3D Surface Generation Algorithm Using Lagrange Basis Functions in CAD/CAM Application

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Abstract:

The objective of this paper is to create an efficient and accurate 3D surface interior data depending on primary initial data based on Lagrangian interpolation concept. The presented algorithm of 3D surface generation is an extended of the conventional Lagrangian interpolation (1D). The interior data of the designed surface have been transformed automatically to a vertical CNC milling machine (Bridge-port) through the serial port (RS232) to machine the designed 3D surfaces, where the toolpath have been generated based on linear interpolation techniques.
The result of the proposed algorithm have been compared with the well know "Hermit, Bezier and B-Spline" 3D surface generation methods.

المقدمة

يهدف البحث لتوليد سطوح ثلاثية الابعاد بشكل دقيق وكفوء بالاعتماد على بيانات اولية لنقاط السيطرة للسطح وباستخدام مفهوم لاكرانج للاستكمال, ان الخوارزمية المقترحة لتمثيل السطوح ثلاثية الابعاد هي امتداد وتوسيع لمفهوم لاكرانج المستخدم في توصيف وتمثيل المنحنيات(بعد واحد). تم في هذا البحث اشتقاق البيانات الداخلية وتوليد السطوح ثم تم تحويل البيانات التصميمية للسطح الى ماكنة تفريز عمودية مبرمجة بشكل مؤتمت من خلال وسيلة نقل البيانات (RS232)وتشغيل عدة سطوح للتاكد من صلاحية وفعالية الأسلوب المقترح. وللتاكد من موثوقية النتائج تمت مقارنة النتائج للمفهوم المقترح مع النتائج المشتقة بالاعتماد على مفاهيم اخرى لتوليد السطوح الثلاثية الابعاد وهي مبدأ هيرمت, بيزر وسبلاين.

KeyWords

Curve interpolation; Surface interpolation; Curve and surface Approximation; Toolpath Generation.

1- Introduction:-

An important area in computer graphics is concerned with the design of curve and surfaces. Frequently this leads to a data-fitting problem: given a number of points through which a curve or surface is to be fitted.

- Paul Mach (2004) presents a number of different methods to represent curves and surfaces. He involves a relatively small number of control points to describe a potentially very detailed curve or surface.[1]
- Xiaogang Guo (1998) represents Bezier and B-spline for creating curve and surface methods. The definition for surface reconstruction maybe related in terms of parametric form.[2]
- Bert Juttler (2004) present a method for approximate parameterization of a planer algebric curve by a relational Bezier (spline) curve.[3]

2- Approximation Using Polynomial Interpolation:

The methods employed in polynomial approximation fall into two fairly distinct classes [4,5]. In the first the functions to be approximated are known, and can be evaluated, at all points in some range of x.
for example, finding the cubic polynomial which is the best approximation to the function e^x over the interval $-1 \le x \le 1$. There are several possible criteria of best fit, and a considerable amount of theory exists on problems of this type. Much of it involves orthogonal polynomials such as Legendre and Chebyshev polynomials.
In the second class, which apply when the function $y(x)$ to be approximated

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is given in a tabulated form, so that y-values are only available for discrete values of (x). The procedure adopted will depend upon whether: [1]

1- The given points (x_i, y_i) are known to contain statistical or other fluctuations, or

2- They are known to be reliable.

Typical example might be (1) points resulting from some experimental procedure which is subject to errors of measurement, and (2) points taken from standard tables such as log table.

In case (2) it is reasonable to construct an interpolating function which actually passes through all the given points (x_i, y_i). In case (1) this could lead to disastrous results;

an interpolating function might well have the effect of amplifying the statistical fluctuations of the data, when what is really required is that these should be smoothed out. When the data points are unreliable, then, seek an approximating function which passes close to all the points but not necessarily through any of them. Since some curve and surface fitting systems employ a preliminary stage of this kind to smooth out anomalies in the data supplied, illustrating a standard method of approach for case (1) in the following section.[7]

2-1- Lagrange's Technique:

Determining a polynomial which passes through just given points which concern the exact fitting of reliable data in two ways: [8]

2-1-1- (1- D) Interpolation (y=F(x)):

Interpolating polynomial of nth degree of a sequence of planar points defined by $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$ where $x_i < x_j$ for $i < j$ can be calculated as: [9]

$$F_n(x) = \sum_{i=1}^{n} y_1 L_{i,n}(x) \quad .(1).$$

where:

$$L_{i,n}(x) = \frac{(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)} \quad .(2)$$

