Solutions of Dynamic Fractional Order Differential Algebraic Equations System

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Received on: 1/11/2009
Accepted on: 6/5/2010

Abstract

In this paper, we are presented the existence and uniqueness theorem, and two proposed methods, based on the theory of Gunwald-Letnikov fractional order derivative. In the first method, the variational approach is implemented, while in the second method, the fractional difference approach is implemented. Dynamic test example is presented to each proposed method, to demonstrate their computational algorithm.

Keywords: Fractional Order Differential Algebraic Equations, Existence and Uniqueness Solution, Variational method, Fractional Difference method, Dynamic System.

حلول أنظمة المعادلات التفاضليه الجبريه الديناميكيه ذات الرتب الكسريه

الخلاصه

في هذا البحث، قدمنا نظريه الوجود ووحدانيه الحل مع طريقتين لحل نظام من المعادلات التفاضليه الجبريه الديناميكيه ذات الرتب الكسريه بالأعتماد على تعريف كراونوولد-ليتنكوف للتفاضلات ذات الرتب الكسريه. في الطريقه الأولى، تم أستخدام أسلوب التغاير بينما استخدمت الطريقه الثانيه أسلوب الفروقات الكسريه. وقد تم تقديم مثال أختباري لكل طريقه، لتوضيح خوارزميه كل طريقه.

1. Introduction

1.1 Differential Algebraic Equations (DAEs) ([3],[12] &[19])

Mathematical models of some engineering, physical, and scientific problems frequently take the following explicit form of a system of ordinary differential equations (ODEs)

$$y' = f(y,,t) \quad \dots (1.1)$$

Where t is time and y is a vector of dependent variables or state variables. The initial value problem for the equation (1.1) is to find the solution of y(t) that satisfies a given initial condition $y(t_0)=y_0$. In some cases, the model also involves dependent variables whose time derivatives do not appear in the equations too. The set of equations which is the combination of both differential and algebraic equations that defines this model is known as a ***differential algebraic equation (DAE) system.*** The most general DAE system is expressed in the fully implicit form as

$$F(y, y', t) = 0 \quad \dots (1.2)$$

Where F is some function. Another way to present a DAE system is to use the following semi-explicit form

$$y' = f(y, y', t) \quad \dots (1.3a)$$

$$0 = g(y, x, t) \quad \dots (1.3b)$$

Where x is another vector of dependent variables.ODE involves differentiations only, while DAE systems are more general than ODE systems, since DAE involves both integrations and differentiations, in

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https://doi.org/10.30684/etj.29.3.16
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which one may hope that performing analytical differentiations to a given system and eliminating, as needed will result in an explicit ODE for all unknowns. This turns out to be true unless the problem is ***singular***. Therefore, a property known as the ***index*** plays a key role in the classification and behavior of DAEs. ***Index*** is defined as the minimum number of times that all or part of DAE system must be differentiated to get a system of ODEs.

1.2 Fractional Calculus: ([5], [15], 16], [19] & [20])

Although fractional derivatives have a long mathematical history, for many years they were not used in many different sciences, but in recent years, growing attention has been focused on the importance of fractional derivatives and integrals in science. Recently, there has been some attempt to solve linear problems with multiple fractional derivatives problems. Not much has been done for the nonlinear problems. A number of definitions have emerged over the years including Riemann-Liouville fractional derivative. Grunwald-Letnikov fractional derivative. Caputo fractional derivative, etc. in this paper, Grunwald-Letnikove fractional derivative is considered.

1.2.1 Properties

Let α > 0, the main properties of fractional derivatives and integrals are the following:

1. If f(t) is an analytical function of t, then its fractional derivative ${}_0D_t^{\alpha} f(t)$ is an analytical function of t, α.
2. For α=n, where n is an integer, the operation ${}_0D_t^{\alpha} f(t)$ gives the same result as classical differentiation of integer order n.
3. For α=0 the operation ${}_0D_t^{\alpha} f(t)$ is the identity operator: ${}_0D_t^{\alpha} f(t) = f(t)$.
4. Fractional differentiation and fractional integration are linear operations ${}_aD_t^{\alpha}\left(\lambda f(t)+\mu g(t)\right)=$ $\lambda_{a\,a}D_t^{\alpha} f(t)+\mu_{a\,a}D_t^{\alpha} g(t)$
5. The additive index law (semigroup property) ${}_0D_t^{\alpha}\,{}_0D_t^{\beta} f(t)=$ ${}_0D_t^{\beta}\,{}_0D_t^{\alpha} f(t)={}_0D_t^{\alpha+\beta} f(t)$ Holds under some reasonable conditions on the function f(t).

