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Solving Second Order Non-Linear Boundary Value Problems by Four Numerical Methods

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Abstract

The boundary value problems for the 2nd order non-linear ordinary differential equations are solved by using four numerical methods. These numerical methods are Rung-Kutta of 4th order, Rung–Kutta Butcher of 6th order, differential transformation method, and the Homotopy perturbation method. Three physical problems from the literature are solved by the four methods for comparing the results. The results were presented in tables and figures. The differential transformation method appeared to be effective and reliable to find the semi numerical-analytical solutions for such type of boundary value problems.

Keywords: Boundary value problem, Rung-Kutta RK4, Rung–Kutta Butcher, Differential transformation method, Homotopy perturbation method.

حل مسائل القيمة الحدية غير الخطية من المرتبة الثانية باستخدام أربعة طرائق عددية مختلفة

الخلاصة

تم في هذا البحث حل مسائل القيمة الحدية للمعادلات التفاضلية غير الخطية من المرتبة الثانية باستخدام أربعة طرائق عددية مختلفة. إن الطرائق العددية المستخدمة هي طريقة رانج كوتا من المرتبة الرابعة و رانج كوتا- بوتشر من المرتبة السادسة وطريقة التحويل التفاضلي وطريقة الاضطراب الهوموتوبي . تم تطبيق الطرائق الأربعة لحل ثلاثة مسائل تطبيقية ومقارنة النتائج. تم عرض النتائج على شكل جداول ومخططات بيانية. لوحظ إن طريقة التحويل التفاضلي ذات كفاءة ودقة عالية في حل مسائل القيمة الحدية للمعادلات التفاضلية غير الخطية من المرتبة الثانية .

Introduction

Several models of mathematical physics and applied mathematics contain boundary value problems BVPs in the 2nd order non-linear ordinary differential equations (ODEs) [1].

Consider the boundary value problem of 2nd order nonlinear (ODEs):

$$\frac{d^2y}{dx^2}+a\ f(x,y)=0, \qquad 0<x<1 \ \ldots(1)$$

$$y(0)=0\ ,\ y(1)=c \qquad \ldots..(2)$$

where $a\geq 0$, c is constant and $f(x,y)$: $(0,1]\times[0,\infty)\rightarrow(0,1]$,continuous and $\frac{\partial f}{\partial y}\geq 0$ exists and continuous. Different analytical and numerical methods were used to solve the 2nd order (ODEs) this can be concerned Agrawal et al. [2-4], Ha and Lee [5] and O' Regan [6,7] using different techniques.

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In this paper four numerical methods are applied containing: Rung-Kutta (RK4) [8], (RK-Butcher) [9], differential transformation method [10] and homotopy perturbation method [11] to solve BVPs.

Rung-Kutta 4th Order Method

Consider the boundary value problem in Eq.(1) under the condition in Eq.(2) , it will be transformed to two 1st (ODEs) by assuming:

$$\frac{dy}{dx} = p = f_{n1}(x, y, p) \quad .... (3)$$

then Eq.(1) becomes:

$$\frac{dp}{dx} = -a(f(x,y) = f_{n2}(x, y, p) \quad \ldots(4)$$

The solution $y(x)$ for this problem from $x_0 = 0$ to $x_n = 1$ after assuming h and n where:

$$h = \frac{x_n - x_0}{n} \quad \ldots.(5)$$

The set of 1st order (ODEs) (3) and (4) are solved together from the following:

$$y^{n+1} = y^n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$p^{n+1} = p^n + \frac{1}{6}(L_1 + 2L_2 + 2L_3 + L_4)$$

where $k_1, k_2, k_3, k_4, L_1, L_2, L_3, L_4$ are calculated as follows :

$$k_1 = h.f_{n1}(x_n, y_n, p_n)$$
$$L_1 = h.f_{n2}(x_n, y_n, p_n)$$
$$k_2 = h.f_{n1}(x_n + h/2, y_n + k_1/2, p_n + L_1/2)$$
$$L_2 = h.f_{n2}(x_n + h/2, y_n + k_1/2, p_n + L_1/2)$$
$$k_3 = h.f_{n1}(x_n + h/2, y_n + k_2/2, p_n + L_2/2)$$
$$L_3 = h.f_{n2}(x + h/2, y + k_2/2, p + L_2/2)$$
$$k_4 = h.f_{n1}(x_n + h, y_n + k_3, p_n + L_3)$$
$$L_4 = h.f_{n2}(x_n + h, y_n + k_3, p_n + L_3)$$

