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Least Squares Method For Solving Integral Equations With Multiple Time Lags

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Abstract

The main purpose of this work is to propose an approximate method to solve integral equation with multiple time lags (IEMTL) namely least squares method with aid of Chebyshev polynomials of (first, second, third, and fourth)kinds. Example is given as an application of least squares method with aid of four kinds of Chebyshev polynomials.

Keywords:-Integral Equation, time lag, Least squares method, Chebychev polynomials

طريقة المربعات الصغرى لحل معادلات تكاملية ذات زمن متعدد التباطؤ

الخلاصة

الغرض الاساسي لهذا العمل هو اقتراح طريقة تقريبية لحل معادلات تكاملية ذات زمن متعدد التباطؤتسمى بطريقة المربعات الصغرى بأستخدام متعددات حدود شبيشيف للانواع (الاولى , الثانية, الثالثة والرابعة).أعطاء مثال كتطبيق لطريقة المربعات الصغرى مع أنواع متعددات حدود شبيشيف

1. Introduction

The most recent kind of equation that worth studying is the delay integral equation. These equations have many applications like: a model to explain the observed periodic out breaks of certain infection diseases [12]. Another application is the electromagnetic inverse scattering problem in a medium with discontinuous change in conductivity [3].

It will be necessary to give simple information about the integral equation with multiple time lags.

2. Integral Equations with Multiple Time Lags [4, 5, 9]

The significance of these equations lies in ability to describe processes with retarded (delay) time which may appear in the unknown function $f(t)$ involved in the integrand or may appear in the unknown function $f(t)$ in the left hand side of the equation or may appear in one of the limits of the integrations.

The integral equation with multiple time lags (which have two lags t_1 and t_2 such that $t_1,\ t_2 \in R, t_1$ and $t_2>0$) having the following cases:-

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- If t_1 appears in the unknown function $f(t)$ inside the integral sign and t_2 appears in the unknown function $f(t)$ out side the integral sign such that

$$h(t)\,f(t-t_2)=g(t)+\int_a^{b(t)}k(t,y)\,f(y-t_1)dy \quad ...(1)$$

- If t_1 appears in the unknown function $f(t)$ inside the integral sign and t_2 appears in one of the limits of integration

$$h(t)\,f(t)=g(t)+\int_a^{t_2}k(t,y)\,f(y-t_1)dy \quad ...(2)$$

or

$$h(t)\,f(t)=g(t)+\int_{t_2}^{b(t)}k(t,y)\,f(y-t_1)dy \quad ...(3)$$

- If t_1 appears in the unknown function $f(t)$ outside the integral sign and t_2 appears in one of the limits of integration:-

$$h(t)f(t-t_1)=g(t)+\int_a^{t_2}k(t,y)\,f(y)dy \quad ...(4)$$

or

$$h(t)\,f(t-t_1)=g(x)+\int_{t_2}^{b(t)}k(t,y)\,f(y)dy \quad ...(5)$$

Remarks [1, 5, 6]:

1. If h(t) =0 these equations are called integral equation with multiple time lags of the first kind.
2. If h(t) =1 the equations are called integral equation with multiple time lags of the second kind
3. If g(t) =0 these equations are called homogenous integral equations with multiple time lags otherwise if g(t)≠0 the equation are called non homogenous integral equation with multiple time lags.
4. If b(t)= t the integral equation is called Volterra integral equation with time lag otherwise when (b(t)= b, b is a constant) it is called Fredholm integral equation with time lag.

3. Chebyshev Polynomials [8, 10]:-

Chebyshev polynomials are of great importance in many area of mathematics particularly approximation theory. Numerous articles and books have been written about this topic. There are several kinds of Chebyshev polynomials. In particular we shall introduce the first, second, third and fourth kind of Chebyshev polynomials.

Some books and many articles use the expression Chebyshev polynomial to refer exclusively to the Chebyshev polynomial $T_n(x)$ of the first kind. Indeed this is by far the most important of the Chebyshev polynomials and when no other qualification is given.

Clearly some definition of Chebyshev polynomials is needed right away so a choice of definitions. However, what gives the various polynomials their power and relevance is their close relationship with the trigonometric functions 'cosine' and 'sine'. It must be a ware of the power of these functions and of their appearance in the description of all kinds of natural phenomena, and this must surely be the key to the versatility of Chebyshev polynomials. It will be necessary to make some primary definitions of these trigonometric relationships.