1.2.2 Fractional Difference

In this paper, we are presenting the fractional difference **due to Grunwald-Letnikov [15],** based on a generalization of the usual differentiation of a function $y(x)$ of order n∈ ℕ of the form

$$y^{(n)}(t)=\lim_{h\to 0}\frac{(\Delta_h^n y)(t)}{h^n} \qquad (1.4)$$

Here $(\Delta_h^n y)(t)$ is a finite difference of order $n\in\mathbb{N}_0$ of function $y(t)$ with a step $h\in\Re$ and centered at the point $x\in\Re$. Property (1.4) is used to define a fractional derivative by directly replacing $n\in\mathbb{N}$ in (1.4) by α > 0. For this, h^n is replaced by h^{α}, while the finite difference $(\Delta_h^n y)(t)$ is replaced by the difference $(\Delta_h^{\alpha} y)(t)$ of fractional order $\alpha\in\Re$ defined by the following infinite series:

$$(\Delta_h^{\alpha} y)(t):=\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}y(t-kh)$$
$$(t,h\in\Re;\alpha>0), \qquad (1.5)$$

where $\binom{\alpha}{k}$ are the binomial coefficients. When h>0 the difference (1.5) is called left-sided difference, while for h<0 it is called a right-sided difference. The series in (1.5) converges absolutely and uniformly for each α>0 and for every bounded function $y(x)$. In particular, when $\alpha=n\in\mathbb{N}$, (1.5) coincides with (1.4):

$$(\Delta_h^n y)(t)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}y(t-kh)$$
$$(t,h\in\Re;n\in \mathrm{N}). \qquad (1.6)$$

Following (1.4), the left- and right-sided Grunwald-Letnikov derivatives $y_+^{\alpha)}(t)$ and $y_-^{\alpha)}(t)$ are defined by

$$y_+^{\alpha)}(t):=\lim_{h\to 0}\frac{(\Delta_h^\alpha y)(t)}{h^\alpha} \qquad (1.7a)$$

$$y_-^{\alpha)}(t):=\lim_{h\to 0}\frac{(\Delta_{-h}^\alpha y)(t)}{h^\alpha} \qquad (1.7b)$$

respectively.

The definition (1.5) of the fractional difference $(\Delta_h^\alpha y)(t)$ assumes that the function $y(t)$ is given at least on the half-axis. For the function $y(t)$ given on finite interval [a, b], such a difference can be defined as follows by a continuation of $y(t)$ as a vanishing function beyond[a, b]:

$$(\Delta_h^\alpha y)(t)=\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}y^*(t-kh)$$
$$(t,h\in\Re;\alpha>0), \qquad (1.8)$$

where

$$y^*(t)=\begin{cases}y(t), & t\in[a,b],\\ 0, & t\notin[a,b].\end{cases}$$

It is acceptable to rewrite the fractional difference (1.8) in terms of the function y(t) itself, avoiding its continuation as a vanishing function, in the forms

$$(\Delta_{h,a+}^\alpha y)(t):=\sum_{k=0}^{\left[\frac{x-a}{h}\right]}(-1)^k\binom{\alpha}{k}y(t-kh)$$
$$(t\in\Re;h>0;\alpha>0). \qquad (1.9a)$$

$$(\Delta_{h,b-}^\alpha y)(t):=\sum_{k=0}^{\left[\frac{b-x}{h}\right]}(-1)^k\binom{\alpha}{k}y(t+kh)$$
$$(t\in\Re;h>0;\alpha>0). \qquad (1.9b)$$

Then, by analogy with (1.7a) and (1.7b), the left-and right-sided Grunwald-Letnikov fractional derivatives of order α>0 on a finite interval [a, b] are defined by

$$y_{a+}^{\alpha)}(t):=\lim_{h\to 0}\frac{(\Delta_{h,a+}^\alpha y)(t)}{h^\alpha} \qquad \ldots (1.10a)$$

and

$$y_{b-}^{\alpha)}(t):=\lim_{h\to 0}\frac{(\Delta_{h,b-}^\alpha y)(t)}{h^\alpha}, \qquad \ldots (1.10b)$$

respectively.

so we define the fractional derivative in the Grunwald-Letnikov sense as:

$$D^\alpha\ y(t)=\lim_{h\to 0}h^{-\alpha}\sum_{j=0}^{[t/h]}(-1)^j\binom{\alpha}{j}y\ \ (t-jh), \qquad \ldots (1.11)$$

where, $[t]$ means the integer part of t and h is the step size.