Rung –Kutta Butcher of 6th order method

The BVPs of second order (ODEs) is solved by Rung – Kutta Butcher of 6th order (RK-Butcher). Applying this method to solve system of 1st order differential equations eqs.(3) and (4) :

$$y^{n+1} = y^n + \frac{h}{90}(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6) \quad \ldots(6)$$

$$p^{n+1} = p^n + \frac{h}{90}(7L_1 + 32L_3 + 12L_4 + 32L_5 + 7L_6) \quad \ldots(7)$$

The Constants $k_1, k_2, k_3, k_4, L_1, L_2, L_3, L_4$ in Eqs.(6) and (7) are calculated from the following :

$$k_1 = f_{n1}(x_n, y_n, p_n)$$
$$L_1 = f_{n2}(x_n, y_n, p_n)$$
$$k_2 = f_{n1}(x_n, y_n, p_n) + \frac{h}{4}L_1$$
$$L_2 = f_{n2}(x_n + \frac{h}{4}, y_n + \frac{hk_1}{4}, p_n + \frac{hL_1}{4})$$
$$k_3 = f_{n1}(x_n, y_n, p_n) + \frac{h}{8}L_1 + \frac{h}{8}L_2$$
$$L_3 = f_{n2}(x_n + \frac{h}{4}, y_n + \frac{hk_1}{8} + \frac{hk_2}{8}, p_n + \frac{hL_1}{8} + \frac{hL_2}{8})$$
$$k_4 = f_{n1}(x_n, y_n, p_n) - \frac{h}{2}L_2 + hL_3$$
$$L_4 = f_{n2}(x_n + \frac{h}{2}, y_n - \frac{hk_2}{2} + hk_3, p_n - \frac{hL_2}{2} + hL_3)$$
$$k_5 = f_{n1}(x_n, y_n, p_n) + \frac{3h}{16}L_1 + \frac{9h}{16}L_4$$
$$L_5 = f_{n2}(x_n + \frac{3h}{4}, y_n + \frac{3hk_1}{16} + \frac{9hk_4}{16}, p_n + \frac{3hL_1}{16} + \frac{9hL_4}{16})$$
$$k_6 = f_{n1}(x_n, y_n, p_n) - \frac{h}{7}(3L_1 - 2L_2 - 12L_3 + 12L_4 - 8L_5)$$

$$L_6 = \begin{cases} f_{n2}[x_n + h, y_n - \frac{h}{7}(3k_1 - 2k_2 - 12k_3 + 12k_4 - 8k_5)], \\ p_n - \frac{h}{7}(3L_1 - 2L_2 - 12L_3 + 12L_4 - 8L_5)] \end{cases}$$

$p_{(0)}$ is predicted from:

$$p^r_{(0)} = [y_{(1)} - y_{(0)}]/h \quad \text{....(8)}$$

Then $p_{(0)}$ is modified by :

$$p^{r+1}_{(0)} = p^r_{(0)} + (y^r_{(1)} - y_{(1)})/h \quad \text{.....(9)}$$

Error is accepted as:

$$\left|y^r_{(1)} - y_{(1)}\right| \le 10^{-5}$$

(or any tolerance needed)

Differential Transformation Method DTM

The differential transformation of the kth derivatives of function $y(x)$ is defined as follows [10]:

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y}{dx^k}\right]_{x=x_0} \quad \text{...(10)}$$

and $y(x)$ is the differential inverse transformation of $Y(k)$ defined as follows:

$$y(x) = \sum_{k=0}^{\infty} Y(k).(x - x_0)^k \quad \text{.... (11)}$$

for finite series of k = N , eq.(11) ca n be written as:

$$y(x) = \sum_{k=0}^{N} Y(k).(x - x_0)^k \quad \text{....(12)}$$

The following theorems that can be deduced from eqs.(10) and (12) [10]:

Theorem 4.1. If $y(x) = g(x) \pm h(x)$,then $Y(k) = G(k) \pm H(k)$.

Theorem 4.2. If $y(x) = a.g(x)$,then $Y(k) = a.G(k)$.

Theorem 4.3. If $y(x) = \frac{dg(x)}{dx}$,then $Y(k) = (k+1).G(k+1)$.

Theorem 4.4. If $y(x) = \frac{d^m g(x)}{dx^m}$,then $Y(k) = ((k+m)!/k!).G(k+m)$.

Theorem 4.5. If $y(x) = g(x).h(x)$,then $Y(k) = \sum_{l=0}^{k} G(l)H(k-l)$.