Short notation for this formula is:

$$F_n(x) = \sum_{i=1}^{n} y_i \prod_{\substack{j=0 \\ j\neq i}}^{n} \left(\frac{x-x_j}{x_i-x_j} \right) \quad \ldots(3)$$

where $\prod$ denotes multiplication of the n factors obtained by varying j from (o to n), excluding $i = j$, it can be observed that the function multiplying y_i is equal to unit when $x = x_i$ but becomes zero when (x) equals any of the other coordinates. [10]

The equation of a line between two points is obtained by setting n=2:

$$F_i(x) = \frac{x-x_2}{x_1-x_2} * y_1 - \frac{x-x_1}{x_2-x_1} * y_2$$

$$y2 = y_1 + (y_2 - y_1)\left(\frac{x_1 - x}{x_1 - x_2} \right) \ldots(4)$$

The Lagrange polynomial has the disadvantage of having the degree of the polynomial tied to the number of points used. The number of points used is increased. The result is polynomial of higher degree subject.

At this work a block diagram Figure (1) and program have been proposed to solve classical Lagrange interpolation problem.

After computing the control points data through lagrangian interpolation the curve equation can be derived as follows:- [13]

Assume $p_1 = (1,\ 1)$ $p_2 = (2,\ 3)$ $p_3 = (3, 1)$

$$L_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$$

$$L_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$$

$$L_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} \quad \ldots(5)$$

$$F(x) = L_1 * y_1 + L_2 * y_2 + L_3 * y_3$$

$$F(x) = -2x^2 + 8x - 5 = y(x)$$

This 2nd degree curve is represented in Figure (2).

Curve with 3rd degree can be presented using Lagrange technique through the assumption that the curve pass through four points as an example:

$p_1 = (1, 1)$ $p_2 = (2, 3)$ $p_3 = (3, 2)$

$p_4 = (4, 3)$ the curve equation can be derived as before [5]:-

$$F(x) = 0.833x^3 - 6.5x^2 + 15.67x - 9 = y(x)$$

The curve representation is illustrated in Figure (3)

2-2-2- (2-D) Interpolation (z=F(x,y)) :

In this work adopted method (2D Lagrange interpolation) of surface data generation has been implemented, tested and evaluated.[10]

The adopted method is an extension to the earlier interpolation method (1D) but by computing both (x) and (y) data to drive the (z) data as in general form of $z = F(x, y)$

The mathematical solution steps of the proposed method can be formulated as follows:

Step-1- Considering a sequence of planar points defined by:-

$(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, (x_n, y_n, z_n)$

where $(x_i, y_k) < (x_j, y_l)$

for $i < j, k < l$.

Step-2- The interpolating polynomial of nth, mth degree can be formulated as:

$$F(x,y) = \sum_{i=1}^{n} L_i(x) * \sum_{k=1}^{m} L_k(y) * F(x_i, y_k)$$

$$L_i(x) = \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{x - x_j}{x_i - x_j} \quad ..(6)$$

$$L_k(y) = \sum_{\substack{k=1 \\ k \neq l}}^{m} \frac{y - y_l}{y_k - y_l} \quad ..(7)$$

Step-3- The determination of Lagrange coefficients in both direction (x,y) can be invested to determine the interior data of the desired surface depending on a few initial data.[10]

Figure (4) illustrates the Lagrangian coefficients $(L(x), L(y))$ of 3D surface. While The block diagram of the adopted technique illustrate in figure (5). [14,15]

In more details the flow chart of the proposed technique illustrated in Figure (6) explain each step of the program that have been built to generate the interior data of the desired 3D surface while this program was linked with Matlab (V.6.5) software to represent the desired 3D surfaces in graphical mode.

3- Testing the Proposed Technique:

Based on the data of initial control points it's able to generate the interior data of the 3D surface and the surface can be represented, with the aid of (Matlab V.6.5) soft-ware.

The following examples have been solved as a test to the adopted technique.

Example (1):
Suppose the initial control points of a desired surface are:-

$p_1 = (1,2,7)$, $p_2 = (1,3,10)$,
$p_3 = (1,4,13)$, $p_4 = (3,2,15)$,
$p_5 = (3,3,18)$, $p_6 = (3,4,21)$,
$p_7 = (5,2,31)$, $p_8 = (5,3,34)$,
$p_9 = (5,4,37)$

Solution steps

- Rearranging the control points in matrix form as follows:

$p_{11} = (1,2,7)$ $p_{12} = (1,3,10)$
$p_{13} = (1,4,13)$ $p_{21} = (3,2,15)$
$p_{22} = (3,3,18)$ $p_{23} = (3,4,21)$
$p_{31} = (5,2,31)$ $p_{32} = (5,3,34)$
$p_{33} = (5,4,37)$

- Determined the Lagrangian coefficient in both (x) and (y) direction

$$L_i(x) = \sum_{\substack{i=1 \\ i \neq j}}^{3} \frac{x - x_j}{x_i - x_j},$$

$$L_k(y) = \sum_{\substack{k=1 \\ k \neq l}}^{3} \frac{y - y_l}{y_k - y_l}$$

- Determined of the surface equation

$$F(x,y) = \sum_{i=1}^{3} L_i(x) * \sum_{k=1}^{3} L_k(y) * F(x_i, y_k)$$

Because the control point matrix is 3×3 then the generated surface is of the 2nd degree that mean the surface equation will be (n-1) degree of the control points.