Next, we recall that the left-handed shifted Grunwald estimate to the left-handed derivative is

$$D^\alpha\ y\ =\sum_{j=0}^{[t/h]}c_j^\alpha\ y\ \ (t-(j-1)h) \qquad \ldots (1.12)$$

The definition of operator in the Grunwald-Letnikov sense (1.12) is equivalent to the definition of operator in the Riemann-Liouville sense. Nevertheless the Grunwald-Letnikov operator is more flexible and most straightforward in numerical calculations.

$$y^{(\alpha)}(t)=\sum_{j=0}^{l}C_j^\alpha y(t-ih) \qquad \ldots (1.13a)$$

Could be written as following

$$y^{(\alpha)}(t_n)=\sum_{j=0}^{l}C_j^\alpha y_{n-j} \qquad \ldots (1.13b)$$

where l is the number of steps, and c_j^α are Grunwald-Letnikov coefficients defined as:

$$c_j^\alpha=\frac{1}{h^\alpha}(-1)^j\,\frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}$$
$$(j=1,2,\ldots). \qquad \ldots (1.14)$$

Where

$$c_0^\alpha=h^{-\alpha},\ c_j^\alpha=\left(1-\frac{1+\alpha}{j}\right)c_{j-1}^\alpha,\ (j=1,2,\ldots).$$

We can compute the coefficients in a simple way. For j=1 we have $c_1^\alpha=\alpha h^{-\alpha}$. For details about the fractional difference and its applications for solving fractional differential equations, see [26].

2. Proposed Problem

In this paper, we are studying more extened structure, hopping to reach the general structures. Consider the following Fractional Order Differential Algebraic Equations FODAEs:

$$Y^{(\alpha)}(t)=F\left(t,Y(t),Y^{(m-1)}(t),X(t)\right) \quad (2.1a)$$

$$0=CY(t)+r(t) \quad (2.1b)$$

Where $Y(t)=\left(y_1(t),...,y_m(t)\right)^T$ is the solution of the system (2.1),

3. Existence and Uniqueness Solution ([4], [9] & [26])

In this section, by Approximating the fractional derivative in (2) by (1.13), and if we considered system (2) in the following form

$$Y^{(\alpha)}(t)=\sum_{j=1}^{m}A_jY^{(j-1)}(t)+BX(t)+q(t), \quad (3.1a)$$

$$0=CY(t)+r(t) \quad (3.1b)$$

Where A_j, B and C are smooth functions of t, $t_0 \le t \le t_f$, $A_j(t) \in R^{nxn}$, j=1,…,m, $B(t) \in R^{nxk}$, $C(t) \in R^{kxn}$, $n \ge 2$, $1 \le k \le n$ and CB is nonsingular (FODAE has index $\alpha+k+1$) except possibly at a finite number of isolated points of t, which in this case, the FODAEs (4.1) have constraint singularity. The inhomogeneties are $q(t)$ and $r(t) \in R$ and $\alpha \ge 0$. From [3] & [12], we can write (3.1a) as

$$X=(CB)^{-1}CY^{(\alpha)}-\sum_{i=1}^{m}\left[A_iY^{(i-1)}+q\right],$$
$$t\in\left[t_0,t_f\right]$$

So the problem (3.1) transforms to the over determined system:

$$\left[I-B(CB)^{-1}C\right]\left[Y^{(\alpha)}-\sum_{i=1}^{m}A_iY^{(i-1)}+q\right],$$
$$t\in\left[t_0,t_f\right] \quad \dots(3.2)$$

which is a DAEs system with *m* equations and *m* unknowns with index m. leads to the numerical solution algorithm described by the following recursive relations(see [19]):

$$y_{i,0}=\beta,\ Y_n=F_n-\sum_{j=1}^{l}C_j^{\alpha_i}Y_{n-j},$$
$$(n=1,2,...) \quad \dots(3.3)$$

