Reliability and Effectiveness of the Differential Transformation Method for Solving Linear and Non-Linear Fourth order Boundary Value Problems

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Abstract

In this paper, differential transformation method is applied to construct analytic solutions of the boundary value problems for linear and non-linear 4^{th} order non-homogenous differential equations. The differential transformation method is tested using three physical model problems. Results are presented in tables and figures. It was appeared in comparing results of the differential transformation method with Rung- Kutta , and RK-Butcher solutions that the differential transformation method is more reliable and effective in solving linear and non-linear differential equations.

Keywords: Differential transformation method, differential equations, boundary value problems

موثوقية وكفاءة استخدام طريقة التحويل التفاضلي في حل المعادلات التفاضلية الخطية وغير الخطية من المرتبة الرابعة لمسائل القيمة الحدية

الخلاصة

تم في هذا البحث استخدام طريقة التحويل التفاضلي في حل مسائل القيمة الحدية للمعادلات التفاضلية الخطية وغير الخطية من المرتبة الرابعة. تم تطبيق طريقة التحويل التفاضلي لحل اربعة مسائل تطبيقية ومقارنة النتائج مع الطرائق العددية الأخرى مثل طريقة رانج كوتا من المرتبة الرابعة و رانج كوتا- بوتشر من المرتبة السادسة. تم عرض النتائج على شكل جداول ومخططات بيانية. لوحظ إن طريقة التحويل التفاضلي ذات كفاءة ودقة عالية في حل مسائل القيمة الحدية للمعادلات التفاضلية الخطية وغير الخطية من المرتبة الرابعة.

Introduction

Several models of mathematical physics and applied mathematics contain boundary value problems BVPs in the 4th order linear and nonlinear ordinary differential equations (ODEs) [1].

Consider the non-homogenous 4th order (ODEs) [1]:

$$\frac{d^4y}{dx^4}+f(x)y(x)=g(x), \qquad a<x<b \qquad (1)$$

subject to the following conditions:

$$\left.\begin{aligned} y(a)=a, y^{/}(a)=b \\ y(b)=g, y^{/}(b)=l \end{aligned}\right\} \qquad (2)$$

where a , b , g , l are constants and $f(x), g(x)$ are continuous on [a , b]. Different analytical and numerical methods were used to solve the 4th order (ODEs) this can be concerned by Kapur [1]. Some of the numerical methods applied by Ortner [2] gave

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an approximate solution. Okey [3] used the GEM (Green Element Method) to solve these problems.

In this paper we apply the differential transformation method DTM to solve BVPs with their condition, Eqs. (1) under the conditions (2). Special program is designed to apply the proposal method. Three physical problems are solved using differential transformation method DTM. Results are presented by tables and figures to compare solutions and errors with Rung-Kutta (RK4) [4] and RK-Butcher [5] methods. It seems from comparison errors, that the differential transformation method is more reliable and effective in solving linear than non-linear differential equations.

Differential Transformation Method DTM

The differential transformation of the kth derivatives of function $y(x)$ is defined as follows [6]:

$$Y(k)=\frac{1}{k!}\left[\frac{d^k y}{dx^k}\right]_{x=x_0} \tag{3}$$

and $y(x)$ is the differential inverse transformation of $Y(k)$ defined as follows:

$$y(x)=\sum_{k=0}^{\infty} Y(k).(x-x_0)^k \tag{4}$$

for finite series of k = N , Eq.(4) can be written as:

$$y(x)=\sum_{k=0}^{N} Y(k).(x-x_0)^k \tag{5}$$

The following theorems that can be deduced from Eqs.(3) and (5)[7]:

Theorem 1. If $y(x)=g(x)\pm h(x)$,then $Y(k)=G(k)\pm H(k)$.

Theorem 2. If $y(x)=a.g(x)$,then $Y(k)=a.G(k)$.

Theorem 3. If $y(x)=\frac{dg(x)}{dx}$,then $Y(k)=(k+1).G(k+1)$.

Theorem 4. If $y(x)=\frac{d^m g(x)}{dx^m}$,then $Y(k)=((k+m)!/k!).G(k+m)$.