Theorem 4.6. If $y(x) = x^m$,then $Y(k) = d(k-m) = \begin{cases} 1 & if \quad k = m \\ 0 & if \quad k \ne m \end{cases}$

Theorem 4.7. If $y(x) = \exp(a.x)$,then $Y(k) = a^k / k!$.

Theorem 4.8. If $y(x) = \sin(a.x + l)$,then $Y(k) = (a^k / k!)\sin(kp/2 + l)$.

Theorem 4.9. If $y(x) = \cos(a.x + l)$,then $Y(k) = (a^k / k!)\cos(kp/2 + l)$.

Theorem 4.10. If $y(x) = \exp(u(x))$, then:

$$Y(k) = \sum_{k=0}^{\infty}\sum_{n=0}^{r} \frac{1}{n!} \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k2=0}^{k3}\sum_{k1=0}^{k2} U(k_1)U(k_2 - k_1)\ldots\times U(k_{n-1} - k_{n-2})U(k - k_{n-1})(x - k_0)^k$$

Homotopy Perturbation Method HPM

Consider the following system of the integral equation [11]:

$$F(t) = G(t) + l\int_0^t K(t,s)F(s)ds \quad \text{....(13)}$$

$$F(t) = (f_1(t), f_2(t), \ldots, f_n(t))^T$$

$$G(t) = (g_1(t), g_2(t), \ldots, g_n(t))^T$$

$$K(t,s) = [K_{ij}(t,s)]$$

$$i = 1,2,3, \ldots, n : j = 1,2,3, \ldots, n$$

Let

$$L(y) = 0 \quad \text{....(14)}$$

Where L an integral or differential operator, and then a convex homotopy is $H(y,p)$ defines by:

$$H(y,p)=(1-p)F(y)+pL(y) \qquad \ldots (15)$$

where $F(y)$ is a functional operator with known solution v_0, which can be obtained easily. It is clear that

$$H(y,p)=0 \qquad \ldots(16)$$

From which we have $H(y,0)=F(y)$, and $H(y,1)=L(y)$

This shows that $H(y,p)$ continuously traces an implicitly defined curve from a starting point $H(v_0,0)$ to a solution $H(f,1)$.The embedding parameter increases monotonically from zero to unit at the problem $F(y)=0$ is continuously deforms the original problem $L(y)=0$. The embedding parameter can be considered as an expanding parameter [11].

The homotopy perturbation method uses the homotopy parameter p as an expanding parameter to obtain:

$$y=\sum_{i=0}^{\infty} p^i y_i = y_0 + y_1 p + y_2 p^2 + y_3 p^3 + y_4 p^4 + y_5 p^5 + \ldots.. \qquad \ldots. (17)$$

If $p \to 1$ then (17) corresponds to (12) becomes the approximate solution of the form

$$f=\lim_{p\to 1} y = \sum_{i=0}^{\infty} y_i \qquad \ldots.(18)$$

It is well known that the series (4.7) is convergent for most of the cases and also the rate of convergent is dependent on $L(y)$, see [12]. We assume that problem (10) has a unique solution.

Consider the ith equation of (10), take

$$f_1(t)=\sum_{i=0}^{\infty} p^i y_i \;, \qquad f_2(t)=\sum_{i=0}^{\infty} p^i v_i \;,$$

$$f_3(t)=\sum_{i=0}^{\infty} p^i w_i \;, \qquad \ldots.(19)$$

The comparison of like powers of p gives solution of various orders.

Numerical Examples

Some of the physical problems are solved to assign the effectiveness and accuracy of the four numerical methods. Results are presented in tables and figures. Computer programs are written to implement the procedures of the four numerical methods.

Example 1

Consider the following boundary value problem of 2nd order non-homogenous non-linear (ODEs)[1]:

$$\frac{d^2y}{dx^2}+2(y')^2+8y=0 \qquad 0<x<1 \qquad ..(20)$$

subject to the boundary conditions

$$y(0)=0\;,\; y(1)=0 \qquad .. (21)$$

This problem was studied by Vedat [1] by applying the differential transformation method to eq.(20) then the recurrence relation can be evaluated using theorems 4.1, 4.2, 4.4, and 4.5 as follows:

$$Y(k+2)=-\frac{[8Y(k)+2\sum_{l=0}^{k}(l+1)(k-l+1)Y(l+1)Y(k-l+1)]}{(k+1)(k+2)} \qquad \ldots (22)$$

The boundary conditions in eq.(21) can be transformed at $x_0=0$ as:

$$Y(0)=0, Y(1)=A \qquad (23)$$

where according to eq.(10) , $y'(0)=A$.