Set $\widehat{D}=I\times C^*(t)$ where $C^*(t)$ is the class of all continuous column vectors Y(t) with the norm

$$\|Y(t)\|=\sum_{i=1}^{m}\|y_i(t)\|=\sum_{i=1}^{m}\max_{i\in I}|y_i(t)|$$

Now, we can state the following theorem:

<u>**Theorem:**</u>

Let $F(t,Y(t)) \in C^*(t)$, where $F(t,Y(t))=(f_1(t, Y(t)), \dots, f_m(t, Y(t)))^T$, i.e. $f_i(t, Y(t)) \in C(\widehat{D})$ for all *i*=1,…,*m*. and each satisfies the Lipschtiz condition

$$\left|f_i(t,Y(t))-f_i(t,X(t))\right|\le k\sum_{l=o}^{m}\left|y_l(t)-x_l(t)\right|$$

for all *i*, … (3.4)

For (t,Y(t)) and (t, X(t)) $\in \widehat{D}$, $k=\min k_i>0$

if $\max C_j^{\alpha_i} \le (1-k)$, … (3.5)

then (3.1) has one and only one solution $Y(t) \in C(I)$ that satisfies $D^{\alpha_i}Y(t)\in C(I)$.

Proof:

If we write

$$TY_n=F(t,Y_n(t))-\sum_{j=1}^{l}C_j^{\alpha_i}Y_{n-j}$$

then for $(t,Y(t))$ *and* $(t,X(t)) \in \widehat{D}$, we get

$$\|TY_n-TX_n\|=\left\|F(t,Y_n(t))-\sum_{j=1}^{l}C_j^{\alpha_i}Y_{n-j}-F(t,X_n(t))-\sum_{j=1}^{l}C_j^{\alpha_i}X_{n-j}\right\|$$
$$\le\left\|F(t,Y_n(t))-F(t,X_n(t))\right\|+\left\|\sum_{j=1}^{l}C_j^{\alpha_i}Y_{n-j}-\sum_{j=1}^{l}C_j^{\alpha_i}X_{n-j}\right\|$$

$$\leq k\sum_{l=o}^{m}\left|y_l(t)-x_l(t)\right|+$$
$$\sum_{j=1}^{l} max\left|C_j^{\alpha_l}\right|\left\|Y_{n-j}-X_{n-j}\right\|$$
$$\leq k\left\|Y_n-X_n\right\|+(1-k)\left\|Y_{n-j}-X_{n-j}\right\|$$
$$=\left\|Y_n-X_n\right\|,\ as\ n\to\infty,$$
$$Y_{n-j}\to Y_n\ \ and\ \ X_{n-j}\to X_n.$$

Hence, the mapping T: $C(\hat{D})\to C(\hat{D})$ is a contraction mapping, and then it has a fixed point Y(t)=T(Y(t)). Providing the condition (3.5) and hence, there exists a unique solution Y(t)$\in C(\hat{D})$ for the system (3.1).

4. The Proposed Methods: ([4], [9], [10], [12], [22] & [25])

The first practical numerical mehod for DAE's was the ***Backward differentiation formulas (BDF)*** introduced by **Gear in 1971** [8]. The method was initially designed for semi-explicit index one DAE's (1.3), where $\frac{\partial g}{\partial x}$ is nonsingular. The algebraic variable x is treated in the same way as the differential variables y in BDF, then the method was soon extended to fully implicit DAE's (1.2). Still, not all DAE's were solved successfully with BDF methods. More details can be found in [4]. While the BDF methods have been successful in solving DAE's, there is a considerable research on solving DAE's with ***Implicit Runge-Kutta (IRK)*** methods. A comprehensive analysis for IRK methods presented in [10], applied to *Hessenberg* index one, two and three systems. In general, IRK methods do not attain the same order of accuracy for DAE's as they do for ODE's, see [17]. Also, *extrapolation methods* may be viewed as*IRK* methods, in which, those methods are an effective way to find the numerical solutions of ***nonstiff and stiff ODE's.*** Many researchers have shown great interest in applying *extrapolation methods to DAE's,* [4]. In this paper, we proposed two approaches, the ***variational and Fractional Difference,*** to constract iterative formulas, to obtain the successive approximation solutions for ***Differential Algebraic Equatins (FODAE's Fractional Order)***.