Theorem 5. If $y(x)=g(x).h(x)$,then $Y(k)=\sum_{l=0}^{k} G(l)H(k-l)$.

Theorem 6. If $y(x)=x^m$,then $Y(k)=d(k-m)=\begin{cases}1 & if \quad k=m\\ 0 & if \quad k\neq m\end{cases}$

.

Theorem 7. If $y(x)=\exp(a.x)$,then $Y(k)=a^k/k!$.

Theorem 8. If $y(x)=\sin(a.x+l)$,then $Y(k)=(a^k/k!)\sin(kp/2+l)$.

Theorem 9. If $y(x)=\cos(a.x+l)$,then $Y(k)=(a^k/k!)\cos(kp/2+l)$.

Numerical applications

Four physical problems of boundary value problems with linear and non-linear fourth order non-homogenous differential equations are solved. Results are presented in tables and figures for comparison solutions and errors between DTM and Exact, RK4, RK-Butcher to assign the effectiveness and accuracy of the Differential transformation method.

Example (1)

Consider the following boundary value problem of 4th order linear (ODEs) :

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$$\frac{d^4y(x)}{dx^4}=y(x)-8.\ x.\ \cos(x)-12\ \sin(x)$$
$$-1<x<1 \tag{6}$$

subject to the boundary conditions:

$$\left.\begin{aligned} y(1)=y(-1)=0 \\ y^{/}(1)=y^{/}(-1)=2.\sin(1) \end{aligned}\right\} \tag{7}$$

This problem was studied by Shahid [8] by using quintic spline as a numerical method. The analytic solution of the given problem is $y=(x^2-1)\sin(x)$. By taking differential transformation of both sides of Eqs.(6) the following recurrence relation is obtained:

$$Y(k+4)=\frac{[Y(k)+8\sum_{l=0}^{k}d(l-1)\frac{1}{k!}Sin(\frac{(k-l)p}{2})+\frac{1}{k!}Cos(\frac{kp}{2})]}{(k+1)(k+2)(k+3)(k+4)} \tag{8}$$

The boundary conditions in Eq.(7) can be transformed at $x_0=0$ as:

$$\left.\begin{aligned} \sum_{k=0}^{N}Y(k)=0\ ,\sum_{k=0}^{N}(-1)^kY(k)=0 \\ \sum_{k=0}^{N}kY(k)=2\sin(1)\ ,\sum_{k=0}^{N}(-1)^{k-1}k.Y(k)=2\sin(1) \end{aligned}\right\} \tag{9}$$

For N =7 and by using the recurrence relations in Eqs.(8) and the transformed boundary conditions in Eqs.(9), the following series solution up to $O(x^8)$ is obtained :

$$y(x)=a_0+a_1.x+a_2.x^2+a_3.x^3+(\frac{a_0}{24})x^4+$$
$$(\frac{a_1}{120}-\frac{1}{6})x^5+(\frac{a_2}{360}).x^6+(\frac{a_3}{840}+\frac{1}{252}).x^7+O(x^8) \tag{10}$$

then from Eqs. (10) a system of equations can be written as follow:

$$-\frac{41}{252}+\frac{25a_0}{24}+\frac{121a_1}{120}+\frac{361a_2}{360}+\frac{841a_3}{840}=0$$

$$+\frac{41}{252}+\frac{25a_0}{24}-\frac{121a_1}{120}+\frac{361a_2}{360}-\frac{841a_3}{840}=0$$

$$-\frac{29}{36}+\frac{a_0}{6}+\frac{25a_1}{24}+\frac{121a_2}{60}+\frac{361a_3}{120}=2.\sin(1)$$

$$-\frac{29}{36}-\frac{a_0}{6}+\frac{25a_1}{24}-\frac{121a_2}{60}+\frac{361a_3}{120}=2.\sin(1)$$

(11)

Solving the system (11), the following constants can be obtained:

$a_0=Y(0)=0,\ a_1=Y(1)=-1.005784$

$a_2=y^{//}(0)/2!=Y(2)=0, a_3=Y(3)=y^{///}(0)/3!=1.175464$

$a_4=Y(4)=y^{(4)}(0)/4!=0,\ a_5=Y(5)=y^{(5)}(0)/5!=-0.175048$

$a_6=Y(6)=y^{(6)}(0)/6!=0,\ a_7=Y(7)=y^{(7)}(0)/7!=0.005367$

Then:

$$y(x)=-1.005784.x+1.175464.x^3-0.1750482.x^5$$
$$+0.005367.x^7+\ldots\ldots \tag{12}$$

Results are summarized in table (1) that represents the comparison between solutions and errors for using DTM, RK4, and RK-Butcher for solving the problem. Figure (1) represents solutions of methods.