3.1 The First- Kind Chebyshev Polynomials $T_n(x)$ [2]

The Chebyshev polynomial $T_n(x)$ of the first kind is a polynomial in x of degree n, and rang of the variable x in the interval $[-1,1]$ then it will be defined by the relation.

$$T_n(x)=2xT_{n-1}(x)-T_{n-2}(x) \qquad n=2,3,\ldots \qquad \ldots(6)$$

which may readily show that.

$$T_0(x)=1, T_1(x)=x, T_2(x)=2x^2-1, T_3(x)=4x^3-3x, \quad T_4(x)=8x^4-8x^2+1 \qquad \ldots(7)$$

which together with the initial conditions

$$T_0(x)=1 \qquad .....(8)$$

$$T_1(x)=x \qquad .....(9)$$

3.2 The Second- kind Chebyshev Polynomial $U_n(x)$ [8, 11]

The Chebyshev polynomial $U_n(x)$ of the second kind is a polynomial of degree n in x and range same as for $T_n(x)$ defined by the recurrence

$$\begin{aligned} &U_n(x)=2xU_{n-1}(x)-U_{n-2}(x) \\ &with \quad U_0(x)=1, U_1(x)=2x \end{aligned} \qquad \ldots(10)$$

so we deduce that

$$U_0(x)=1, U_1(x)=2x, U_2(x)$$
$$=4x^2-1, U_3(x)=8x^3-4x$$

3.3 The Third and Fourth –kind Chebyshev Polynomial $V_n(x)$ and $W_n(x)$ (the airfoil polynomial) [8]

Two other families of polynomials $V_n(x)$ and $W_n(x)$ may be constructed, which are related to $T_n(x)$ and $U_n(x)$, but which have trigonometric definitions involving the half angle $q/2$ (where $x=\cos q$ as before).These polynomials are same time referred to as the 'airfoil polynomials', which is rather a appropriately named third and fourth-kind Chebyshev polynomials.

3.3.1 The Chebyshev polynomials $V_n(x)$ and $W_n(x)$ of the third and fourth kinds are polynomials of degree n in x:-

These polynomials too may be efficiently generated by the use of a recurrence relation.

$$\begin{aligned} &V_n(x)=2x\, V_{n-1}(x)-V_{n-2}(x) \\ &with \quad V_0(x)=1, V_1(x)=2x-1 \end{aligned} \qquad \ldots(11)$$

$$\begin{aligned} &W_n(x)=2xW_{n-1}(x)-W_{n-2}(x) \\ &with \quad W_0(x)=1, W_1(x)=2x+1 \end{aligned} \qquad \ldots(12)$$

where n=2,3,4,…

we may readily show that

$$V_0(x)=1,\ V_1(x)=2x-1,\ V_2(x)=4x^2-2x-1,\ V_3(x)$$
$$=8x^3-4x^2-4x+1$$

and

$$W_0(x)=1,\ W_1(x)=2x+1,\ W_2(x)$$
$$=4x^2+2x-1, W_3(x)=8x^3+4x^2-4x-1,\ldots$$

Thus $V_n(x)$ and $W_n(x)$ share precisely the same recurrence relation as $T_n(x)$,$U_n(x)$ and their generation differs only in the prescriptions of the initial condition n=1.

4. Chebyshev Polynomials for the General Range $[a,b]$ [8]

For more generally Chebyshev polynomials may define appropriate to any given finite range $[a,b]$ of x, by making this range correspond to the range $[-1,1]$

of a new variable t under the linear transformation

$$t = \frac{2x-(a+b)}{b-a}$$

4.1 Shifted Chebyshev Polynomials [8]:-

The shifted polynomials $T_n^*(x), U_n^*(x), V_n^*(x), W_n^*(x)$ since the range $[0,1]$ is quite often more convenient to use tan the range $[-1,1]$ we sometimes map the independent variable x in $[0,1]$ to the variable t in $[-1,1]$ by the transformation

$$t = 2x - 1 \quad or \quad x = \tfrac{1}{2}(t+1)$$

and this lead to a shifted polynomial (of the first kind) $T_n^*(x)$ of degree n in x on $[0,1]$ given by

$$T_n^*(x) = T_n(t) = T_n(2x-1) \quad \ldots(13)$$

The recurrence relation for T_n^* may deduce in the form

$$T_n^*(x) = 2(2x-1)T_{n-1}^*(x) - T_{n-2}^*(x) \quad \ldots(14)$$

with the initial conditions

$$T_0^*(x) = 1, \quad T_1^*(x) = 2x - 1 \quad \ldots(15)$$

shifted polynomials $U_n^*(x), V_n^*(x), W_n^*(x)$ of the second, third and fourth kinds may be defined in precisely analogous ways:-

$$U_n^*(x) = U_n(2x-1)$$

$$V_n^*(x) = V_n(2x-1),$$

$$W_n^*(x) = W_n(2x-1) \quad \ldots.(16)$$

5. The Least Squares Method for Solving IEMTL [3, 7]:-

This method is one of the approximate methods used to solve the integral equations without time lag.