For N =8 and by using the recurrence relations in Eq.(22) and the transformed boundary conditions in

Eq.(23), the following series solution up to $O(x^9)$ is obtained :

$$y(x)=Ax-A^2x^2+(-\frac{4A}{3}+\frac{4A^3}{3})x^3+(2A^2-2A^4)x^4$$
$$+(\frac{8A}{15}-\frac{56A^3}{15}+\frac{16A}{5})x^5+(-\frac{88A^2}{45}+\frac{328A^4}{45}-\frac{16A^6}{3})x^6$$
$$+(-\frac{32A}{315}+\frac{1696A^3}{315}-\frac{4544A^5}{315}+\frac{64A^7}{7})x^7$$
$$+(\frac{344A^2}{315}-\frac{1448A^4}{105}+\frac{1808A^6}{63}-16A^8)x^8+O(x^9)$$

…(24)

Applying the boundary conditions at x=1, then A= 1 and equation (24) can be written as:

$$y(x)=x-x^2 \qquad \ldots(25)$$

This was the exact solution of the problem. The same result found by Wazwaz [13].

If we want to solve the BVPs in eq(20) by applying the Homotopy perturbation method, we can rewrite the second order boundary value problem as a system of two differential equations:

$$dy/dx=q(x)$$
$$dq/dx=-2q^2(x)-8y(x) \qquad \ldots..(26)$$

with $y(0)=0$ and $q(0)=A$

This can be written as a system of integral equations:

$$y(x)=0+\int_0^x q(t).dt$$

$$q(x)=A+\int_0^x [-2q^2(t)-8t]dt$$

(27)

Using (16) and (18) for (27) we have:

$$y_0+py_1+p^2y_2+p^3y_3+........=0+p\int_0^x (q_0+pq_1+p^2q_2+p^3q_3+........)dt$$

$$q_0+pq_1+p^2q_2+p^3q_3+........=A+p\int_0^x [-2(q_0+pq_1+p^2q_2+p^3q_3+.....)^2-8t]dt$$

….(28)

Comparing the coefficient of like powers of p, we have

$$p^{(0)}:\begin{cases} y_0=0 \\ q_0=A \end{cases}$$

$$p^{(1)}:\begin{cases} y_1=Ax \\ q_1=-2A^2x-4x^2 \end{cases}$$

, ,

$$p^{(2)}:\begin{cases} y_2=-A^2x^2-\frac{4x^3}{3} \\ q_2=4A^3x^2+\frac{16Ax^3}{3} \end{cases}$$

$$p^{(3)}:\begin{cases} y_3=\frac{4A^3x^3}{3}+\frac{4Ax^4}{3} \\ q_3=-8A^4x^3-\frac{40A^2x^4}{3}-\frac{32x^5}{5} \end{cases}$$

$$p^{(4)}:\begin{cases} y_4=-2A^4x^4-\frac{8A^2x^5}{3}-\frac{16x^6}{15} \\ q_4=16A^5x^4+32A^3x^5+\frac{832Ax^6}{45} \end{cases}$$

$$p^{(5)}:\begin{cases} y_5=\frac{16A^5x^5}{5}+\frac{16A^3x^6}{3}+\frac{832Ax^7}{315}) \\ q_5=..... \end{cases}$$

….(29)

Adding all the terms, (29) gives:

$$y(x)=Ax-A^2x^2+(\frac{4A^3}{3}-\frac{4}{3})x^3+(\frac{4A}{3}-2A^4)x^4$$
$$+(\frac{16A^5}{5}-\frac{8A^2}{3})x^5+O(x^6)$$

… (30)

Using the boundary conditions at x=1, we have $A=0.99998$ then:

$$y(x)=0.99998x-0.99996x^2-0.000079998x^3-0.666533x^4+0.533121x^5 \quad \ldots(31)$$

Results are presented in table (1), while errors of the four numerical methods are summarized in table (2). Figure (1) presents the solution of problem in example (1) by the four numerical methods, RK4, RK-Butcher, DTM, and HPM.