Example (2)

Consider the following boundary value problem of 4th order linear (ODEs) :

$$\frac{d^4y(x)}{dx^4}=y^{//}(x)+y(x)+e^x(x-3) \tag{13}$$

subject to the boundary conditions:

$$\left.\begin{aligned} y(0)=1,\quad y^{/}(0)=0 \\ y(1)=0,\quad y^{/}(1)=-e \end{aligned}\right\} \tag{14}$$

This problem was studied by Sayed Tauseef [9] by applying the

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homotopy perturbation method. The analytic solution of the given problem is $y=(1-x)e^x$. By taking differential transformation of both sides of Eq.(3.8) the following recurrence relation is obtained:

$$Y(k+4)=\frac{(k+1)(k+2)Y(k+2)+Y(k)+\sum_{z=0}^{k}(z-1).\frac{1}{(k-z)!}-\frac{3}{k!}}{(k+1)(k+2)(k+3)(k+4)} \tag{15}$$

For N=9, the following definition to y(x) can be obtained:

$$\begin{aligned} y(x) &= a_0+a_1.x+a_2.x^2+a_3.x^3+\frac{1}{4!}(-3+a_0+2a_2)x^4 \\ &+\frac{1}{5!}(-2+a_1+6a_3)x^5+\frac{1}{6!}(-3+4a_2)x^6+\frac{1}{7!}(-2+12a_3)x^7 \\ &+\frac{1}{8!}(-4+6a_2)x^8+\frac{1}{9!}(-4+12a_3)x^9+O(x^{10}) \end{aligned} \tag{16}$$

The boundary conditions in Eq.(14) can be transformed at $x_0=0$ as:

$$\left. Y(0)=1,\ \ Y(1)=0,\ \sum_{k=0}^{N}Y(k)=0,\ \ \sum_{k=0}^{N}(k+1)Y(k+1)=-e \right\} \tag{17}$$

For N = 9, we get the following equations:

$$\begin{aligned} \frac{-324894}{362880} &= \frac{4391}{4032}Y(2)+\frac{38190}{36288}Y(3) \\ \frac{1796}{40320}-e &= \frac{11934}{5040}Y(2)+\frac{131724}{40320}Y(3) \end{aligned} \tag{18}$$

By solving Eqs. (18) then the solution can be obtained :

$$\begin{aligned} y(x) &= 1-0.5000165x^2-0.3333195x^3-0.1250014x^4 \\ &-0.3333263x^5-0.00694453x^6-0.00119044x^7 \\ &-0.0001736136x^8-0.00002204539x^9+O(x^{10}) \end{aligned} \tag{19}$$

Results are summarized in table (2) that represents the comparison between solutions and errors for using DTM, RK4, and RK-Butcher for solving the problem. Figure (2) represents solutions of methods.

Example 3

Consider the boundary value problem of linear 4th order (ODEs) represented by:

$$\frac{d^4y(x)}{dx^4}+4y(x)=1 \qquad -1\le x\le 1 \tag{20}$$

subject to the boundary conditions:

$$\left. \begin{aligned} & y(-1)=y(1)=0 \\ & y'(-1)=-y'(1)=\frac{\sinh(2)-\sin(2)}{4[\cosh(2)+\cos(2)]} \end{aligned} \right\} \tag{21}$$

This problem was studied by [8] by applying the spline method. The exact solution of the given problem is:

$$\begin{aligned} y(x) &= 0.25[1-2[\sin(1)\sinh(1)\sin(x)\sinh(x) \\ &+\cos(1)\cosh(1)\cos(x)\cosh(x)]/(\cos(2)+\cosh(2))] \end{aligned} \tag{22}$$