4.1 Variational Approach

In this section, the variational iteration method is applied for finding the solution of linear and nonlinear fractional order differential algebraic equations FODAE's. The functional of the fractional order derivatives is constructed by a general Lagrange multiplier. The successive approximation will be obtained, by the sequence of such functional. A tested problem is presented to demonstrate the performance of the proposed method. For simplicity, we consider problem (3.1) when m=1 (problem has index 2), n=2,3 and k=1,2.Also, if we suppose that DAE is nonsingular, i.e. CB(t)≠0, $t\in\left[t_0,t_f\right]$, then by theorems presented in [3] & [12], the given index-2 problem is equivalent to the index-1 DAE system problem

$$\bar{A}y^{\alpha}+\bar{B}y=\hat{q} \qquad \ldots(4.1a)$$

And

$$x=(CB)^{-1}C\left[y^{\alpha}-Ay-q\right] \qquad \ldots(4.1b)$$

Such that

$$\bar{A}=\begin{bmatrix} b_1a_{21}-b_2a_{11} & b_1a_{22}-b_1a_{12} \\ c_1 & c_2 \end{bmatrix},$$
$$\bar{B}=\begin{bmatrix} b_2 & b_1 \\ 0 & 0 \end{bmatrix},\ \hat{q}=\begin{bmatrix} b_2q_1-b_1q_2 \\ -r \end{bmatrix}$$

Now, we proposed a modification of a general ***Lagrange multiplier method*** [16]. In the variational iteration method, the differential equation

L[$\boldsymbol{u}(\boldsymbol{t})$]+$\boldsymbol{N}$[$\boldsymbol{u}(\boldsymbol{t})$]=$\boldsymbol{h}(\boldsymbol{t})$

is considered, where **L** and **N** are linear and nonlinear operators, respectively and ***h*(*t*)** is an inhomogeneous term. The correction functional

$$u_{m+1}(t) = u_m(t) + \int_0^t \lambda\big[[u_m(t)] + N[\tilde{u}_m(s)]\big) - h(s)\big]ds \quad \ldots(4.2)$$

Is considered, where λ is a general Lagrange multiplier, u_m is the m^{th} approximate solution and $\tilde{u}_m$ is a restricted variation which means $\delta\tilde{u}_m$, $m\geq 0$, see[6].

The main difficulties when dealing with fractional derivatives arise during the computations such as in the fractional calculus of variation, while we are applying the fundamental theorem, the following property is needed (see [14]&[15]), called Classical product rule for Riemann-Liouville derivatives, for all α>0:

$$\int_a^b D_+^{\alpha} f(t)g(t)\,dt - \int_a^b f(t)D_-^{\alpha} g(t)\,dt \quad \ldots(4.3)$$

As long as f(a)=f(b) and g(a)=g(b). Also, we are presenting the definition and some properties of classical Mittag-Leffer function, denoted by $E_\alpha(.)$. More details can be found in [27]. The function $E_\alpha(t)$ defined by

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+1)}$$

In particular, when α=1 and α=2, we have $E_1=e^t$ and $E_2(t) = cosh(\sqrt{t})$.

In this proposed method, first we determine the Lagrange multiplier λ that can be identified via variational theory, i.e. the multiplier should be chosen such that the correction functional is stationary, i.e. $\delta u_{m+1}(u_m(t), t)=0$. Then the successive approximation u_m, $m \geq 0$ of the solution u will be obtained by using any selective initial function u_0 and calculated Lagrange multiplier λ.

Consequently $u = \lim_{m\to\infty} u_m$. It means that, by the correction functional (4.2) several approximations will be obtained and therefore, the exact solution emerges at the limit of the resulting successive approximations.

To demonstrate the performance of our proposed method, we consider the linear index-2 FODAE problem:

$y^{(\alpha)}=Ay+Bx+q$
(Ia)
$0=Cy+r$
(Ib)
With 0≤t≤1 and

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1+2t \end{bmatrix},$$

$$q = \begin{bmatrix} -sin(t) \\ 0 \end{bmatrix},\ C^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$r(t)=-(e^{-t}+sin(t))$

with $y_1(0)=1$, $y_2(0)=0$, with exact solution, when α=1(i.e. it is DAE's) $y_1(t) = e^{-t}$, $y_2(t) = sin(t)$ and

$$x(t) = \frac{cos\,t}{1+2t}$$

The tested problem (I) can be transform to the following index-1 DAE:

