}\mathbf{I} + \left(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right)\mathbf{B} - \frac{\partial W}{\partial I_2}\mathbf{B}^2\right) \quad \text{.....(12)}$$

where **B** is the left Cauchy-Green tensor.

5. Incompressible Hyperelastic Materials

Incompressible hyperelastic materials are materials that can sustain finite deformations with approximately no volume changes, and only isochoric motions are possible. For many cases, this is a common idealization and accepted assumption often invoked in continuum and computational mechanics. Incompressible hyperelastic materials are characterized by the incompressibility constraint $J = 1$ or $\det\mathbf{F} = 1$. In order to derive the general constitutive equations for incompressible hyperelastic materials, the strain energy function may be expressed as:

$$W = W(\mathbf{F}) - p(J - 1) \quad \text{... (13)}$$

where the scalar p is an indeterminate Lagrange multiplier which can be identified as a hydrostatic pressure.

A general constitutive equation for the first Piola-Kirchhoff stress tensors is deduced by differentiating equation (13) with respect to the deformation gradient **F** as:

$$S_I = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} - p\frac{\partial J}{\partial \mathbf{F}} = -p\mathbf{F}^{-T} + \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} \quad \text{....(14)}$$

and the second Piola-Kirchhoff stress tensors may be expressed as [8],

$$S_{II} = -p\,\mathbf{F}^{-1}\mathbf{F}^{-T} + \mathbf{F}^{-1}\frac{\partial W(\mathbf{F})}{\partial \mathbf{F}}$$
$$= -p\,\mathbf{C}^{-1} + 2\frac{\partial W(\mathbf{C})}{\partial \mathbf{C}} \quad \text{….(15)}$$

Similarly, for the Cauchy stress tensor, using (14) as:

$$[s] = -p\,\mathbf{I} + \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}}\mathbf{F}^{T} \quad \text{..(16)}$$

6. Constitutive Equations in Terms of Principal Stretches

When the constitutive relationship is expressed in terms of the strain energy density function, **W**, the stress-stretch behaviour is found by differentiation with respect to the stretch. For the case of incompressibility, the principal Cauchy (true) stresses, σ_i are found by differentiating with respect to the principal stretches, λ_i [9]:

$$s_i = l_i \frac{\partial W}{\partial l_i} + p \quad \text{…(17)}$$

where p is the pressure determined by satisfying boundary conditions. If $W = \hat{W}(I_1, I_2)$, this may be written as:

$$s_i = l_i\left[\frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial l_i} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial l_i}\right] + p \quad \text{..(18)}$$

7. Constitutive Models for Hyperelastic Materials

7.1 Mooney–Rivlin Model

The earliest significant phenomenological theory of large elastic deformations, which has played a dominant part in all later work in the field, is that of Mooney [10]. Actually, Mooney's theory was developed in two forms, a special and a general. The theory is based on the following assumptions:

1. The rubber is incompressible, and isotropic in the unstrained state;
2. Hooke's law is obeyed in simple shear. The more general theory is based on an arbitrary, non-linear, stress-strain relation in shear.

On the basis of these assumptions Mooney derived, by purely mathematical arguments involving considerations of symmetry, the strain-energy function

$$W = c_{01}\left(l_1^2 + l_2^2 + l_3^2 - 3\right) + c_{10}\left(\frac{1}{l_1^2} + \frac{1}{l_2^2} + \frac{1}{l_3^2} - 3\right) \quad \text{..(19)}$$

which contains the two elastic constants c_{01} and c_{10}

Rivlin [11] generalized the work of Mooney by putting the strain energy function in terms of strain invariants. He took as his basic assumptions that the material is incompressible and that it is isotropic in the unstrained state. The condition for isotropy requires that the function ***W*** shall be *symmetrical* with respect to the three principal extension ratios l_1^2, l_2^2, and l_3^2. Furthermore, since the strain energy is unaltered by a change of sign of two of the stretch ratio l_i, corresponding to a rotation of the body through 180°, Rivlin argued that the strain-energy function must depend only on the *even* powers of the l_i. The three simplest possible even-powered functions which satisfy these requirements are the following:

$$I_1 = l_1^2 + l_2^2 + l_3^2$$
$$I_2 = l_1^2 l_2^2 + l_2^2 l_3^2 + l_3^2 l_1^2 \qquad \ldots(20)$$
$$I_3 = l_1^2 l_2^2 l_3^2$$

These three expressions, being independent of the particular choice of coordinate axes, are termed *strain invariants*. Any more complex even-powered function of the l_i can always be expressed in terms of these three basic forms.