This method can be used to find an approximate solution for the integral equations with multiple time lags. To do this, consider the linear Fredholm integral equation with multiple time lags:-

$$f(t-t_1) = g(t) + \int_a^b k(t,y) f(y-t_2)\,dy \quad \ldots(17)$$

This method is based on approximating the unknown function f as:-

- $f(t) \cong \sum_{i=0}^{n} c_i Q_i(t)$

where $Q_i(t)$ is chosen to be one of the four kind of chebyshev polynomials $(T_i(t), U_i(t), V_i(t)$ or $W_i(t)$ by substituting these solutions into equ. (17). One can obtain

$$\sum_{i=0}^{n} c_i Q(t-t_1) - g(t) - \int_a^b k(t,y)\left[\sum_{i=0}^{n} c_i Q(y-t_2)\right]dy = R(t,c_0,c_1,c_2,\ldots,c_n) \quad \ldots(18)$$

let

$$M(c_0,c_1,\ldots,c_n) = \int_a^b \left[R(t,c_0,c_1,\ldots,c_n)\right]^2 w(t)\,dt$$

Where w(t) is any positive function defined on the interval $[a,b]$. It is usually called the weight function. In this work we take $w(t)=1$ for simplicity.

Thus

$$M(c_0,c_1,\ldots,c_n) = \int_a^b \left[\sum_{i=0}^{n} c_i Q(t-t_1) - g(t) - \int_a^b k(t,y)\left[\sum_{i=0}^{n} c_i Q(y-t_2)\right]dy\right]^2 dt$$

So, finding the values of c_i i=0,1,2,…,n which minimize M is equivalent to finding the best approximation for the solution of the integral equation given by eq.(17).

The minimum value of M is obtained by setting:-

$$\frac{\partial M}{\partial c_j} = 0 \text{, j=0,1,2,…,n}$$

$$\int_a^b \{ \sum_{i=0}^{n} c_i Q_i(t-t_1) - g(t) - \int_a^b k(t,y) \left[\sum_{i=0}^{n} c_i Q_i(y-t_2) \right] dy \} \left\{ Q_j(t-t_1) - \int_a^b k(t,y) Q_j(y-t_2) dy \right\} dt = 0$$

; therefore

$$\int_a^b R(t,c_0,c_1,\ldots,c_n) \left\{ Q_j(t-t_1) - \int_a^b k(t,y) Q_j(y-t_2) dy \right\} dt = 0$$

By evaluating the above equation for j=0,1,…,n one can obtain a system of n+1 linear equations with n+1 unknown c_i's. This system can be formed by using matrices form as follows:

A=

$$\begin{bmatrix} \int_a^b R(t,c_0)h_0 dt & \int_a^b R(t,c_1)h_0 dt & \cdots & \int_a^b R(t,c_n)h_0 dt \\ \int_a^b R(t,c_0)h_1 dt & \int_a^b R(t,c_1)h_1 dt & \cdots & \int_a^b R(t,c_n)h_1 dt \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^b R(t,c_0)h_n dt & \int_a^b R(t,c_1)h_n dt & \cdots & \int_a^b R(t,c_n)h_n dt \end{bmatrix}$$

$$, \; b = \begin{bmatrix} \int_a^b g(t) h_0 dt \\ \int_a^b g(t) h_1 dt \\ \vdots \\ \int_a^b g(t) h_n dt \end{bmatrix}$$

where

$$h_j = Q_j(t-t_1) - \int_a^b k(t,y) Q_j(y-t_2) dy$$

j=0,1,2,…,n

and

$$\int_a^b \sum_{i=0}^{n} c_i Q_i(t-t_1) - \int_a^b k(t,y) \left[\sum_{i=0}^{n} c_i Q_i(y-t_2) \right] dy = R(t,c_i)$$

for i=0,1,…,n

By solving the above system using Gauss elimination method to get an approximation solution to the equ.(17).