By taking differential transformation of both sides of Eq.(20) the following recurrence relation is obtained:

$$Y(k+4)=\frac{-4Y(k)+d(k)}{(k+1)(k+2)(k+3)(k+4)} \tag{23}$$

for N=8, the following definition to y(x) can be obtained :

$$\begin{aligned} y(x) &= a_0+a_1x+a_2x^2+a_3x^3+(\frac{1}{24}-\frac{a_0}{6})x^4 \\ &-(\frac{a_1}{30})x^5-(\frac{a_2}{90})x^6-(\frac{a_3}{210})x^7+(\frac{1}{10080}+\frac{a_0}{420})x^8 \\ &+O(x^9) \end{aligned} \tag{24}$$

The boundary conditions in Eq.(22) can be transformed at $x_0=0$ as:

$$\left. \sum_{k=0}^{N}Y(k)=0,\ \sum_{k=0}^{N}(-1)^{k-1}Y(k+1)=0 \right\} \tag{25}$$

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$$\frac{421}{10080}+\frac{351a_0}{420}+\frac{29a_1}{30}+\frac{89a_2}{90}+\frac{219a_3}{210}=0$$

$$\frac{421}{10080}+\frac{351a_0}{420}-\frac{29a_1}{30}+\frac{89a_2}{90}-\frac{219a_3}{210}=0$$

$$+\frac{1}{6}-\frac{2a_0}{3}+\frac{5a_1}{6}+\frac{29a_2}{15}+\frac{89a_3}{30}=-\frac{\sinh(2)-\sin(2)}{4[\cosh(2)+\cos(2)]}$$

$$-\frac{1}{6}+\frac{2a_0}{3}+\frac{5a_1}{6}-\frac{29a_2}{15}+\frac{89a_3}{30}=\frac{\sinh(2)-\sin(2)}{4[\cosh(2)+\cos(2)]}$$

(26)

It seems from Eqs. (26) that $a_1=0$, and $a_3=0$, but $a_0=0$ and $a_2=0$ are calculated from the following equations:

$$\frac{351}{420}a_0+\frac{89}{90}a_2=\frac{-421}{10080}$$

$$\frac{2}{3}a_0-\frac{29}{15}a_2=\frac{1}{6}+\frac{\sinh(2)-\sin(2)}{4[\cosh(2)+\cos(2)]}$$

(27)

by solving Eqs.(27) then :

$a_0 = y(0) = Y(0) = 0.1252121$,

$a_1 = y^{/}(0) = Y(1) = 0$,

$a_2 = y^{//}(0)/2! = Y(2) = -0.1480524$,

$a_3 = Y(3) = y^{///}(0)/3! = 0$,

$a_4 = Y(4) = y^{(4)}(0)/4! = 0.0207979$,

$a_5 = Y(5) = y^{(5)}(0)/5! = 0$

$a_6 = Y(6) = y^{(6)}(0)/6! = 0.001645$,

$a_7 = Y(7) = y^{(7)}(0)/7! = 0$,

$a_8 = Y(8) = y^{(8)}(0)/8! = 0.0003973$

and the solution :

$$y(x)=0.1252121-0.1480524x^2 -0.0207979x^4+0.001645x^6 +0.0003973x^8+.........$$

(28)

Results are summarized in table (3) that represents the comparison between solutions and errors for using DTM, RK4, and RK-Butcher for solving the problem. Figure (3) represents solutions of methods.

Example (4)

Consider the following 4th order non-linear boundary value problem [1]:

$$\frac{d^{(4)}y}{dx^4}=e^{-x}.y^2(x) \qquad 0<x<1$$

(29)

subject to the boundary conditions

$$y(0)=y^{/}(0)=1\ ,\ y(1)=y^{/}(1)=e$$

(30)

The exact solution is given by

$$y(x)=e^x$$

(31)

This problem was studied by [10] by applying the variational iteration decomposition method (VIDM).