The condition for incompressibility or constancy of volume during deformation introduces the further relation;

$$I_3 = l_1 l_2 l_3 = 1 \qquad ..(21)$$

which enables the remaining two strain invariants to be written in the form;

$$I_1 = l_1^2 + l_2^2 + l_3^2$$
$$I_2 = 1/l_1^2 + 1/l_2^2 + 1/l_3^2 \qquad \ldots(22)$$

The quantities I_1 and I_2 may be regarded as two independent variables which are determined by the three extension ratios (of which, for an incompressible material, only two are independent). The general Rivlin strain-energy function for an incompressible isotropic elastic material may therefore be expressed as the sum of a series of terms;

$$W = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}(I_1 - 3)^i (I_2 - 3)^j \qquad \ldots.(23)$$

involving powers of $(I_1 - 3)$ and $(I_2 - 3)$. These quantities are chosen in preference to I_1 and I_2 in order that **W** shall vanish automatically at zero strain ($I_1 = I_2 = 3$); for the same reason c_{00}=0.

When only the first term is retained, one obtains;

$$W_{NH} = c_{10}(I_1 - 3) \qquad \ldots(24)$$

which is often called the neo-Hookean model.

7.2 Ogden Model

The Ogden model for incompressible materials formulate the strain energy function in terms of principal stretches l_1, l_2, and l_3. This model has been shown to be of excellent accuracy in spite of a relatively complicated numerical realization [12, 5]. The strain energy function of this model is expressed as:

$$W = \sum_{n=1}^{\infty} \frac{m_n}{a_n}\left(l_1^{a_n} + l_2^{a_n} + l_3^{a_n} - 3\right) \qquad \ldots(25)$$

μ_n are material constants and α_n are dimensionless constants (determined experimentally). For practical purposes the sum in the Ogden model, equation (25), is restricted to a finite number of terms *N*, where *N* is a positive integer, while, for consistency with linear theory the parameter μ denotes the classical shear modulus and material constants μ_n and α_n are related by

$$\sum_{n=1}^{N} m_n a_n = 2m \qquad \ldots.(26)$$

The principal Cauchy stresses corresponding to the strain energy function (25) are of the form

$$s_i = \sum_{n=1}^{N} m_n l_i^{a_n} - p$$
…(27)

where p is an arbitrary hydrostatic stress. The indeterminacy associated with the arbitrary pressure p is a consequence of the assumption of incompressibility and does not appear in the equations for the differences of principal stresses. These are of the form;

$$s_1 - s_2 = \sum_{n=1}^{N} m_n \left(l_1^{a_n} - l_2^{a_n} \right) \quad ..(28)$$

8. Stress Matrix Using Mooney-Rivlin Model

The compressible form of Mooney-Rivlin material model is as [5]:

$$W = c_{10}(\bar{I}_1 - 3) + c_{01}(\bar{I}_2 - 3) + \frac{1}{2}K(J-1)^2 \quad .....(29)$$

The 2nd Piola-Kirchhoff stress can be written as follows [8]:

$$S_{II} = \frac{\partial W}{\partial e_G} = 2\frac{\partial W}{\partial \mathbf{C}} \quad .....(30)$$

Differentiating equation (29),

$$\frac{\partial W}{\partial \mathbf{C}} = c_{10}\frac{\partial \bar{I}_1}{\partial \mathbf{C}} + c_{01}\frac{\partial \bar{I}_2}{\partial \mathbf{C}} + K(J-1)\frac{\partial J}{\partial \mathbf{C}} \quad .... (31)$$

The partial differentials of $\bar{I}_1$ and $\bar{I}_2$ are as follows:

$$\frac{\partial \bar{I}_1}{\partial \mathbf{C}} = I_3^{-\frac{1}{3}}\left(\mathbf{I} - \frac{1}{3}I_1\mathbf{C}^{-1} \right) \quad .... (32)$$

and,

$$\frac{\partial \bar{I}_2}{\partial \mathbf{C}} = I_3^{-\frac{2}{3}}\left(I_1\mathbf{I} - \mathbf{C} - \frac{2}{3}I_2\mathbf{C}^{-1} \right) \quad ...(33)$$