Example 2

Consider the following boundary value problem of 2nd order non-homogenous non-linear (ODEs)[1]:

$$\frac{d^2y(x)}{dx^2}+y^2(x)-x^4-2=0 \qquad 0<x<1 \quad \ldots(32)$$

subject to the boundary conditions

$$y(0)=0\ ,\ y(1)=1 \quad (33)$$

This problem was studied by Vedat [1]. By applying the differential transformation method of eq.(32) using theorems 4.1, 4.2, 4.4, and 4.5 as follows:

$$Y(k+2)=\frac{[-\sum_{l=0}^{k}Y(l)Y(k-l)+d(k-4)+2d(k)]}{(k+1)(k+2)} \quad \ldots..(34)$$

The boundary conditions in eq.(33) can be transformed at $x_0=0$ as:

$$Y(0)=0, Y(1)=A \quad \ldots.(35)$$

where according to Eq.(10) , $y'(0)=A$.
For k = 9 and by using the recurrence relations in Eq.(34) and the transformed boundary conditions in Eq.(35), the following series solution up to $O(x^{10})$ is obtained :

$$y(x)=Ax+x^2-\frac{A^2}{12}x^4-\frac{A}{10}x^5+\frac{A^3}{252}x^7+\frac{11A^2}{1680}x^8+\frac{A}{360}x^9+O(x^{10})$$

(36)

Applying the boundary conditions at x=1, then A= 0 and eq.(36) can be written as:

$$y(x)=x^2 \quad \ldots.(37)$$

Which is the exact of Eq.(32). This result was agreement with Jafari [14].

If we want to solve the BVPs in eq.(32) by applying the Homotopy perturbation method, we can rewrite the second order boundary value problem as a system of two differential equations:

$$dy/dx=q(x)$$
$$dq/dx=-y^2(x)+x^4+2 \quad \ldots.(38)$$

with $y(0)=0$ and $q(0)=A$
This can be written as a system of integral equations:

$$y(x)=0+\int_0^x q(t).dt$$
$$q(x)=A+\int_0^x[-y^2(t)+t^4+2]dt \quad \ldots.(39)$$

Using (16) and (18) for (39) we have:

$$y_0+py_1+p^2y_2+p^3y_3+\ldots.$$
$$=0+p\int_0^x(q_0+pq_1+p^2q_2+p^3q_3+\ldots)\,dt$$
$$q_0+pq_1+p^2q_2+p^3q_3+\ldots.$$
$$=A+p\int_0^x[-(y_0+py_1+p^2y_2+p^3y_3+\ldots)^2+t^4+2]dt \quad \ldots.(40)$$

Comparing the coefficient of like powers of p, we have

$$p^{(0)}:\begin{cases} y_0 = 0 \\ q_0 = A \end{cases}$$

$$p^{(1)}:\begin{cases} y_1 = Ax \\ q_1 = \dfrac{x^5}{5} + 2x \end{cases},$$

$$p^{(2)}:\begin{cases} y_2 = \dfrac{x^6}{30} + x^2 \\ q_2 = 0 \end{cases},$$

$$p^{(3)}:\begin{cases} y_3 = 0 \\ q_3 = -\dfrac{A^2x^3}{3} \end{cases},$$

$$p^{(4)}:\begin{cases} y_4 = -\dfrac{A^2x^4}{12} \\ q_4 = -\dfrac{Ax^8}{24} - \dfrac{Ax^4}{2} \end{cases},$$

$$p^{(5)}:\begin{cases} y_5 = -\dfrac{Ax^9}{216} - \dfrac{Ax^5}{10} \\ q_5 = -\dfrac{x^{13}}{11700} - \dfrac{x^9}{135} - \dfrac{x^5}{5} \end{cases}$$

$$p^{(6)}:\begin{cases} y_6 = -\dfrac{x^{14}}{163800} - \dfrac{x^{10}}{1350} - \dfrac{x^6}{30} \\ q_6 = \ldots \end{cases}$$

…. (41)

Adding all the terms, (41) gives:

$$y(x) = Ax + x^2 - \frac{A^2}{12}x^4 - \frac{A}{10}x^5 - \frac{A}{1080}x^9 - \frac{1}{1350}x^{10} - \frac{1}{163800}x^{14} + O(x^{15})$$

….(42)

Using the boundary conditions at x=1, we have $A = -0.0016616$ then:

$$y(x) = -.00166162x + x^2 - 0.00000023x^4 + 0.000166162x^5 + 0.00000153x^9 - 0.00047x^{10} - 0.0000061x^{14} + O(x^{15})$$

…(43)

By using the special computer programs solutions for the four numerical are evaluated and presented in table (3), while errors of the four numerical methods are summarized in table (4). Figure (2) presents the solution of problem in example (2) by the four numerical methods RK4, RK-Butcher, DTM, and HPM .

Results by the four methods are compatible, so that solution by Homotopy perturbation method in eq(43) is agreed with the exact solution in eq.(37) determined by the differential transformation method.