By applying the differential transformation method using theorems 1, 2, 4, and 5 to Eq. (29) the recurrence relation can be evaluated as follows:

$$Y(k+4)=\frac{k!}{(k+4)!}[\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{(-1)^{(k-k_2)}}{(k-k_2)!} Y(k_1)Y(k_2-k_1)]$$

(32)

The boundary conditions in Eqs. (30) can be transformed at $x_0=0$ as:

$Y(0)=1, Y(1)=1$ (33)

Then:

$$y(x)=1+x+Y(2)x^2+Y(3)x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5 +\frac{1}{6!}[4Y(2)-1]x^6+\frac{1}{7!}[12Y(3)-1]x^7+O(x^8)$$

(34)

$$\sum_{k=0}^{N}Y(k)=e, \qquad (35)$$

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$$\sum_{k=0}^{N} (k+1)!Y(k+1)/k! = e \qquad (36)$$

For N=8 and by using the recurrence relations in Eq. (32) and the transformed boundary conditions in Eqs. (35) and (36), the following set of equations obtained:

$$\begin{bmatrix} \frac{181}{180} & \frac{421}{420} \\ \frac{61}{30} & \frac{181}{60} \end{bmatrix} \cdot \begin{bmatrix} Y(2) \\ Y(3) \end{bmatrix} = \begin{bmatrix} e-\frac{2581}{1260} \\ e-\frac{863}{720} \end{bmatrix} \qquad (37)$$

By solving Eqs. (37) we get $Y(2) = 0.4998566, Y(3) = 0.1668383$

Then:

$$y(x) = 1 + x + 0.4998566\,x^2 + 0.1668383\,x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + 0.0013881\,x^6 + 0.0001988\,x^7 + O(x^8) \qquad (38)$$

Results are summarized in table (4) that represents the comparison between solutions and errors for using DTM, RK4, and RK-Butcher for solving the problem. Figure (4) represents solutions of methods.

Results and Conclusion

The differential transformation method was studied for solving boundary value problems of 4th order non-homogenous linear and non-linear Differential Equations. Differential transformation method gave good agreements and reliable for solving differential equations.

This result appeared when comparing errors of the differential transformation method with RK4, and RK-Butcher methods, for solving problems in the Numerical Examples. Sometimes errors of DTM are greater than errors of RK4 and RK-Butcher that is due to truncation errors of the required order of the solution.

Finally differential transformation method was an effective and reliable technique in solving Boundary Value Problems of 4th order Differential Equations in the required conditions.

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Table (1) Results of solving the problem in example (1).

X	Exact Solution	DTM (N=9) Solution	DTM Error	RK4 Solution h=0.1	RK4 Error	RK-Butcher Solution h=0.1	RK-Butcher Error
0.1	.9946538	.9946536	1.788139E-07	.9946542	3.576E-07	.9946581	4.351E-06
0.2	.9771222	.9771218	4.172325E-07	.977123	7.748E-07	.9771916	6.967E-05
0.3	.9449012	.9449	1.132488E-06	.9449024	1.251E-06	.94526	3.592E-04
0.4	.8950948	.895093	1.847744E-06	.8950967	1.907E-06	.8962532	1.158E-03
0.5	.8243606	.8243582	2.384186E-06	.8243633	2.682E-06	.8272503	2.890E-03
0.6	.7288475	.7288443	3.159046E-06	.7288511	3.635E-06	.7349727	6.126E-03
0.7	.6041257	.6041222	3.516674E-06	.6041306	4.827E-06	.6157312	1.160E-02
0.8	.4451081	.4451046	3.457069E-06	.4451143	6.258E-06	.4653665	2.026E-02
0.9	.2459601	.2459573	2.846122E-06	.2459681	8.016E-06	.279182	3.322E-02

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Table (2) Results of solving the problem in example (2).