Substitute equations (32) and (33) into (31) gives:

$$\frac{\partial W}{\partial \mathbf{C}} = c_{10}I_3^{-\frac{1}{3}}\left(\mathbf{I} - \frac{1}{3}I_1\mathbf{C}^{-1} \right) + c_{01}I_3^{-\frac{2}{3}}\left(I_1\mathbf{I} - \mathbf{C} - \frac{2}{3}I_2\mathbf{C}^{-1} \right) + \frac{K}{2}(J-1)I_3^{\frac{1}{2}}\mathbf{C}^{-1} \quad ....(34)$$

Rearranging,

$$\frac{\partial W}{\partial \mathbf{C}} = \left(c_{10}I_3^{-\frac{1}{3}} + c_{01}I_3^{-\frac{2}{3}}I_1 \right)\mathbf{I} - c_{01}I_3^{-\frac{2}{3}}\mathbf{C} - \left(\frac{1}{3}c_{10}I_1I_3^{-\frac{1}{3}} + \frac{2}{3}c_{01}I_2I_3^{-\frac{2}{3}} \right)\mathbf{C}^{-1} + \frac{1}{2}KJ(J-1)\mathbf{C}^{-1} \quad .....(35)$$

Hence, the 2nd Piola-Kirchhoff stress may be written as:

$$S_{II} = 2\left(c_{10}I_3^{-\frac{1}{3}} + c_{01}I_3^{-\frac{2}{3}}I_1 \right)\mathbf{I} - 2c_{10}I_3^{-\frac{2}{3}}\mathbf{C} - 2\left(\frac{1}{3}c_{10}I_1I_3^{-\frac{1}{3}} + \frac{2}{3}c_{01}I_2I_3^{-\frac{2}{3}} \right)\mathbf{C}^{-1} + KJ(J-1)\mathbf{C}^{-1}$$

Or as;

$$S_{II} = B_1\mathbf{I} - B_2\mathbf{C} - B_3\mathbf{C}^{-1} + KJ(J-1)\mathbf{C}^{-1} \quad (36)$$

Where:

$$B_1 = 2\left(c_{10}I_3^{-\frac{1}{3}} + c_{01}I_3^{-\frac{2}{3}}I_1 \right)$$

$$B_2 = 2c_{10}I_3^{-\frac{2}{3}}$$

$$B_2 = 2\left(\frac{1}{3}c_{10}I_1I_3^{-\frac{1}{3}} + \frac{2}{3}c_{01}I_2I_3^{-\frac{2}{3}} \right)$$

9. Calibration and Numerical Examples

9.1 Fitting Mooney-Rivlin Material Model to Experimental Data

The incompressible form of Mooney-Rivlin material model is as:

$$W = c_{10}\left(l_1^2 + l_2^2 + l_3^2 - 3\right) + c_{01}\left(\frac{1}{l_1^2} + \frac{1}{l_2^2} + \frac{1}{l_3^2} - 3\right) \quad \ldots(37)$$

For the case of uniaxial tension or uniaxial compression the change in strain energy can be expressed in terms of the work done by external forces:

$$W = f_1\, d l_1 \qquad \ldots..(38)$$

where f_1 is the force acting on the specimen.

Equation (38) can be written in variational form as:

$$dW = \left(\frac{\partial W}{\partial l_1}\right) d l_1 \quad \ldots.(39)$$

Differentiating equation (37) with respect to $d l_1$ gives:

$$\frac{\partial W}{\partial l_1} = 2c_{10}\, l_1 - 2c_{01}\frac{1}{l_1^3} \quad \ldots.(40)$$

For the case of incompressibility, the principal Cauchy (true) stresses, σ_i are given by equation (17), therefore using equation (40), the true stresses can be written as:

$$s_i = 2\left[c_{10}\, l_i^2 - c_{01}\frac{1}{l_i^2}\right] + p \quad \ldots.(41)$$

Thus for simple uniaxial tension and complete incompressibility $l_1 = l$ and $l_2 = l_3 = l^{-1/2}$. For this special case $s_2 = s_3 = 0$, therefore, from equation (41) an expression for the hydrostatic pressure can be obtained as:

$$p = -2\left[c_{10}\frac{1}{l} - c_{01}\, l\right] \quad \ldots.(42)$$

Substituting equation (42) into equation (41) gives:

$$s = 2\left(l^2 - \frac{1}{l}\right)\left[c_{10} + \frac{c_{01}}{l}\right] \ldots. (43)$$

In order to best fit the experimental data to the constitutive model coefficient, the least squares approach has been used, writing equation (43) as [2]:

$$s_i = A_i\, c_{10} + B_i\, c_{01} \qquad \ldots.(44)$$

Where:

$$A_i = 2\left(l_i^2 - \frac{1}{l_i}\right), \qquad B_i = 2\left(l_i - \frac{1}{l_i^2}\right)$$

.