The mathematical representation of the desired surface can be derived:

$(z = x^2 + 3y)$

With the aid of Matlab software, the graphical mode of the desired surface can be presented as shown in Fig. (6).

Example (2)
Suppose the initial control points of a desired surface are:-

$p_1 = (1,1,0)$, $p_2 = (1,2,0)$,
$p_3 = (1,3,0)$, $p_4 = (1,4,0)$,
$p_5 = (2,1,0)$, $p_6 = (2,2,2)$,
$p_7 = (2,3,2)$, $p_8 = (2,4,0)$,
$p_9 = (3,1,0)$, $p_{10} = (3,2,2)$,
$p_{11} = (3,3,2)$, $p_{12} = (3,4,0)$,
$p_{13} = (4,1,0)$, $p_{14} = (4,2,0)$,
$p_{15} = (4,3,0)$, $p_{16} = (4,4,0)$:

Similar to previous solution steps by arranging the control points in a matrix form then determining Lagrangian coefficient in both direction (x) and (y) then the surface equation can be determined:

$p_{11} = (1,1,0)$ $p_{12} = (1,2,0)$
$p_{13} = (1,3,0)$
$p_{14} = (1,4,0)$ $p_{21} = (2,1,0)$
$p_{22} = (2,2,2)$
$p_{23} = (2,3,2)$ $p_{24} = (2,4,0)$
$p_{31} = (3,1,0)$
$p_{32} = (3,2,2)$ $p_{33} = (3,3,2)$
$p_{34} = (3,4,0)$
$p_{41} = (4,1,0)$ $p_{42} = (4,2,0)$
$p_{43} = (4,3,0)$ $p_{44} = (4,4,0)$

$$L_i(x) = \sum_{\substack{i=1 \\ i \neq j}}^{4} \frac{x - x_j}{x_i - x_j}$$

$$L_k(y) = \sum_{\substack{k=1 \\ k \neq l}}^{4} \frac{y - y_l}{y_k - y_l}$$

$$F(x,y) = \sum_{i=1}^{4} L_i(x) * \sum_{k=1}^{4} L_k(y) * F(x_i, y_k)$$

The derived mathematical equation of the desired surface is:

$(z = 0.5x^2y^2 - 2.5xy^2 + 2y^2 - 2.5x^2y + 12.5xy - 10y + 2x^2 - 10x + 8)$

While the desired surface can be represented in graphical mode as shown in Figure (7)

Example (3)

Suppose the initial control points of a desired surface are:-

$p_1 = (1,1,1)$, $p_2 = (1,2,2)$,
$p_3 = (1,3,1)$, $p_4 = (1,4,1)$,
$p_5 = (2,1,1)$, $p_6 = (2,2,2)$,
$p_7 = (2,3,1)$, $p_8 = (2,4,2)$,
$p_9 = (3,1,1)$, $p_{10} = (3,2,1)$,
$p_{11} = (3,3,1)$, $p_{12} = (3,4,2)$,
$p_{13} = (4,1,1)$, $p_{14} = (4,2,1)$,
$p_{15} = (4,3,1)$, $p_{16} = (4,4,1)$

In similar manner

Rearranging the initial control points in matrix form:

$p_{11} = (1,1,1)$ $p_{12} = (1,2,2)$
$p_{13} = (1,3,1)$ $p_{14} = (1,4,1)$
$p_{21} = (2,1,1)$ $p_{22} = (2,2,2)$
$p_{23} = (2,3,1)$ $p_{24} = (2,4,2)$
$p_{31} = (3,1,1)$ $p_{32} = (3,2,1)$
$p_{33} = (3,3,1)$ $p_{34} = (3,4,2)$
$p_{41} = (4,1,1)$ $p_{42} = (4,2,1)$
$p_{43} = (4,3,1)$ $p_{44} = (4,4,1)$

Determined Lagrangian coefficients:

$$L_i(x) = \sum_{\substack{i=1 \\ i \neq j}}^{4} \frac{x - x_j}{x_i - x_j},$$

$$L_k(y) = \sum_{\substack{k=1 \\ k \neq l}}^{4} \frac{y - y_l}{y_k - y_l}$$

The surface equation can be determined

$$F(x,y) = \sum_{i=1}^{4} L_i(x) * \sum_{k=1}^{4} L_k(y) * F(x_i, y_k)$$

The derived mathematical representation of the desired surface:

$(z = 0.1667x^3y^3 - 1.333x^3y^2 + 3.1667x^3y - 2x^3 - 1.333x^2y^3 + 3xy^3 - 1.333y^3 + 10.5x^2y^2 - 24.667x^2y + 15.5x^2 - 23.1667xy + 10y^2 + 53.667xy - 33.5x - 22.667y + 15)$

While the desired surface can be represented in graphical mode as shown in Figure (8).