X	Exact Solution	DTM (N=7) Solution	DTM Error	RK4 Solution h=0.1	RK4 Error	RK-Butcher Solution h=0.1	RK-Butcher Error
0.1	-.098835	-.09940	5.66E-04	-.0988333	1.7E-06	-.0988350	9.611E-07
0.2	-.190722	-.191809	1.06E-03	-.1907191	3.4E-06	-.1907226	8.493E-07
0.3	-.268923	-.2704219	1.48E-03	-.2689183	5.1E-06	-.2689234	6.854E-07
0.4	-.327111	-.3288676	1.76E-03	-.3271048	6.6E-06	-.3271115	4.172E-07
0.5	-.359569	-.3613873	1.88E-03	-.3595611	8.0E-06	-.3595693	8.940E-08
0.6	-.361371	-.3630317	1.60E-03	-.361362	9.2E-06	-.3613713	2.980E-07
0.7	-.328551	-.329843	1.21E-03	-.3285408	1.0E-05	-.3285513	7.748E-07
0.8	-.258248	-.2590237	7.75E-04	-.2582373	1.0E-05	-.2582486	1.341E-06
0.9	-.148832	-.1490891	2.5E-04	-.1488207	1.1E-05	-.1488327	2.011E-06

Table(3) Results of solving the problem in example (3).

X	Exact Solution	DTM (N=8) Solution	DTM Error	RK4 Solution h=0.1	RK4 Error	RK-Butcher Solution h=0.1	RK-Butcher Error
0.1	.1239401	.1237337	2.06E-04	0.1239401	7.450581E-9	0.1239401	2.980232E-8
0.2	.1195382	.1193234	2.14E-04	0.1195382	1.490116E-8	0.1195382	6.705523E-8
0.3	.1122857	.1120571	2.28E-04	0.1122857	5.960464E-8	0.1122857	1.043081E-7
0.4	.1023106	.1020631	2.47E-04	0.1023105	1.192093E-7	0.1023106	1.192093E-7
0.5	.0897962	.0895261	2.70E-04	8.979601E-2	2.011657E-7	8.979622E-2	1.490116E-7
0.6	.0749849	.0746920	2.92E-04	7.498469E-2	2.831221E-7	7.498498E-2	1.713634E-7
0.7	.0581836	.0578764	3.07E-04	5.818331E-2	3.762543E-7	0.0581837	1.937151E-7
0.8	.0397692	.0394753	2.93E-04	3.976877E-2	4.731119E-7	3.976926E-2	2.123415E-7
0.9	.0201953	.0199804	2.14E-04	2.019481E-2	5.550683E-7	2.019539E-2	2.253801E-7

Table (4) Results of solving the problem in example (4).

X	Exact Solution	DTM (N=8) Solution	DTM Error	RK4 Solution h=0.1	RK4 Error	RK-Butcher Solution h=0.1	RK-Butcher Error
0.1	1.105171	1.10517	1.192093E-6	1.105171	1.192093E-7	1.105167	5.364418E-6
0.2	1.221403	1.221398	4.410744E-6	1.221403	2.384186E-7	1.221333	7.05719E-5
0.3	1.349859	1.34985	8.46386E-6	1.349858	4.768372E-7	1.3495	3.601313E-4
0.4	1.491825	1.491813	1.192093E-5	1.491824	5.960464E-7	1.490667	1.15943E-3
0.5	1.648721	1.648707	1.430511E-5	1.648721	7.152557E-7	1.645833	2.889514E-3
0.6	1.822119	1.822104	1.478195E-5	1.822118	1.072884E-6	1.816	6.120682E-3
0.7	2.013753	2.013741	1.144409E-5	2.013751	1.430511E-6	2.002167	1.1588E-02
0.8	2.225541	2.225537	4.053116E-6	2.225539	2.145767E-6	2.205333	2.020979E-2
0.9	2.459603	2.459612	8.583069E-6	2.459601	2.622604E-6	2.4265	3.310585E-2