Applying the least squares approach to the error between the empirical test data and the analytical expression given by equation (44), hence:

$$Error = \sum_{i=1}^{N}\left[A_i\, c_{10} + B_i\, c_{01} - s_i\right]^2 \ldots.(45)$$

Therefore:

$$\frac{\partial Error}{\partial c_{10}} = \sum_{i=1}^{N} 2\left[A_i\, c_{10} + B_i\, c_{01} - s_i\right] A_i = 0 \quad \ldots(46)$$

and,

$$\frac{\partial Error}{\partial c_{01}}=\sum_{i=1}^{N}2[A_i c_{10}+B_i c_{01}-s_i]B_i=0 \quad (47)$$

Equation (46) and equation

….(47)

can be written in a matrix form

$$\begin{bmatrix}\sum A_i^2 & \sum A_i B_i \\ \sum A_i B_i & \sum B_i^2\end{bmatrix}\begin{bmatrix}c_{10}\\ c_{01}\end{bmatrix}=\begin{bmatrix}\sum A_i s_i\\ \sum B_i s_i\end{bmatrix} \quad ..(48)$$

Solving for c_{10} and c_{01} leads to:

$$c_{10}=\frac{\sum A_i s_i\cdot\sum B_i^2-\sum B_i s_i\cdot\sum A_i B_i}{\sum A_i^2\cdot\sum B_i^2-\left(\sum A_i B_i\right)^2}$$

....(49)

And,

$$c_{01}=\frac{\sum A_i^2\cdot\sum B_i s_i-\sum A_i s_i\cdot\sum A_i B_i}{\sum A_i^2\cdot\sum B_i^2-\left(\sum A_i B_i\right)^2} \quad ..(50)$$

From the experimental data of Treloar [3], Figure 1, the Mooney-Rivlin coefficient c_{01} and c_{10} can be found using the above mentioned procedure. Fitting these data to that of Treloar simple extension, the coefficients of Mooney-Rivlin model are 0.03168 and 0.03470, respectively. Figure2 shows the plot of both data of Treloar [3] and fitted curve of Mooney-Rivlin model. The maximum error deviation is 8%. When stretch ratio is less than 2, i.e. the extension is less than 100%, the maximum error deviations is less than 0.5 %. Therefore this model is more suitable for small to moderate deformations.

9.2 Fitting Ogden Material Model to Experimental Data

A very sophisticated development for simulating incompressible rubber-like materials in the phenomenological contest is due to Ogden. The postulated strain energy is a function of the principal stretches λ_i, i=1, 2, 3 is computationally simple, and plays a crucial role in the theory of finite elasticity and has the form:

$$W=\sum_{n=1}^{\infty}\frac{m_n}{a_n}\left(l_1^{a_n}+l_2^{a_n}+l_3^{a_n}-3\right)$$

For the case of *simple tension*, let $\lambda_i=\lambda$ be the stretch ratio in the direction of elongation and $\sigma_1=\sigma$ be the corresponding Cauchy stress. Here, $\sigma_2=\sigma_3=0$. From incompressibility constraint, $\lambda_2=\lambda_3=\lambda^{-1/2}$. Hence,

$$s=s_i=l_i\frac{\partial W}{\partial l_i}+p$$

$$=\sum_{n=1}^{N}m_n l^{a_n}+p \quad …. (51)$$

For σ_2 or σ_3

$$0=\sum_{n=1}^{N}m_n l^{-a_n/2}+p$$

Eliminating p,

$$s=\sum_{n=1}^{N}m_n\left(l^{a_n}-l^{-a_n/2}\right) \quad …(52)$$

For the case of *pure shear*, one of the principal extension ratios is fixed, say λ_i=1. Setting, $\lambda_1=\lambda_3=\lambda^{-1}$, therefore,

$$s_1=\sum_{n=1}^{N}m_n l^{a_n}+p,$$

$$s_2=\sum_{n=1}^{N}m_n+p,$$

$$0=\sum_{n=1}^{N}m_n l^{-a_n}+p, \text{ and}$$

$$s = \sum_{n=1}^{N} m_n \left(l^{a_n - 1} - l^{-(a_n + 1)} \right) \quad \ldots$$
(53)