$$y_1 + y_2 = e^{-t} + sin(t) \quad \ldots\text{(IIa)}$$

$$y_1^{\alpha} + y_1 - y_2 + sin(t) = 0 \quad \ldots\text{(IIb)}$$

To solve the new problem, we transform the algebraic equation (IIa) in the iterative form with respect to y_2 and by the variational iteration method and using (4.2), we construct the correction functional in y_1-direction for the differential equation (IIb). Therefore, we obtain the following system:

$$y_{2,n+1}(t) = e^{-t} + sin(t) - y_{1,n}(t) \quad \ldots\text{(IIIa)}$$

$$y_{1,n+1}(t) = y_{1,n}(t) + \int_0^t \lambda(s)\left[y_{1,n}^{\alpha}(s) + y_{1,n}(s) - \tilde{y}_{2,n}(s) + \sin(s)\right]ds \qquad \text{...(IIIb)}$$

Where $\tilde{y}_{2,n}$ is considered as a restricted variation, i.e. $\delta\tilde{y}_{2,n} = 0$.

By taking the variation from both sides of the correction functional (IIIb) and using the property (4.3), we have

$$\delta y_{1,n+1}(t) = \delta y_{1,n}(t) + \int_0^t \left[\left[\lambda^{\alpha}(s) - \lambda(s)\right]\delta y_{1,n}(s)\right]ds$$

By imposing $\delta y_{1,n+1}(t) = 0$, we obtain the stationary condition

$$\lambda^{\alpha}(s) + \lambda(s) = 0 \qquad \text{... (IV)}$$

Therefore

$$\lambda(t) = e_{\alpha}^{-t} \qquad \text{... (V)}$$

where

$e_{\alpha}^{-t} = t^{\alpha-1}E_{\alpha,\alpha}(-t) = \sum_{k=0}^{\infty} \frac{t}{\Gamma(k+1)\alpha}$, see [27].

By substituting the optimal value (V) into functional (IIIb), we obtain the following iteration formula:

$$y_{2,n+1}(t) = e^{-t} + sin(t) - y_{1,n}(t) \qquad \text{... (VIa)}$$

$$y_{1,n+1}(t) = y_1^{(n)}(t) - \int_0^t e_{\alpha}^{-s}\left[y_1^{(\alpha)}(s) + y_{1,n}(s) - y_{2,n}(s) + \sin(s)\right]ds \qquad \text{...(VIb)}$$

with n=1,2,…, $y_{1,0}$=y_1(0)=1 and $y_{2,0}$=y_2(0)=0.

Now, we expand the coefficient functions e^{-t} and sin (t) at t=0 and e^{s-t} at t =s, by Taylor series expansion, We obtain

$$y_{2,1}(t) = \frac{1}{2}t^2 - \frac{1}{3}t^3 + \frac{1}{24}t^4 + ...$$

$$y_{1,1}(t) = 1 - t + \frac{1}{6}t^3 - \frac{1}{24}t^4 + ...$$

$$y_{2,10}(t) = t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{1}{5040}t^7 + \frac{1}{362}t^9 - \frac{1}{4989600}t^{11} + ...$$

$$y_{1,10}(t) = 1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4 - \frac{1}{120}t^5 + \frac{1}{720}t^6 - \frac{1}{5040}t^7 + ...$$

and continuing as n tends to infinity, the obtained series are the Taylor series expansion when α=1, y_1(t)=e^{-t}, y_2(t)=sin (t) and $x(t) = \frac{\cos t}{1+2t}$.

The exact and approximated results are presented tables (1) & (2), and figures (1) & (2).

4.2 Fractional Difference Approach

In this section, we are constructing an approximation solution to the fractional order differential equations system FODAEs, using the fractional difference method, based on the definition of Grunwald-Letnikove sense. A simple non-fractional order recurrence formula is constructed. The testing problem is presented to describe the ability of the algorithm, and the simplicity computational performance.

The importance and desirability of working directly with (1.2) have been recognized for years by scientists and engineers in many areas. There has been considerable researches on numerical methods for DAE's. Most of the numerical analysis literatures on DAE's to date has dealt with DAE's of indices less than three, and often assumed the DAE system to have a special structure. See [4], [6], [9], [10] &[19]).

Now, system (3.1a) can be transformed to a system of DAEs, by the classical idea, in studying fractional order derivative, using the definition of the Grunwald-Letnikov fractional derivatives.

