Design of Free – Form Planar Closed Curves Using Four Point Interpolation

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Abstract:

It is desirable to construct a free-form closed curve, since large numbers of mechanical elements follow these curves. In this work we present a parametric interpolation for construction a free form closed curves using four point interpolation technique. The problems and solutions approach for closing open or disjoint curves is demonstrated taking C^1 continuity as a criterion for the solution of these problems. When the coordinates of any number of control points are introduced, the proposed algorithm constructs and closed the intended curve. The designer is free to vary and manipulate these control points to change the shape of the desired curve.

Key Words: Free-Form Curves, Interpolation, CAD.

الخلاصة

هنالك العديد من الاجزاء الميكانيكية التي يتم وصفها بواسطة المنحنيات الحرة المغلقة . في هذة الدراسة تم استخدام طريقة الاستكمال رباعي النقاط لتصميم المنحنيات الحرة المغلقة. تم اعتماد الاستمرارية من نوع (C^1) كمعيار لغلق المنحنيات المفتوحة وكحل لمشكلة عدم ارتباط هـذة المنحنيـات . تـم اعـداد خوارزمية لرسم اي منحني ولاي عدد من النقاط وغلق هذة المنحنيات بمعلومية احداثيات النقاط المكونة للمنحني. تتيح هذة الطريقة للمصمم الحرية بتغير ومناقلة النقاط المكونة للمنحني للتغيير والتحكم بشكل المنحني.

1. Introduction

Interpolation referees to the initiations of polynomial curve segment throughout given discrete control points [1, 2]. The degree of such polynomial is based on the number of control points (degree = number of control points -1). The oscillatory nature of the polynomials is related to the degree of them. Higher the degree, the more undulating the curve is likely to be. The maximum degree used in CAD/CAM systems is usually seven, so that the maximum numbers of control points are eight [3]. However, polynomials of higher degree will increasingly cause ripples in the curve as shown in Fig(1.a). One possible solution to construct more complex curves without increasing the degree of them is to join more than one curve which fit the design requirements and overcome the oscillatory nature of the higher degree curves as shown in Fig(1.b).

The parametric four point interpolation (PFPI) is the technique used in this paper to construct a cubic parametric polynomial that passing through each four successive control points.

The preferred way to represent shapes in computer aided geometric design (CAGD) is with parametric equations. There are many rezones for this [4-8].

1. Free form curves of geometric modeling are non planar and bounded, therefore, are not easily represented by non-parametric function.
2. The parametric representation is axes independent, compared with explicit function in which the mathematical models have values of

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infinity for some slopes or ill-defined mathematical properties.

3. Fully closed space curve can be expressed by the vector function with single parametric [7].
4. The parametric format greatly reduced the number and complexity of subroutines of computer programs, since those parametric functions are easy to express in matrix or vector tools.
5. Evaluation of tool path and cutter offset curve can be simplified using parametric function.

The parametric free form curve can be defined by [9,10]:

$\mathbf{P}(u) = [\mathbf{x}(u)\ \mathbf{y}(u)\ \mathbf{z}(u)]$ …..... (1)

Where u is an independent parameter takes the values from $-\infty$ to $+\infty$. But of course it is not possible to plot the curve for all values of $-\infty$ to $+\infty$. The designer should select a proper interval that has some significance to the modeling situation and that has computational convenience. The curve shown in fig(2) is plotted for all interval on u [0, 1], for:

$x = 2(1-u)^3 + 6u(1-u) + 24u^2 (1-u) + 8u^3$

$y = 5(1-u)^3 + 6u(1-u) + 30u^2 (1-u) + 8u^3$ …………………(2)

Substituting a specific values of u in equation (2) produces specific values of x and y. Each value of u generates a point on the curve, e.g. the points (2,5) and (8,8) in the above fig. is generated by substituting u=0 and u=1 in equation (2). The curves (u,x) and (u,y) can also be generated in parameter space using a successive values of u from 0 to 1 with a certain increment Δu of (0.1) as shown in fig(2).