Example 3

Consider Troesch's problem of 2nd order non-homogenous non-linear (ODEs) [1]:

$$\frac{d^2y(x)}{dx^2} = \sinh(y(x)) \qquad 0 < x < 1$$

…(44)

subject to the boundary conditions

$$y(0) = 0 \,,\; y(1) = 1 \qquad \ldots(45)$$

where:

$$\sinh(y(x)) = \frac{e^y - e^{-y}}{2} \approx y + \frac{1}{3!}y^3 + \frac{1}{5!}y^5$$

then eq(44) became:

$$\frac{d^2y(x)}{dx^2} = y + \frac{1}{3!}y^3 + \frac{1}{5!}y^5$$

….(46)

This problem was studied by Vedat [1] by applying the differential transformation method to eq.(46) using theorems 4.1, 4.2, 4.4, and 4.5 as follows:

$$Y(k+2) = \frac{1}{(k+1)(k+2)} \times [Y(k) + \frac{1}{3!}\sum_{k2=0}^{k}\sum_{k1=0}^{k2} Y(k_1) Y(k_2 - k_1)Y(k - k_2) + \frac{1}{5!}\sum_{k4=0}^{k}\sum_{k3=0}^{k4}\sum_{k2=0}^{k3}\sum_{k1=0}^{k2} Y(k_1) Y(k_2 - k_1)Y(k_3 - k_2)Y(k_4 - k_3)Y(k - k_4)]$$

…(47)

The boundary conditions in eq (45) can be transformed at $x_0 = 0$ as:

$$Y(0) = 0, Y(1) = A \qquad (48)$$

where according to eq (10) , $y'(0) = A$.

For N = 9 and by using the recurrence relations in eq(47) and the transformed boundary conditions in eq.(48), the following series solution up to $O(x^{10})$ is obtained :

$$y(x) = Ax + \frac{A}{3!}x^3 + \frac{(A+A^3)}{5!}x^5 + \frac{(A+11A^3+A^5)}{7!}x^7 + (\frac{9A+102A^3+57A^5}{9!})x^9 + O(x^{11}) \quad ..(49)$$

Applying the boundary conditions at x=1, then A= 0.84524 and equation (49) can be written as:

$$y(x) = 0.84524x + 0.140873x^3 + 0.0120759x^5 + 0.00157126x^7 + 0.000239832x^9 + O(x^{11}) \quad .. (50)$$

If we want to solve the BVPs in eq.(44) by applying the Homotopy perturbation method, we can rewrite the second order boundary value problem as a system of two differential equations:

$$dy/dx = q(x)$$
$$dq/dx = \sinh(y(x))$$
…(51)

With $y(0)=0$ and $q(0)=A$

This can be written as a system of integral equations:

$$y(x) = 0 + \int_0^x q(t).dt$$
$$q(x) = A + \int_0^x [\sinh(y(t))]dt$$
…(52)

If sinh[y(x)] is expanded around $y_0 = 0$, then

$$\sinh(y) = y + \frac{1}{6}y^3 + \frac{1}{120}y^5$$

Using (16) and (18) for (52) we have:

$$y_0 + py_1 + p^2y_2 + p^3y_3 + . \; . \; . \; .$$
$$= 0 + p\int_0^x (q_0 + pq_1 + p^2q_2 + . \; . \; . \; . \;).dt$$
$$q_0 + pq_1 + p^2q_2 + p^3q_3 + . \; . \; . \; .$$
$$= A + p\int_0^x [y + \frac{1}{6}y^3 + \frac{1}{120}y^5]dt \quad .. (53)$$

Comparing the coefficient of like powers of p, we have

$$p^{(0)} : \begin{cases} y_0 = 0 \\ q_0 = A \end{cases}, \quad p^{(1)} : \begin{cases} y_1 = Ax \\ q_1 = 0 \end{cases},$$
$$p^{(2)} : \begin{cases} y_2 = 0 \\ q_2 = \frac{Ax^2}{2} \end{cases}, \quad p^{(3)} : \begin{cases} y_3 = \frac{Ax^3}{6} \\ q_3 = 0 \end{cases}$$
$$p^{(4)} : \begin{cases} y_4 = 0 \\ q_4 = \frac{(A+A^3)x^4}{24} \end{cases},$$
$$p^{(5)} : \begin{cases} y_5 = \frac{(A+A^3)x^5}{120} \\ q_5 = 0 \end{cases} \quad .. (54)$$

Adding all the terms, (54) gives:

$$y(x) = Ax + \frac{A}{6}x^3 + (\frac{A}{120} + \frac{A^3}{20})x^5 + (\frac{A+61A^3}{5040})x^7 + \frac{1}{72}(\frac{A^2}{60} + \frac{13}{180})x^9 + O(x^{11}) \quad (55)$$

Using the boundary conditions at x=1, we have $A = 0.821162$ then:

$$y(x) = 0.821162x + 0.1368603333x^3 + 0.0345287x^5 + 0.00686464x^7 + 0.0011591x^9 + O(x^{11}) \quad .... (56)$$

By using the special computer programs solutions for the four numerical are evaluated and presented in table (5), while errors of the four numerical methods are summarized in table (6). Figure (3) presents the solution of problem in example (3) by the four numerical methods, RK4, RK-Butcher, DTM, and HPM.