Example (4)

Suppose the initial control points of a desired surface are:-

$p_1 = (1,1,1)$ $p_2 = (1,2,1.06)$,
$p_3 = (1,3,1.13)$, $p_4 = (1,4,1.15)$,
$p_5 = (1,5,1.06)$, $p_6 = (2,1,1.04)$
$p_7 = (2,2,2.63)$, $p_8 = (2,3,2.64)$,
$p_9 = (2,4,2.29)$, $p_{10} = (2,5,2.81)$,
$p_{11} = (3,1,1.05)$, $p_{12} = (3,2,3.15)$,
$p_{13} = (3,3,3.14)$, $p_{14} = (3,4,2.67)$,
$p_{15} = (3,5,3.39)$, $p_{16} = (4,1,1.04)$,
$p_{17} = (4,2,1.06)$, $p_{18} = (4,3,1.13)$,
$p_{19} = (4,4,1.15)$, $p_{20} = (4,5,1.06)$,
$p_{21} = (5,1,1)$, $p_{22} = (5,2,1.06)$,
$p_{23} = (5,3,1.13)$, $p_{24} = (5,4,1.15)$,
$p_{25} = (5,5,1.06)$

In similar manner

Rearranging the initial control points in matrix form:

$p_{11} = (1,1,1)$ $p_{12} = (1,2,1.06)$
$p_{13} = (1,3,1.13)$ $p_{14} = (1,4,1.15)$
$p_{15} = (1,5,1.06)$
$p_{21} = (2,1,1.04)$ $p_{22} = (2,2,2.63)$
$p_{23} = (2,3,2.64)$ $p_{24} = (2,4,2.29)$
$p_{25} = (2,5,2.81)$
$p_{31} = (3,1,1.05)$ $p_{32} = (3,2,3.15)$
$p_{33} = (3,3,3.14)$ $p_{34} = (3,4,2.67)$
$p_{35} = (3,5,3.39)$

$p_{41} = (4,1,1.04)$ $p_{42} = (4,2,1.06)$
$p_{43} = (4,3,1.13)$ $p_{44} = (4,4,1.15)$
$p_{45} = (4,5,1.06)$
$p_{51} = (5,1,1)$ $p_{52} = (5,2,1.06)$
$p_{53} = (5,3,1.13)$ $p_{54} = (5,4,1.15)$
$p_{55} = (5,5,1.06)$

Determined Lagrangian coefficients:

$$L_i(x) = \sum_{\substack{i=1 \\ i \neq j}}^{5} \frac{x - x_j}{x_i - x_j}, \quad L_k(y) = \sum_{\substack{k=1 \\ k \neq l}}^{5} \frac{y - y_l}{y_k - y_l}$$

The surface equation can be determined

$$F(x,y) = \sum_{i=1}^{5} L_i(x) * \sum_{k=1}^{5} L_k(y) * F(x_i, y_k)$$

The derived equation representation of the desired surface is:

$$(z = -0.0357x^2y^3 + 0.214xy^3 - 0.183y^3 - 0.0357x^2y^3 + 0.214xy^3 - 0.183y^3 + 0.694x^2y^2 - 2.09x^2y + 1.457x^2 - 4.1632xy^2 + 3.52y^2 + 1255xy - 8.74x - 10.5y + 8.28)$$

This surface can be represented in graphical mode as shown in Figure (9).

The interior data of the desired surfaces that generated from the proposal technique have been compared with three techniques which are:-

- Surface data of Bezier technique.
- Surface data of B-Spline technique.
- Surface data of Hermite technique.

For each comparison the following parameters have been taken:-

- Deviation
- Deviation rate
- Maximum and minimum Deviation and location
- Deviation percentage

z_a =value of (z-dir.) in approximation or interpolation

z_f = value of (z-dir.) in function.

Deviation= z_f - z_a

Deviation rate= z_a / z_f

Deviation percentage = (deviation / number of points)*100%

The interior data of Lagrange surface compared with interior data of surfaces generated according to (Hermite, Bezier, B-Spline techniques) are illustrated in Figure (10), while the results of the comparison of these surfaces are presented in Table (1).