For the case of *equibiaxial tension*, two principal stresses are equal, $\sigma_2=\sigma_3=\sigma$ while the third is zero. Correspondingly, $\lambda_2=\lambda_3=\lambda$, and $\lambda_1=\lambda^{-2}$. As before, elimination of p yields

$$s = \sum_{n=1}^{N} m_n \left(l^{a_n} - l^{-2a_n} \right)]$$
… (54)

Best fitting of the experimental data to Ogden constitutive model are not straight forward, because the coefficients are nonlinear, therefore the nonlinear least squares approach has to be used. In this work MATLAB ver. 7 with trusted region TD has been used to find these coefficients. Hence the resulted fitting is plotted against Treloar experimental work, as shown in Figure 3.

10. Conclusions

Hyperelastic constitutive model predict the mechanical response of rubbery material in the equilibrium state. There are many constitutive models available in literatures; the more suitable constitutive model is that the one which represents the real behaviour of matter under different loading conditions. In this work two constitutive model are used: Mooney-Rivlin and Ogden material constitutive models. From which the following conclusions have been deduced:

1. Using Mooney-Rivlin, the material parameters are linear; therefore, the linear least squares method has been used. While nonlinear least squares approach has been used for Ogden model because the material coefficients are nonlinear.
2. The present mathematical formulations are proved to be correct and valid for modeling rubbery materials behaviour via comparing the results with that of Treloar [3] experimental data on natural rubber. The comparison shows good agreement and the validity of the present formulations have been confirmed.
3. The analysis shows that Mooney-Rivlin model is simple and more suitable when the deformation is not to exceed 100%. Whereas, for more extreme conditions, when deformation exceed 100%, the Ogden model is more suitable.
4. As a final point, the number of required material coefficients (terms) is increased as the degree of non-linearity in material behaviour increases.

11. References

[1]BERGSTROM, J.S., AND BOYCE, M.C., “Constitutive modeling of the large strain time-dependent behavior of elastomers”, *J. Mech. Phys. Solids 46*, 931-954, 1998.

[2]GENT, A.N., *Engineering with Rubber: How to Design Rubber Components*, Hanser Publishers, 1992.

[3]TRELOAR, L. R. G., *The Physics of Rubber Elasticity*, 3rd Edition, Clarendon Press, Oxford, 1975.

[4]MULLINS, L., AND TOBIN, N.R., “Stress softening in rubber vulcanizates. Part I. Use of a strain amplification factor to describe the elastic behavior of filler reinforced vulcanized rubber”, *J. Appl. Polym. Sci. 9*, 2993-3009, 1965.

[5]BOYCE, M.C., ARRUDA, E.M., "Constitutive models of rubber elasticity: A Review", *Rubber Chemistry and Technology*, 73, 504-523, 2000.

[6]GREEN, A.E., *Theoretical Elasticity*, 2nd Edition, Oxford University Press, Ely House, London, 1968.

[7]MASE, G.E., *Continuum Mechanics*, Schaum's Outline series, McGraw-Hill, 1971.

[8] CALVO, B., MARTINEZ, M.A., DOBLARE, M., "On solving large strain hyperelastic problems with the natural element method", *Int. J. Numer. Meth. Engng*, 62: 159-185, 2005.

[9]BAUER, R.F. AND A.H. CROSSLAND. "The Resolution of Elastomer Blend Properties by Stress-Strain Modeling—An Extension of the Model to Carbon-Black-Loaded Elastomers," *Rubber Chemistry and Technology*, Vol. 63, No. 5, pp. 779-791, 1990.

[10MOONEY, M, "A theory of large elastic deformation", *J. Appl. Phys. 11*, 582-592, 1940.

[11]RIVLIN, R.S., "Large elastic deformations of isotropic materials IV. Further developments of the general theory", *Phil. Trans. R. Soc. A241*, 379-397, 1948.

[12]OGDEN, R.W., "Large elastic isotropic elasticity-on the correlation of the theory and experiment for the incompressible rubber-like solids". *Proc. of Roy. Soc. London A326*, 565-584, 1972.