Conclusion

In this paper, four numerical methods RK4, RK-Butcher, DTM, and HPM, were applied for finding the solution of a class of two-point nonlinear boundary value problems. It may be concluded that all methods gave the same trend of solutions. It is obvious that all solution in the three examples are positive solutions in $0 < x < 1$. That was appeared from figures(1) - (3), the differential transformation method is an effective tool in finding the semi numerical-analytical solutions to this type of boundary value problems.

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Table (1) Results of the problem in example (1).

X	(DTM)*Exact Solution*	*RK4 solution*	*RK-Butcher solution*	*HPM solution*
0.1	0.09	9.000599E-2	9.000006E-2	0.089972
0.2	0.16	0.1600111	0.1600001	0.1596373
0.3	0.21	0.2100154	0.2100001	0.208596
0.4	0.24	0.2400192	0.2400001	0.236928
0.5	0.25	0.2500225	0.2500001	0.2458333
0.6	0.24	0.2400254	0.2400001	0.238272
0.7	0.21	0.2100277	0.2100001	0.219604
0.8	0.16	0.1600295	0.1600001	0.1638
0.9	.08999992	9.003057E-2	9.000009E-2	0.09212
1.0	0	3.074731E-5	3.607737E-8	3.60773E-5

Table (2) Errors of the four numerical methods of example (1).

X	(DTM)*Exact Solution*	*RK4 error*	*RK-Butcher Error*	*HPM Error*
				0
0.1	0.09	5.990267E-6	5.960464E-8	2.799928E-5
0.2	0.16	1.105666E-5	1.043081E-7	3.626645E-4
0.3	0.21	1.54078E-5	1.192093E-7	1.404002E-3
0.4	0.24	1.92225E-5	1.490116E-7	3.071994E-3
0.5	0.25	2.253056E-5	1.490116E-7	4.166663E-3
0.6	0.24	2.537668E-5	1.490116E-7	1.727998E-3
0.7	0.21	2.773106E-5	1.639128E-7	9.604007E-3
0.8	0.16	2.95341E-5	1.788139E-7	3.822401E-3
0.9	.08999992	3.065169E-5	1.713634E-7	9.622807E-2
1.0	0	3.074731E-5	3.607737E-8	3.607731E-5

Table (3) Results of the problem in example (2).

X	(DTM)*Exact Solution*	*RK4 solution*	*RK-Butcher solution*	*HPM solution*
0.1	0.01	9.999979E-3	0.01	0.0101661
0.2	0.04	3.999992E-2	0.04	0.0403322
0.3	0.09	8.999982E-2	0.09	0.0904980
0.4	0.16	0.1599997	0.16	0.1606626
0.5	0.25	0.2499995	0.25	0.2508220
0.6	0.36	0.3599993	0.36	0.3609615
0.7	0.49	0.489999	0.49	0.4910293
0.8	0.64	0.6399987	0.64	0.6408696
0.9	0.81	0.8099983	0.810001	0.8100682
1.0	1	0.9999979	1	0.9976317

Table (4) Errors of the four numerical methods of example (2).

X	(DTM)*Exact Solution*	*RK4 Error*	*RK-Butcher Error*	*HPM Error*
0.1	0.01	2.142042E-8	0	1.661601E-4
0.2	0.04	8.568168E-8	0	3.322698E-4
0.3	0.09	1.862645E-7	0	4.980639E-4
0.4	0.16	3.427267E-7	0	6.625503E-4
0.5	0.25	5.066395E-7	0	8.219779E-4
0.6	0.36	7.450581E-7	0	9.614825E-4
0.7	0.49	1.072884E-6	5.960464E-8	1.029253E-3
0.8	0.64	1.430511E-6	5.960464E-8	8.69453E-4
0.9	0.81	1.907349E-6	1.192093E-7	6.80089E-5
1.0	1	2.384186E-6	0	2.368569E-3

Table (5) Results of the problem in example (3).