4-LagrangianTechnique Limitations:

Although the proposed technique gives good results but there are some limitations which are:-

1. The value of (x) in (1D) must be increased or decreased $x_1 < x < x_n$ or $x_1 > x > x_n$ and the value of x and y in (2D) must be increase or decrease.
2. The error increases when the incremental value in x-direction (Δx) according to the incremental value in y-direction (Δy) is too large and the contrary is right.
3. The order of the Lagrange's equation depends on the number of control points.
4. When the degree of the Lagrange equations is more than 6th order the Z-deviation of surface will be increased greatly. As shown in Figure (11).

5- 3D Surface Machining:

The technique has been tested through designing several different surfaces depending on a few data of control points, the design have been implemented to machined two surfaces through a vertical CNC milling machine where the design data have been automatically transferred from the PC to the CNC machine through [RS 232 serial port].

The input data as a wireframe have been manipulated to generate the interior data of the surface. Then, the surfaces have been represented and the toolpath generated then this data are transformed to the CNC machine tool.

The tool path and the machined surface are illustrated in Fig. (12, 13, 14, 15).

6- Conclusions

It has been proved that the interior data of 3D surfaces can be successfully derived depending on a few control points using Lagrange techniques then the desired surface can be constructed and represented were the adopted technique coded in graphic software like Matlab that used at this paper.

1. Adopted 2D Lagrangian technique not only successfully derived the interior data of the 3D surfaces but define the desired surfaces according to its mathematical equation and this play significant rule in reducing the part program when machining that surface using variable programming techniques.
2. The range of similarity of the adopted 2D Lagrange techniques is higher then any similarity concluded with the other techniques.
3. Adopted techniques give the best results in construction of 3D surfaces but it is not active for all engineering applications because of its limitation of the basic rule that $x_0 < x_1 < x_2$ and $y_0 < y_1 < y_2$.
4. The RS-232 serial interfacing was successfully used in this research to automate the data transmission between the PC and CNC vertical milling machine "Bridge port CNC milling machine".
5. Although the resolution of the machined models is limited "depended on the side step and forward step" the developed technique proved sufficient success through all the stages.

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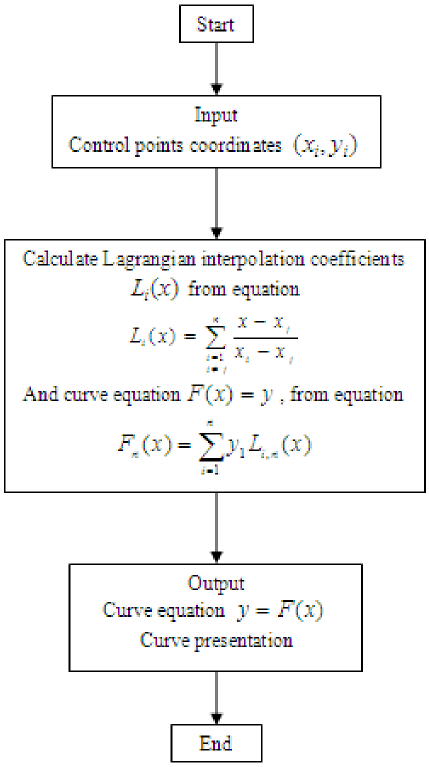