6. Numerical Example:-

Consider the Volterra integral equation with multiple time lags

$$f(t-t_1) = ((t-1)^2 - \tfrac{1}{4}t(t^4-1) - \tfrac{1}{2}t(t^2-1)) + \int_{t_2}^{t} t y f(y) dy \quad t \in [1,2]$$

Where $t_1 = 1$ *and* $t_2 = 1$ with the exact solution $f(t) = (t^2 + 1)$

Assume that the approximate solution is

1. $f(t) \cong f_2(t) = \sum_{i=0}^{2}{}' c_i T_i(2t-3)$
2. $f(t) \cong f_2(t) = \sum_{i=0}^{2}{}' c_i U_i(2t-3)$
3. $f(t) \cong f_2(t) = \sum_{i=0}^{2}{}' c_i V_i(2t-3)$
4. $f(t) \cong f_2(t) = \sum_{i=0}^{2}{}' c_i W_i(2t-3)$

The least squares method with shifted Chebychev (1^{st}, 2^{nd}, 3^{rd}, 4^{th}) approximation used to solve this problem. Their results are obtained by using matlab program. The following table(1) shows a comparison between the exact and the approximated results by absolute error (Abs.E) and least square error (L.S.E) between the methods.

6. Conclusions

1. The results obtained by using least squares method are very accurate in a comparison with the exact solution of worked examples.
2. If we were asked for a pecking order of these four chebyshev polynomials $T_n(x), U_n(x), V_n(x), W_n(x)$ then we say:-

(a) $T_n(x)$ is clearly the most important and $T_n(x)$ generally leads to simplest formulae, whereas results for the other polynomials may involve slight complications.

(b) All four polynomials have their role. For example $U_n(x)$ is useful in numerical integration, while $V_n(x)$ and $W_n(x)$ can be useful in situations in which singularities occur at one end points $(+1\ or\ -1)$ but not at the other.

3. The results and the required duration of the least squares method with the aid of four kinds of chebyshev polynomials (1st, 2nd, 3rd, 4th) declare that the least square method with aid of (3rd, 4th) kinds gives more accurate results than the others.
4. A disadvantage of the least squares method with aid of four kinds of chebyshev polynomials is dependence on a free parameter (n) which gives the smallest least square error.

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Table (1)

t	Exact	Least Square							
		first Cheby	Abs error	second Cheby	Abs error	third Cheby	Abs error	fourth Cheby	Abs error
1	2	1.9701	0.0299	1.9633	0.0367	2.0067	0.0067	2.000	0.0000
1.1	2.21	2.1819	0.0281	2.1747	0.0353	2.2164	0.0064	2.2101	0.0001
1.2	2.44	2.41399	0.02601	2.4072	0.0328	2.4459	0.0059	2.4400	0.0000
1.3	2.69	2.667	0.023	2.6608	0.0292	2.6952	0.0052	2.6901	0.0001
1.4	2.96	2.9407	0.0193	2.9354	0.0246	2.9643	0.0043	2.9600	0.0000
1.5	3.25	3.2458	0.0042	3.23105	0.01895	3.2533	0.0033	3.2500	0.0000
1.6	3.56	3.5509	0.0091	3.5478	0.0122	3.5620	0.002	3.5601	0.0001
1.7	3.89	3.8873	0.0027	3.8856	0.0044	3.8906	0.0006	3.8900	0.0000
1.8	4.24	4.2446	0.0046	4.2444	0.0044	4.2389	0.0011	4.2401	0.0001
1.9	4.61	4.6227	0.0127	4.6243	0.0143	4.6071	0.0029	4.6100	0.0000
2	5.00	5.0217	0.0217	5.0253	0.0253	4.9951	0.0049	5.0000	0.0000
L.S.E		4.023e-3		6.518e-3		2.15e-4		4E-08	

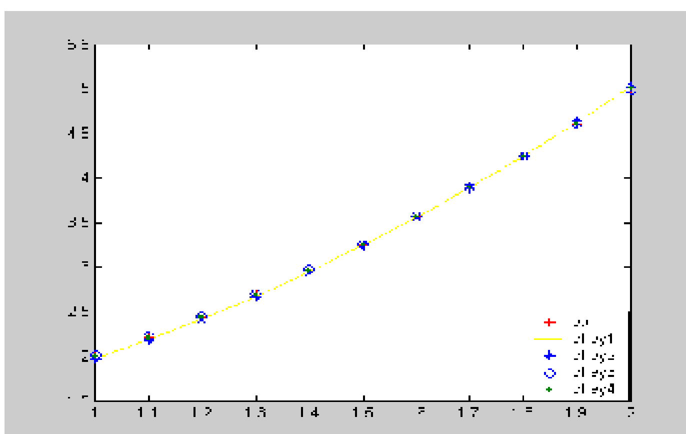

Figure (1) shows the comparison between these results