X	(DTM)*Exact Solution*	*RK4 solution*	*RK-Butcher solution*	*HPM solution*
0	0	0	0	0
0.1	8.466499E-2	8.466491E-2	8.466499E-2	8.466525E-2
0.2	0.1701789	0.1701787	0.1701789	0.170187
0.3	0.2574054	0.257405	0.2574053	0.2574677
0.4	0.347239	0.3472379	0.3472382	0.3475057
0.5	0.4406234	0.4406189	0.4406193	0.4414536
0.6	0.5385728	0.5385576	0.5385582	0.5406895
0.7	0.6421996	0.6421563	0.642157	0.6469099
0.8	0.7527484	0.7526405	0.7526415	0.7622502
0.9	0.8716399	0.8714001	0.8714013	0.8894458
1.0	1.000527	1.000043	1.000045	1.032048

Table (6) Errors of the four numerical methods of example (3).

X	(DTM)*Exact Solution*	*RK4 error*	*RK-Butcher Error*	*HPM Error*
0	0	0	0	0
0.1	8.466499E-2	8.195639E-8	0	2.533197E-7
0.2	0.1701789	1.639128E-7	0	8.121133E-6
0.3	0.2574054	3.576279E-7	1.192093E-7	6.234646E-5
0.4	0.347239	1.192093E-6	8.642673E-7	2.66701E-4
0.5	0.4406234	4.529953E-6	4.11272E-6	8.301735E-4
0.6	0.5385728	1.519918E-5	1.466274E-5	2.116621E-3
0.7	0.6421996	4.333258E-5	4.261732E-5	4.710257E-3
0.8	0.7527484	1.078844E-4	1.069307E-4	9.501755E-3
0.9	0.8716399	2.397895E-4	2.385974E-4	1.780593E-2
1.0	1.000527	4.839897E-4	4.825592E-4	3.152049E-2

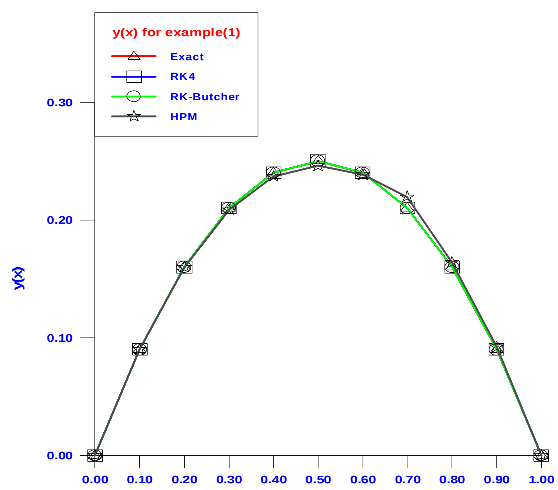

Figure (1) Exact solution of problem in example (1)

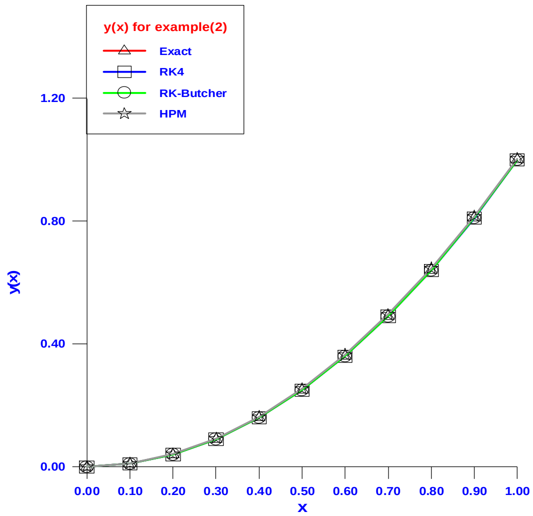

Figure (2) Exact solution of problem in example(2)

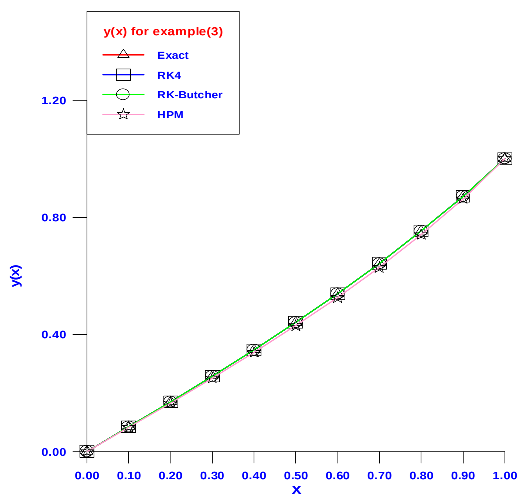

Figure (3) Exact solution of problem in example (3)