Figure (1): Block diagram of 1D Lagrange interpolation method

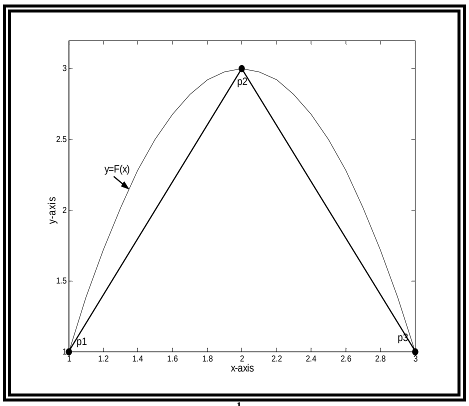


Figure (2): 2nd degree curve representation

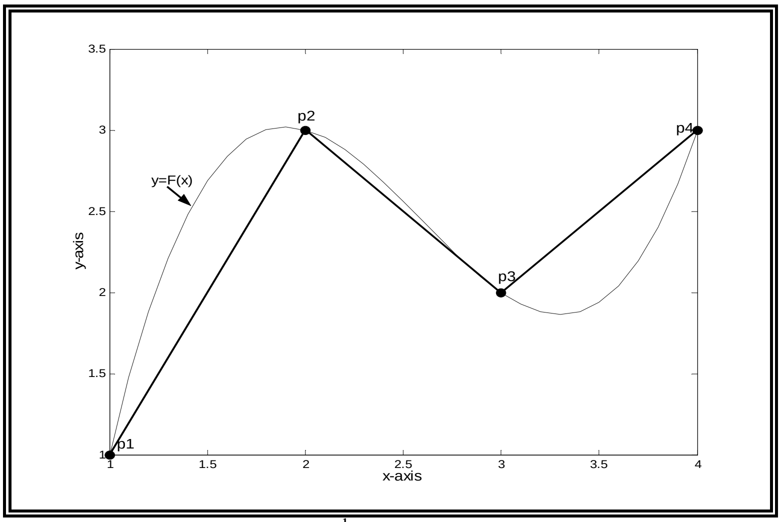
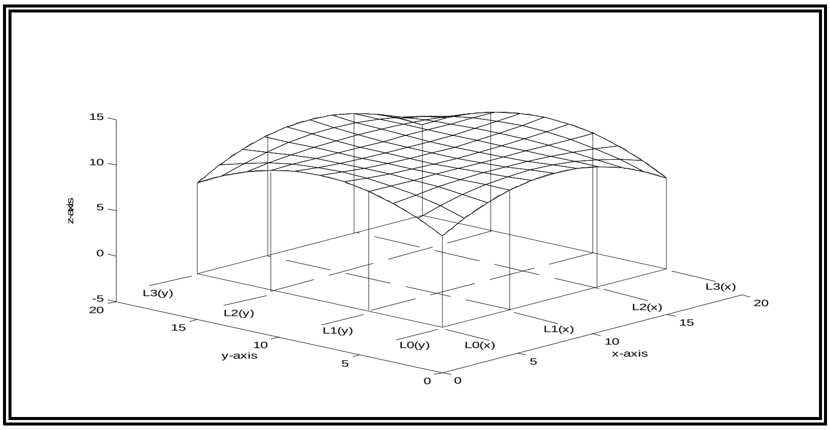


Figure (3): 3rd degree curve representation



Figure(4): Lagrangian Coefficient ($L(x), L(y)$) of a 3D surface [16]

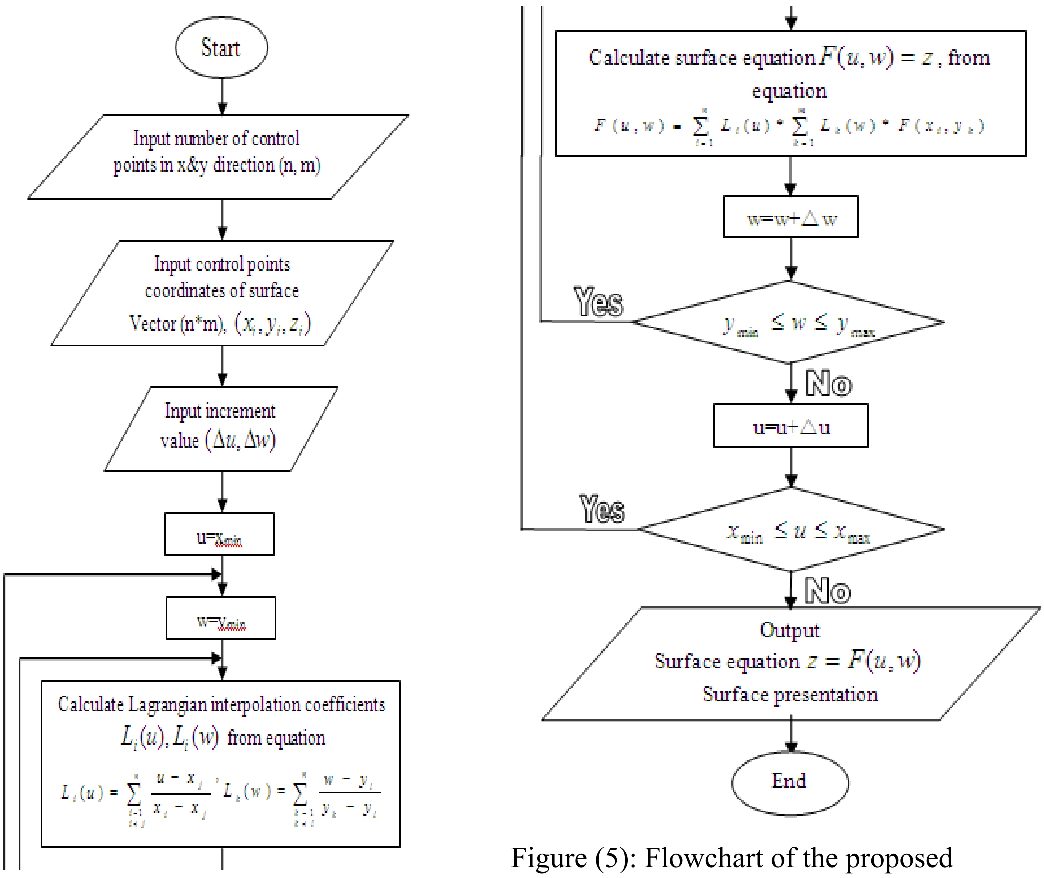
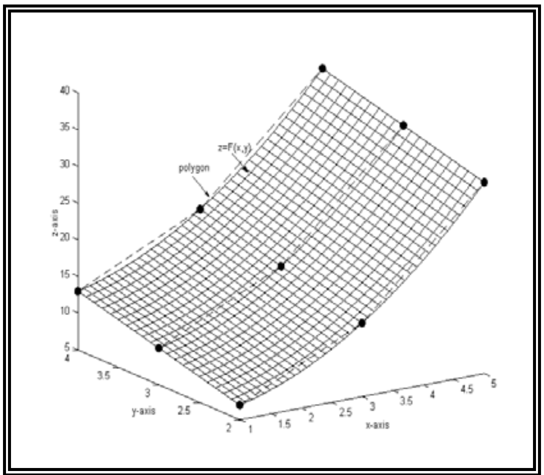