Substituting (2.1b) into the first sum term of (3.1a), we obtain

$$y_0 = \beta,$$

$$y_n = \left(\sum_{i=1}^{m} A_i \left(C^{-1} r(t)\right)^{(i-1)} + q - \sum_{j=1}^{l} \left(C^{\alpha}_{j-1} y_{n-j}\right) \right),$$

(n=1,2, …) … (4.1)

In this proposed method, the successive approximation y_n will be obtained. The algorithm is simple for computational performance for all values of α.

To demonstrate the performance of our proposed method, we consider the FODAE problem:

$$y^{(\alpha)} = A(1-y)^4,$$

$$0 = y - r$$

With $0 \le t \le 1$.The exact solution is given by

$$y(t) = \frac{1 - 3t - \left(1 + 6t + 9t^2\right)^{1/3}}{1 + 3t}$$

when α=1, A=1 & r=0.8.

Now, with initial condition y_0=0, and step length h =0.01, we obtain: Start with

n=1, $y_1 = \frac{1}{\alpha\Gamma(\alpha)} t^{\alpha}$

n=2, $y_2 = \frac{-4}{(\alpha+\alpha^2)\Gamma(\alpha)} t^{\alpha+1}$

n=3, $y_3 = \frac{28}{(2\alpha+3\alpha^2+\alpha^3)\Gamma(\alpha)} t^{\alpha+2}$

n=4, $y_4 = \frac{-280}{\Gamma(\alpha+4)} t^{\alpha+3}$ …

The other components were also determined, in which y(t) was evaluated to have the following expansion

$$y(t) = \frac{1}{\alpha\Gamma(\alpha)} t^{\alpha} + \frac{-4}{(\alpha+\alpha^2)\Gamma(\alpha)} t^{\alpha+1} + \frac{28}{(2\alpha+3\alpha^2+\alpha^3)\Gamma(\alpha)} t^{\alpha+2} + \frac{-280}{\Gamma(\alpha+4)} t^{\alpha+3} + \ldots$$

In which, setting α=1, in the above expansion, we obtain the approximate solution to the exact solution in a series form as

$$y(t) = 1 - 2t^2 + \tfrac{14}{3} t^3 - \tfrac{35}{3} t^4 + \tfrac{91}{3} t^5 - \tfrac{728}{9} t^6 + \tfrac{1976}{9} t^7 - \tfrac{5434}{9} t^8 + \ldots$$

and using Pade' approximation [28], we have

$$y(t) = \frac{t + 4.5t^2 + 5.595t^3 + 1.642t^4}{1 + 6.5t + 1.9t^2 + 10.8t^3 + 2.1t^4}$$

In which the exact and approximated solutions are presented in table (3) and figure (3).

5. Discussion

The main objective of this work has been to construct an approximation solution to the Fractional Order Differential Algebraic Equations (FODAE's), by transforming FODAE's into FODE's, and using two proposed methods, the variational method and fractional difference method. The two methods were implemented without using linearization, perturbation or restrictive assumptions. There are two points to make here. **First,** in the variational method, the lagrange multiplier should be determined such that the constructed correction functional is in stationary state, and several approximations will be obtained and the exact solution will be reached at the limit of the resulting successive approximations. **Second,** in the fractional difference method, the fractional derivative approximation and Pade' approximants was combined, and successfully implemented achieved to be promising.

Also, one could see some differences between the exact and the approximated results in the tables and in the figures, due to the terms truncation in the forms of the final approximated solutions.

In future works, we will focus an to implement a new analytical approach, without transforming FODAE's into FODE's.

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Table (1) Solution of $y_1(t)$ by using Variational Approach and the exact of $y_1(t)$ for t=0, …, 2

t	$Y_1(t)$ Exact	$Y_1(t)$ Approximate
0	1	0.833753
0.181818	0.833753	0.695144
0.363636	0.695144	0.579581
0.545455	0.579578	0.483245
0.727273	0.483225	0.402982
0.909091	0.40289	0.336231
1.090909	0.335911	0.280991
1.272727	0.280067	0.235813
1.454545	0.233506	0.199844
1.636364	0.194687	0.172896
1.818182	0.162321	0.155556
2	0.135335	0.833753