[13]GUO, Z., SLUYS, L.J., "Application of a new constitutive model for the description of rubber-like materials under monotonic loading", *Int. J. of Solids and Struct. 43*, 2799-2819, 2006.

[14]CHAGNON, G., VERRON, E., MARCKMANN, G., and GARNET, L., "Development of new constitutive equations for the Mullins effect in rubber using the network alteration theory", *Int. Journal of Solids and Structures 43*, 6817-6831, 2006.

[15]SHI SHOUXIA AND YANG JIALING, "Large deformation of incompressible rubber cylinder under plane strain," *ACTA Mechanica Solida Sinica*, Vol. 12, No. 4, December 1999.

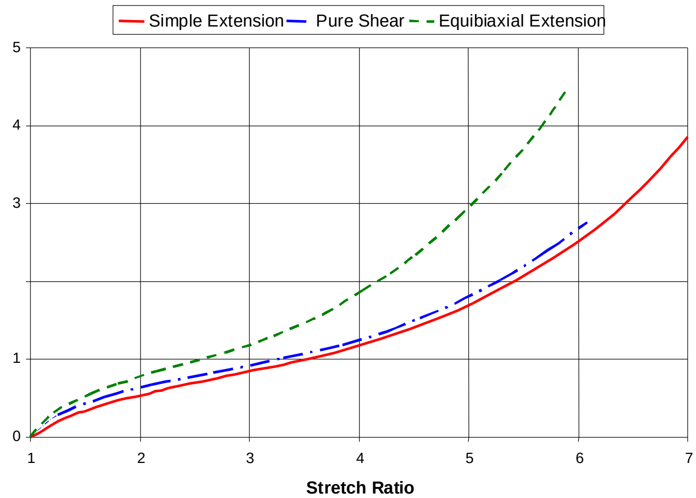

Figure 1: Experimental data for simple extension, equibiaxial extension, and pure shear [3].

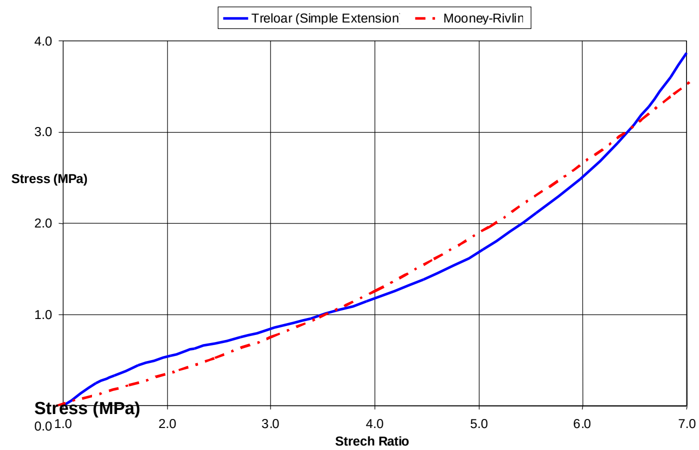

Figure 2: Treloar [3] simple extension and Mooney-Rivlin model.

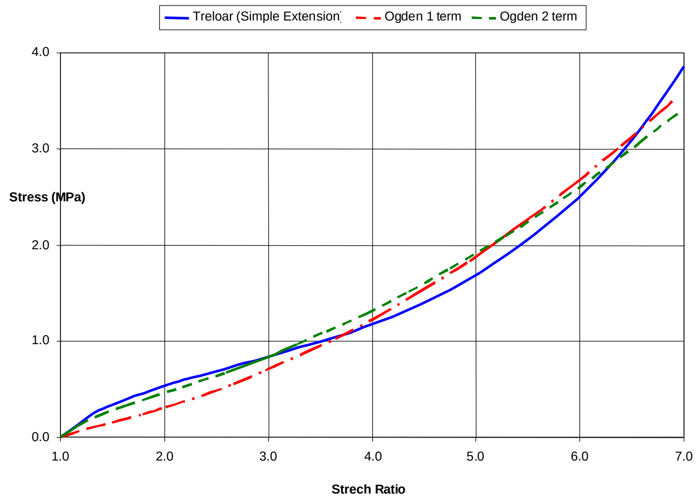

Figure 3: Treloar [3] simple extension and Ogden model.