Figure (5): Flowchart of the proposed program



Figure(6): 2nd degree surface representation of (3×3) control points

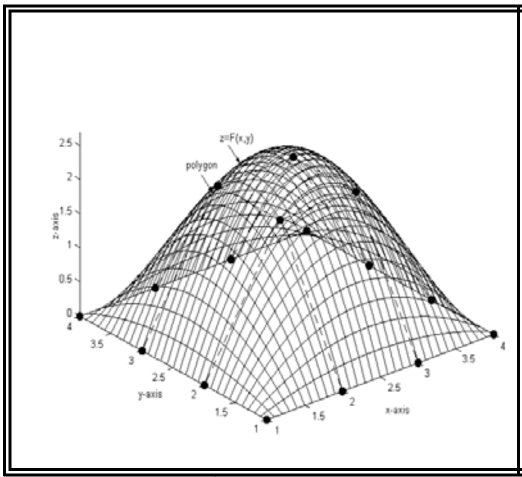
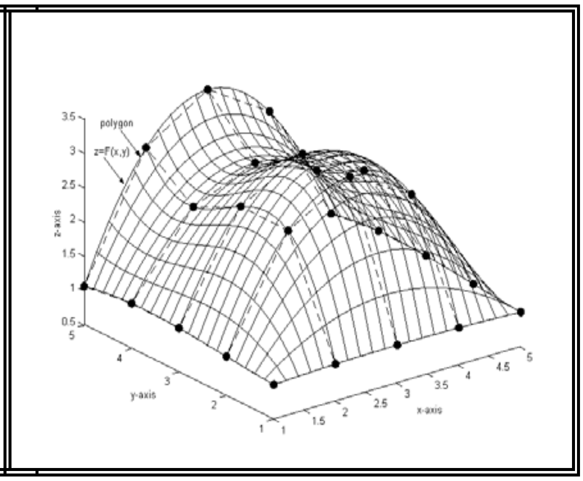