2. Curve Description

The four-point interpolation formula is the technique used for designing any curve segment is adopted in this paper. The four-point interpolation is a special case of cubic hermit curve that passes through four given points. We begin by specifying four distinct points in x-y plane [P1 P2 P3 P4], assigning to each successive u values, so that u1=0 which gives P1, u2=1/3 which gives P2, u3=2/3 which gives P3 and finally u=1 which gives P4. The parametric equation in vector from for four-point interpolation is given by [4]:

$P(u) = G1(u)\ P1 + G2(u)P2 + G3(u)P3 + G4(u)P4$ ….....…....(3)

Where G1, G2, G3 and G4 are the basis function given by:

$G1(u)=-4.5u^3 + 9u^2 - 5.5u + 1$

$G2(u) = 13.5u^3 - 22.5u^2 + 9u$

$G3 (u) = -13.5u^3 + 18u^2-4.5u$

$G4(u)=2-4.5u^3-4.5u^2+u$…(4)

The condition of four point interpolation basis function is that these basis functions process the equal partition of unity characteristic i.e.

2.1 Elements of Parametric Closed Curve

The elements of parametric closed curve are the constituents of such curves, which are point, tangent vector and curve segment.

2.1.1 The point

The components of a vector P (u) for any value of u on the curve is denoted by a point or position vector [3, 4]. Fig. (4) illustrates this and other important vector elements of a curve.

2.1.2 Tangent Vector

The tangent vector at any point on the curve is given by the first derivative of the parametric equation defining the curve [3,4]. Mathematically the tangent vector is denoted by Pu (u) and from equation (3) [4]:

Pu (u) = G1u (u) P1 + G2u (u) P2 + G3u (u) P3 + G4u (u) P4……….. (5)

Where:

G1u (u) P1= -13.5u^3 + 18u-5.5

G2u (u) P2= 41.5u^2 -45u +9

G3u (u) P3= -41.5u^2 + 36u-4.5

G4u (u) P4= 13.5u^2 -9u+1…….. (6)

Substituting any value of u from 0 to 1 in the above equation with specified increment of Δu, we can find the tangent vector at any point on the curve as shown in Fig (3).

2.1.3 Curve Segment

The curve segment is the curve that passes through each four successive data points. Given the data points P1 P2 P3 and P4 any curve segment can be constructed using equations (3) and (4) as maintained previously (see Fig.(3)). If (N) represents the number of curve segments and (n) represents the number of data points and form Figs. (4,a), (4,b) and (4,c) a simple relationship between N and n can be derived that is:

N = (i + 1) ……………………… (7)

n= 3 (i + 1) …………………….. (8)

or N= n / 3 …..………………… (9)

where: i = 0, 1, 2, 3, …

Since we use the four point interpolation technique and according to equation (9) each closed curve with the number of data points which takes the formula of equation (8), the fourth point (P4) should have the same coordinates of the first point (P1), otherwise the curve will be opened or unclosed, substituting i = 0, 1, 2, 3… in equation (7) and (8) give n = 3, 6, 9…. And N = 1, 2, 3, respectively. Figures (5, a), (5, b) and (5, c) show curves for n = 3, 6, 9 respectively.

3. Continuity

Continuity refers to the smoothness of the joints between adjacent polynomial pieces [11]. The most important two types of continuity between curved segments are [4,7]:

-Position Continuity C^0. The curve joined at a common end point have at least C^0 continuity at their junctions. These two curves form the simplest kind of composite curves.

-Gradient Continuity C^1. This continuity requires a continuity of the common tangent line at their junctions in addition to the position continuity.

The two above mentioned continuity between any two disjoined curved segments are adopted in this study as will be seen in section (5).

4. Problem Statements of Closing Curves

Each closed curve consists of one or more than one curve segment as stated in pervious section. The problems with the design of closed curves are introduced with incapability of closing open curves. The reasons for this incapability is expressed as follows:

(1) When the number of data points (n) takes one of the following formula:

(1.a) n = 3i + 4 …………………. (10)

(1.b) n = 3i + 5 ……………….... (11)

(2) When the number of data points takes the formula stated in equation (8) i.e., n = 3 (I + 1) if the first and last points are not duplicated i.e., $P1^1 \neq P J^4$ Where (J) is any real number greater than one (J> 1).

In these two cases, the curve can't be closed, since a missing data points should be addes between first and last point in order to close the curve.

4.1 Solution Approach

- When the number of data points takes the formula of equation (10), in this case mainly there are two probabilities:

(3.1.a) When (i = 0), substitute this in equation (10) yields:

$n = 3i + 4$
or $n = 4$ $(i = 0)$
In this case the curve can't be closed unless $P1^1 = P J^4$ as shown in Fig. (4a, b, c).