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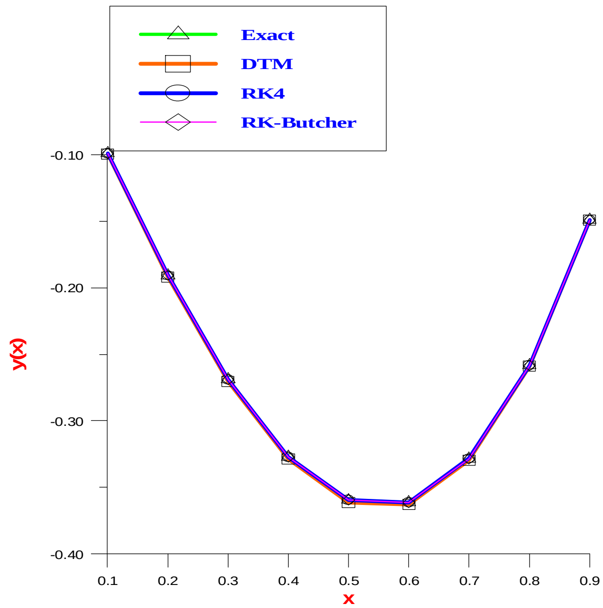
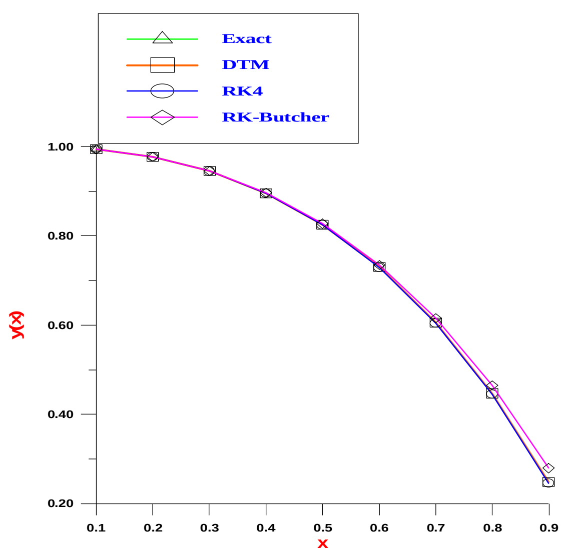


Figure (1) Solutions of methods for example (1)



Figure(2) Solutions of methods for example(2)

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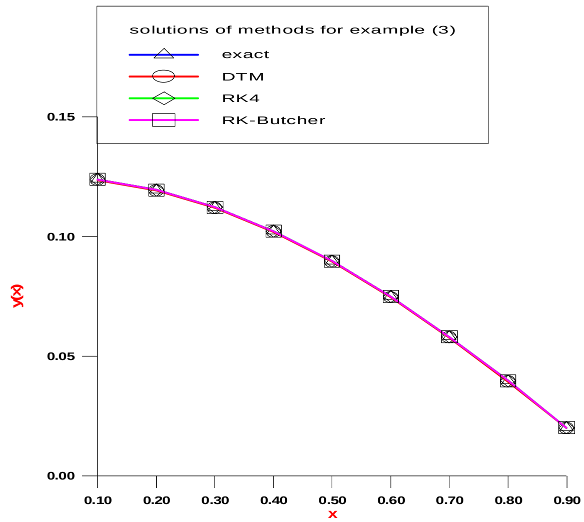

Figure(3) Solutions of methods of example(3)

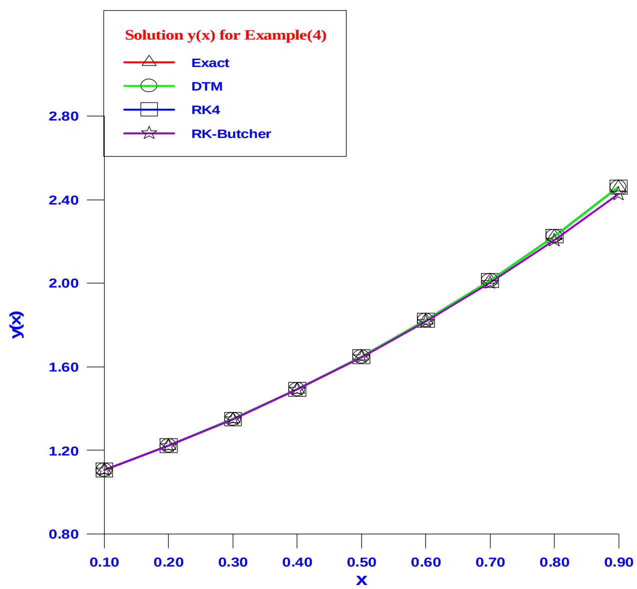

Figure (4) Solutions of methods of example(4)

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