Figure (7): 3rd degree surface representation of (4×4) control points



Figure(9): 4th degree surface representation of (5×5) control points

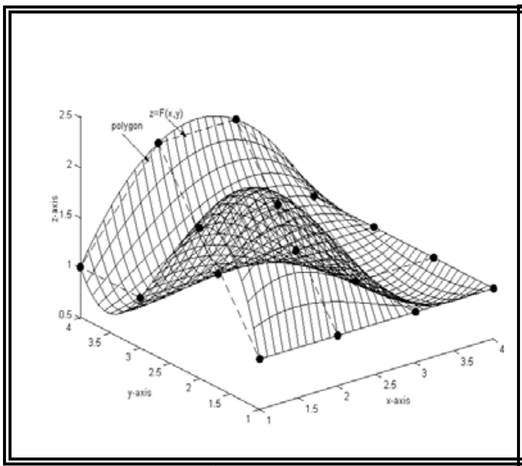


Figure (8): 3rd degree surface representation of (4×4) control points

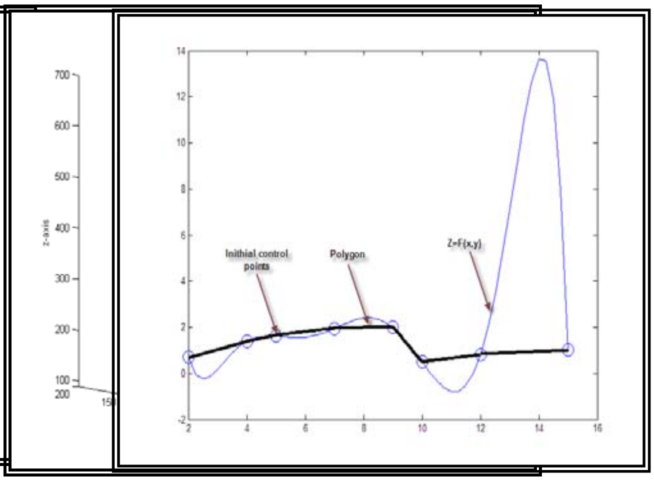


Figure (10): Surfaces representation Comparison Between four Technique.

Table 1: Comparison results with Parabola Functions.

Comparison with Lagrang Interpolation			
Parameters	approximation		
	Hermite	Bezier	B-spline
Deviation rate	0.5916	0.859	0.859
Maximum deviation (mm)	108.0572	51.2383	51.2383
Location in (I,J)direction	(8,21)	(8,19)	(8,19)
Minimum deviation (mm)	0	0	0
Number of coincided points	4	4	4
deviation percentage	66.2265	31.4031	31.4031

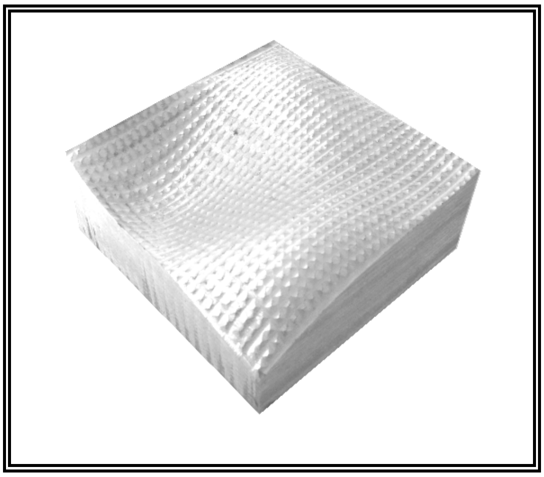


Figure (13): Lagrange machined part (1)

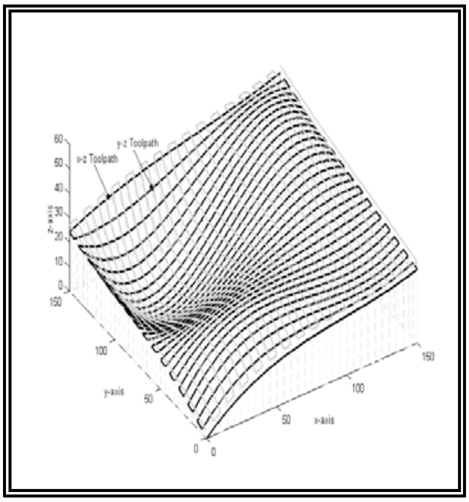


Figure (12): Lagrange Tool path (1)

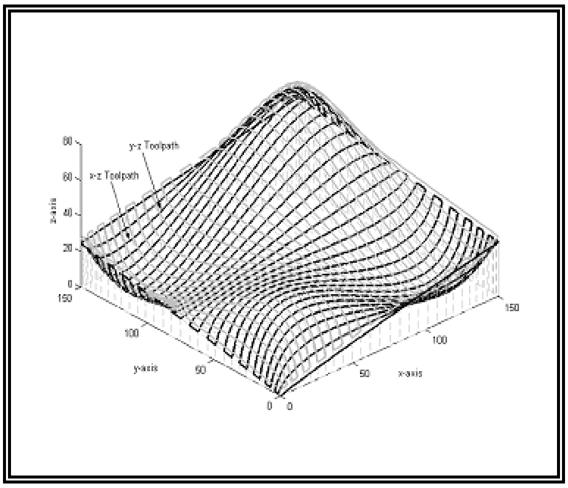


Figure (14): Lagrange Tool path (2)

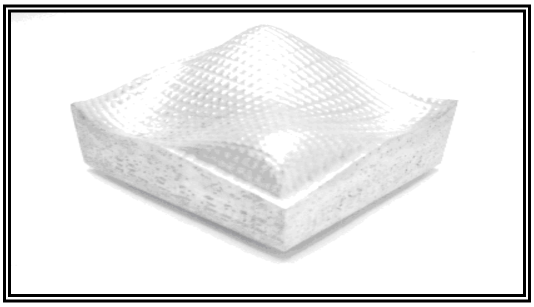


Figure (15): Lagrange machined part (2)