- When $(i > 0)$ i.e., $(i = 1,2,3,\ldots)$, substitute these in equation (10) yields:
  $n = 3i + 4$
  or $n = 7, 10, 13, \ldots$ $(i = 1,2,3,\ldots)$

In this case, if $P1^1 \neq P J^4$ as shown in Fig.(5) we need to add two missing data point namely ($P3^2$) and ($P3^3$) between ($P1^1 = P3^4$) and ($P2^4 = P3^1$) to close the curve.
The evaluation of these two missing data points is dependent on the geometry continuity condition between the disjoint curved segments.
The blending curve technique is the approach adopted in this paper as the solution of this problem, as will be discussed later in section (4).

- When the number of data points takes the formula of equation (11), for example: if I = 0,1,2,3,…… equation (11) yields:
  $n = 3i + 5$
  or $n = 5, 8, 11, 14, \ldots$

This problem is schematically represented in Fig.(6.a). in this case in order to apply blending curve technique, the last point (PE) must be eliminated (that is if the designer need) as shown in Fig. (6.b), in this case the number of data points will be (n = 4, 7, 10, 13, …) which is coincident with the problem formula of equation (10) solved previously. But, if the designer is constrained with the given data points, in this case curve can't be closed.

- When the number of data points takes the formula of equation (8) i.e., $n = 3i + 3$. In this case the curve can be closed if $P1^1 \neq P J^4$ as shown in previous Figs. (4.a,b,c), otherwise, the curve can't be closed.

5. Blending Curves Technique

Inserting a new curve segment between two existing or disjoint curve segments to form a more complex composite closed curve or to close any open curve is called Blending Curve [12]. The most important conditions for blending a curve segment between two disjoint curves are the conditions of continuity at a joint between the two disjoint curves, this condition is termed as geometric continuity, denoted as G^n. the first order geometric continuity G^1 which is adopted in this paper require the following conditions [4,6] see Fig (7):

1. $P1(1) = P2(0)$
   $P3(0) = P2(1)$ ………………. (12)
2. $P^u2(0) = P^u1(1)$ …..………… (13)
   $P^u2(1) = P^u3(0)$ …………….. (14)

Form the above equation it is observed that the tangent vectors have the same direction but not the same magnitude.
Solving equations (3-6) and equations (12-14) give the coordinates of the two missing data points as follows:

$P2^3 = A\acute{C} + BP2^1 - \hat{C}\check{C} - DP2^4$ …………………….. (15)
$P2^2 = E\acute{C} + EP2^1 - F\, P2^3 + GP2^4$ …………………… (16)
Where:
$A = 0.0842$; $B = 0.4386$; $\hat{C} = 0.0249$
$D = 0.1449$; $E = 0.1818$; $F = 1.8100$
$G = 1.0000$
$\acute{C} = - P1^1 + 5.5\, P1^2 - 10\, P1^3 + 5.5\, P1^4$
$\check{C} = - 5.5\, P3^1 + 9\, P3^2 - 4.5\, P3^3 + P3^4$

5.1 Algorithm for Closing Curves Defined by Four Point Interpolation Approach.

The new algorithm for closing open curves defined by four point interpolation technique may be summarized as follows:

1- Input the number of data points (n).

2- Input the x-y coordinates of the data points.
3- Evaluate (i) from equation (8).
3.a. If (i) real and positive number then.
3.a.1 If $P1^1 = P J^4$ the curve is already closed.
3.a.2 Otherwise, then curve can't be closed, and Go To step (3.b).
3.b. If (i) not real number or negative value. Go To step (4).
4- Evaluate (i) from equation (11).
4.a. If (i) not real number or has a negative value then the curve can't be closed, GoTo step (4.b).
4.b. If (i) real and positive number then:
4.b.1 Would you like to eliminate the last point?
- If yes, evaluate the blended curve segment using equations (12-16) to closed the curve.
- Otherwise, Otherwise, the curve can't be closed, Go To step (5).

5- Evaluate (i) from equation (10).
5.a. If (i = 0) then.
5.a.1 If $P1^1 = P J^4$ the curve is already closed.
5.a.2 Otherwise, the curve can't be closed, Go To step (5.b).
5.b. If (i) real and positive number, then evaluate the blended curve segment,otherwise Go To step (4).
5.c. If (i) not real number or has a negative value, then the curve can't be closed, Go To step (6).
6- Stop.