Table (2) Solution of y2(t) by using Variational Approach and the exact of y2(t) for t=0, …, 1.2

t	$Y_2(t)$ Exact	$Y_2(t)$ Approximate
0	0	0
0.109091	0.108875	0.107108
0.218182	0.216455	0.210252
0.327273	0.321462	0.309453
0.436364	0.422647	0.40476
0.545455	0.518807	0.49627
0.654545	0.608799	0.584142
0.763636	0.691553	0.66861
0.872727	0.766085	0.750004
0.981818	0.831509	0.82876
1.090909	0.887047	0.905437
1.2	0.932039	0.980736

Table (3) Solution of y(t) by using Fractional Difference Approach and the exact of y(t) for t=0, …, 1.2

t	y(t) Exact	y(t) Approximate
0	0	0
0.109091	0.02746	0.009464
0.218182	0.063174	0.032045
0.327273	0.086277	0.060203
0.436364	0.095101	0.086067
0.545455	0.091769	0.102778
0.654545	0.078945	0.105544
0.763636	0.059012	0.09165
0.872727	0.033896	0.059909
0.981818	0.00508	0.010059
1.090909	-0.026321	-0.057705
1.2	-0.059479	-0.143007

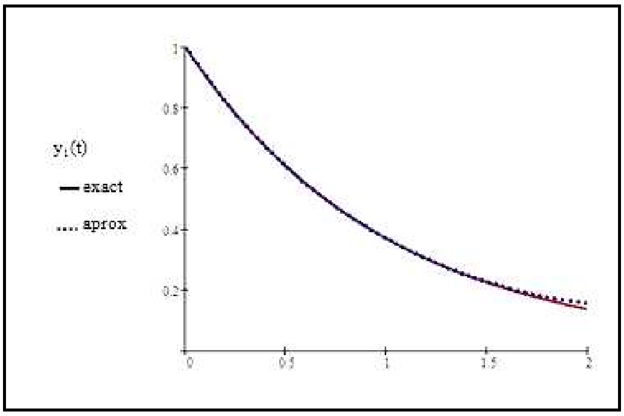

Figure (1)Graph of y_1 (t) by using Variational Approach and the exact of y_1 (t)

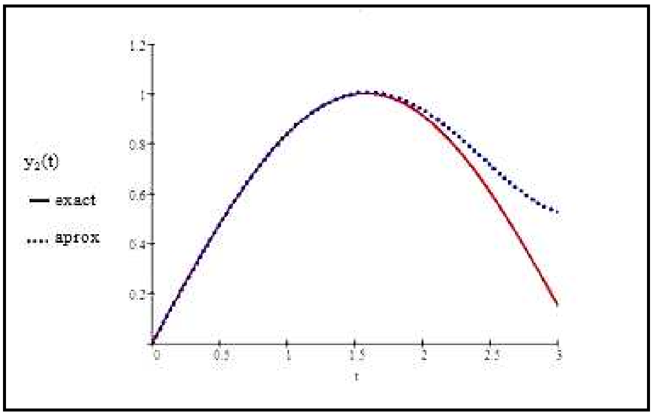

Figure (2)Graph of y_2 (t) by using Variational Approach and the exact of y_2 (t)

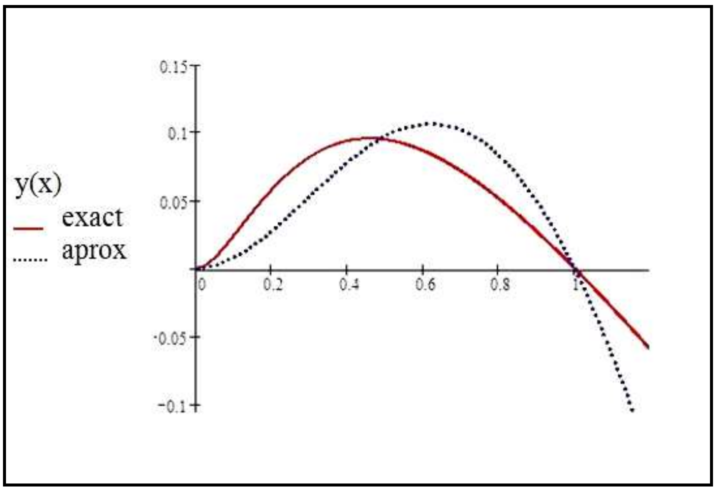

Figure (3) Graph of y (t) by using Fractional Difference Approach and the exact of y (t)