6. Examples

we have implemented our algorithm using MATLAB programming language [13] on a number of parametric open or disjoint curves . we cite two examples here to show the efficiency of our algorithm.

In example (1) we construct a parametric composite curve of glass lens, the initial coordinates of control points are listed in table (1) which consists of ten control points and three curved segments in addition to the fourth blended curved segments with i =2.blending technique is done between first and third segment to construct the final closed curve as shown in Fig.(9).

Table(1) Control Points of Glass Lens

X	Y	X	Y
0.5	5.0	11.2	4.1
2.1	3.0	13.0	5.0
4.0	2.5	10.0	7.0
6.0	2.7	6.0	7.2
8.5	3.2	3.0	7.0

Example (2) shows the hot air Gun. The elimination of the final point problem and closing the curve using blending technique are shown in Fig.(10). The coordinates of the 17 control points for i = 4 and 5 curved segments in addition to the blended segment are listed in table (2).

Table (2) Control Points of Hot Air Gun

X	Y	X	Y
0.5	12.0	16.8	2.0
4.8	11.8	23.0	8.0
6.75	10.2	18.0	12.0
7.2	9.0	20.5	14.0
9.0	8.0	22.0	16.0
12.0	7.0	18.3	16.2
14.0	2.0	15.0	16.3
16.2	1.0	17.5	

7. Conclusion and Discussion

In this paper we present a practical algorithm for construction a free form planar closed curved by using parametric four-point interpolation technique applying C^0 and C^1 continuity conditions between the disjoint curve segments. To overcome the limited descriptive power of a single parametric polynomial, we employ here the technique of blending new curve segment between existing curves, to form more complex continuous curves, regardless of the degree of the polynomial.

In the first example, the C^0 and C^1 continuity was archived through the given control points. Accordingly, there is no need to eliminate any control point from the existing points .The position of the two blended points (see fig. 9) were "selected" by the adopted algorithm to guarantee continuity ; since that the curved segment joining 1st and 10th points exhibits high curvature through small X- axis range (from X= 0.5 to 3).

The problem of control point elimination has been conducted in the second example .The eliminated point was eliminated since that the continuity conditions (Eqs.12-14) can't be achieved across the curved segments. To show the validity of the proposed method, the two examples were conducted using the well known IRONCAD software programming system. The control points of the two examples were used to implement the glass lens and hot air gun. Figures 11 and 12 show the two examples implemented using IRONCAD soft ware system respectively .The boundary coordinates of the glass lens and hot air gun were exported to the IRONCAD system. The deviation between the desired boundary which was conducted using IRONCAD and the actual boundary which was implemented using the proposed method was shown in figure 13 and 14 respective. The maximum deviation is seen to be 0.23 mm between the boundary implemented by IRONCAD system and that implemented using our algorithm. Mean while 0.175 mm is the maximum error of the hot air gun due to its complexity.

The probabilities of closing any open curve and their problems are discussed via their simple solutions.

Computer run for different examples is conducted showing the effectiveness of our algorithm.

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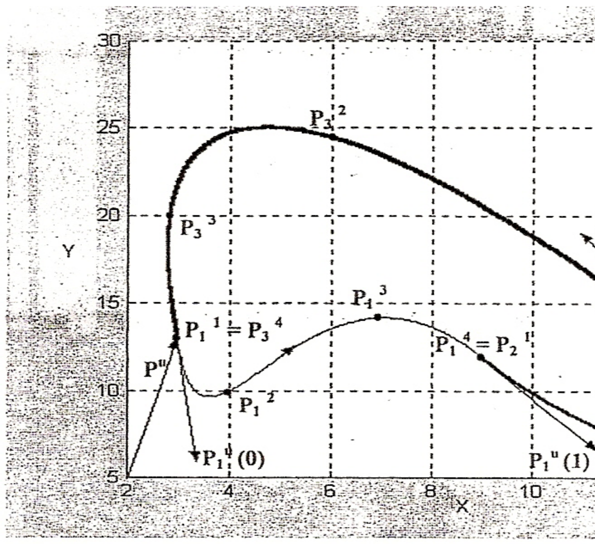
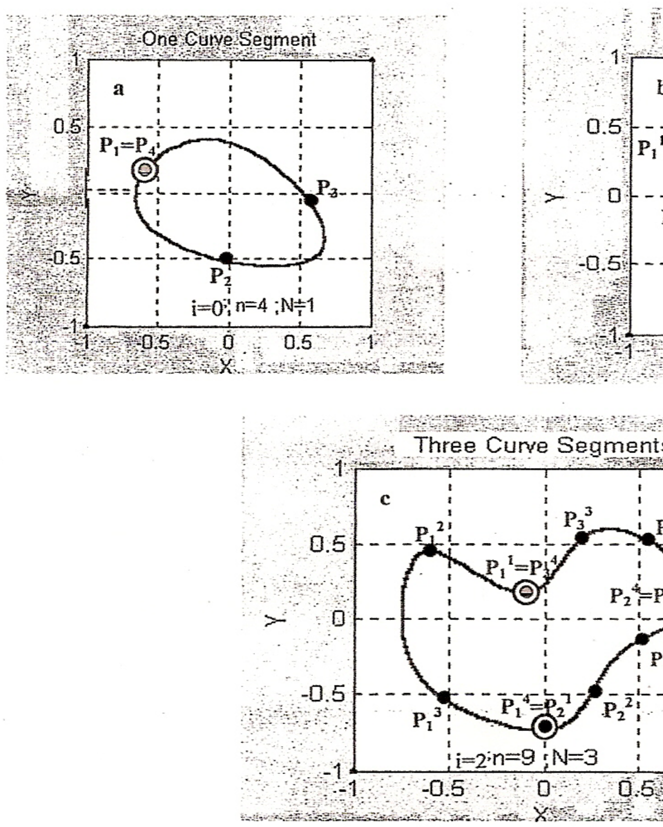


Figure (4) Basic Elements of Closed Free-Form
Curved Segments.



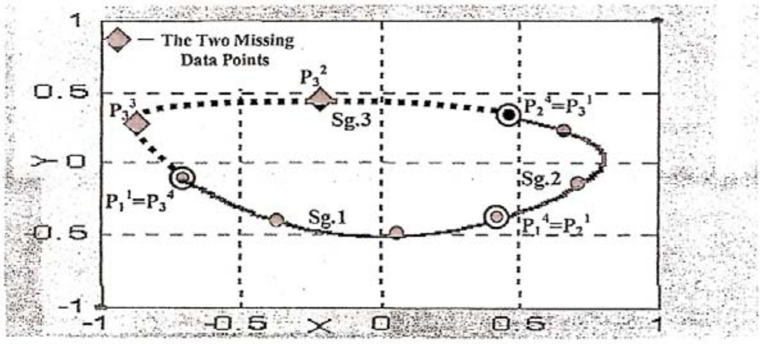
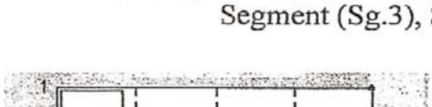
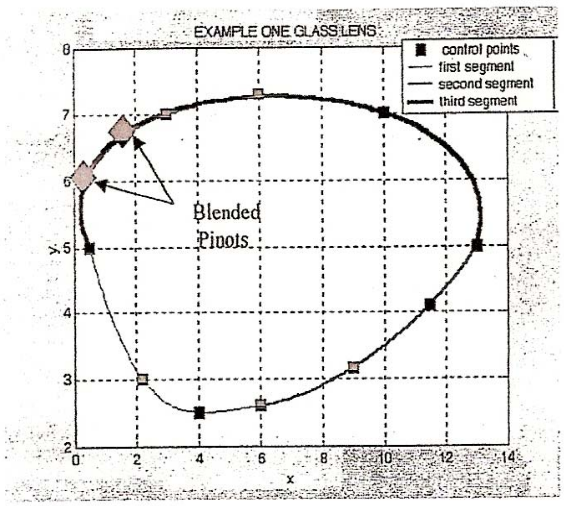


Figure (6) The Two Missing Data Points Added Between Two Curve Segments (Sg.1) and (Sg.2) in Order to Find the Missing Segment (Sg.3), Since $P_1^{\,1} \neq P_2^{\,4}$.





Figure(9) Glass Lens Construction

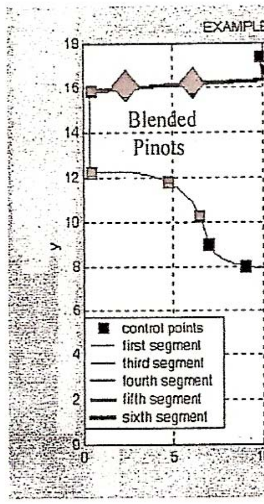


Figure (10) Hot Air Gun C