The Estimation of Random Response of a Coupled Cylindrical-Conical Shell System Using Statistical Energy Analysis

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Abstract

A composite, coupled, thin cylindrical-conical shell system made of polyester resin reinforced by uniformly distributed, chopped, E-glass is analyzed using Statistical Energy Analysis. Response displacement estimate of the two subsystems are obtained due the excitation of the cylinder by a broadband white noise of constant spectral density function. The paper is an attempt to study the validity of the SEA hypothesis as applied to coupled built-up structures. This is carried out by the comparison of response estimates of the coupled system in different 1/3rd octave frequency bands with those obtained from Finite Element method. The outcome of this work shows that SEA is a powerful tool for the vibration analysis of coupled systems at high frequencies when the number of interacting coupled resonant modes is high. Percentage error obtained from the comparison of results drops sharply as one goes further in frequency. This paper recommends that the user of SEA must always be aware of the uncertainty of the results obtained. The uncertainty may arise from the improper selection of subsystems, coupling loss factors, and the number of interacting resonant modes of the coupled system.

تقدير الأستجابة العشوائية لنظام اسطواني-مخروطي باستخدام طريقة تحليل الطاقة الأحصائي

الخلاصة

في هذا البحث تم تحليل نظام مخروط-اسطوانة, قشري, مركب, مرتبط مصنوع من مادة البوليستر الرابطة والمقواة بالالياف الزجاجية المقطعة نوع E والموزعة توزيعا منتظما ولقد تم التحليل باستخدام طريقة تحليل الطاقة الاحصائي. تم الحصول على قيم تقديرية لازاحات الانظمة الثانوية نتيجة اثارة الاسطوانة بقوة عشوائية عبارة عن ضوضاء ابيض بحزمة ترددات واسعة ذات دالة طيفية ثابتة. البحث الحالي هو عبارة عن محاولة لدراسة صلاحية فرضية طريقة تحليل الطاقة الاحصائي عند تطبيقها على الانظمة المرتبطة والمركبة. لقد تمت الدراسة بمقارنة الاستجابات التقديرية للانظمة المرتبطة في حزم ترددات ذات فترات ثلثية, والمحسوبة بهذه الطريقة, مع تلك المحسوبة بطريقة العناصر المحددة. لقد بينت النتائج بان الطريقة (SEA) هي اداة فعالة عند الترددات العالية اي عندما يكون هنالك عدد كبير من الاطوار الرنينية المتداخلة. لقد وجد بان نسبة الخطا عند مقارنة النتائج المستخلصة من الطريقتين تنخفض بشكل حاد كلما زادت قيمة التردد. يوصي البحث الحالي مستخدمي طريقة تحليل الطاقة الاحصائي بالانتباه الى موضوع عدم التاكد من صحة النتائج الماخوذة من هذه الطريقة. ان عدم التاكد او الوثوقية بالنتائج قد ياتي من عدم الاختيار الصحيح للانظمة الثانوية, معاملات الارتباط وعدد الاطوار الرنينية المتداخلة للنظام المرتبط.

Keywords: Statistical Energy Analysis (SEA), Finite Element method (FEM)

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Symbol	Description (Units)
$P_{in,i}$	Input power to subsystem I (Watts)
$P_{diss,\ i}$	Power dissipated in subsystem I (Watts)
$P_{trans,\ i,\ j}$	Power transmitted from subsystem i to subsystem j (Watts)
E_i	Total time averaged energy of subsystem i (Jouls)
$n(\omega)$	modal density (modes per unit frequency)
ω	Frequency (rad/sec)
η_i	Dissipation loss factor of subsystem i
C_L	Longitudinal wave speed (m/sec)
S	Surface area of flat plate (m)
t	Thickness of flat plate (m)
Y	Young's modulus (N/m^2)
ν	Poisson's ratio
f	Frequency (Hz)
f_{ring}	Ring frequency of cylinder (Hz)
F	Frequency band factor
Z_m	Mechanical impedance
η_{ij}	Coupling loss factor between subsystems i and j
ρ_a	Mass per unit area (kg/m^2)
C_B	Bending wave speed (m/sec)
τ_{12}	Transmission coefficient between subsystem 1 and 2
X	Thickness ratio
m_i	Total mass of subsystem (kg)

Introduction

Today, Statistical Energy Analysis (SEA) is considered as one of the most famous energy-based methods that provides estimate of large and complex structures to broadband forces in terms of gross system parameters in simple explicit formulae without going into great technical details. The word"Statistical" means, that the variables are drawn from statistical population and all results are expected values.

"Energy" refers to the independent dynamical response variable which is common to the acoustical and mechanical systems; in principle, all other mean-squared quantities can be derived from it. The word "Analysis" means that SEA is more a general approach rather than a particular technique. The principle idea in SEA is that the structure under broadband excitation is partitioned into coupled "subsystems" and the stored and exchanged energies are analyzed [1], [2], [3]. Three main assumptions are made in the hypothesis of SEA. First, the average of power flow between two coupled multi-mode systems excited by wide band forces is proportional to the difference in their average total energies. This is an *ad hoc* extension of an exact relationship for two coupled

oscillators [4]. Second, the rate of energy dissipation by a subsystem is proportional to the stored energy of that subsystem. Third, the energy stored in a subsystem is distributed uniformly among the modes. In this work a coupled, composite, cone-cylinder shell structure made of three layers of polyester resin reinforced by uniformly distributed, chopped, E-glass is analyzed using SEA in a two subsystems model. The system is clamped at the free end of the cylinder. The structure is assumed to be isotropic due to the selected type of reinforcement [10]. The configuration of the coupled system used in the present study is chosen because of the following reasons; first, it has wide range of applications in the mechanical, marine, aeronautical, chemical, civil and power Engineering and second, no similar previous work, in particular, is shown in the literature. The effectiveness of SEA, as an analysis tool, will be verified by the comparison of results with those obtained from Finite Element method.

Statistical Energy Analysis model

A composite cylindrical shell of 2 mm thickness is welded to a composite conical shell of the same thickness to form a two-subsystem coupled structure. The cylinder (subsystem 1) has a length of 300 mm and a mean radius of 75 mm. The conical part (subsystem 2) has a length of 70 mm and small mean radius of 25 mm. The coupled system is shown in Figure (1). The cylindrical part (subsystem1) is excited by a broadband white noise of constant power spectral density function in different 1/3rd octave frequency bands. The SEA model is constructed using the following relations:

The subsystems energies and input powers are related by the following matrix equation [1],

$$\{\overline{P}_{in}\} = \omega\,[A]\{\overline{E}\} \quad ..(1)$$

Equation (1) gives the relationship between the power injected vector $\{\overline{P_{in}}\}$, and the vector of total subsystems energies of vibration $\{\overline{E}\}$. $[A]$ is a matrix of energy influence coefficients in the relevant frequency band having a center frequency ω, and the symbol (¯) indicate time averaged quantity. In SEA a power balance equation is written for each sub-system, so that, for sub-system i,

$$\overline{P}_{in,i} = \overline{P}_{diss,i} + \overline{P}_{trans,ij} \quad . (2)$$

The power balance equation means that the input power into the sub-system i equals to the summation of power dissipated due to damping in sub-system i, $\overline{P}_{diss,i}$ and the power transmitted from sub-systems i to sub-system j, $\overline{P}_{trans,ij}$, through the coupling between sub-systems i and j. The power dissipated by damping at any subsystem can be written as:-

$$\overline{P}_{diss,i} = \omega\eta_i \overline{E}_i \quad \ldots (3)$$

Where, η_i is the damping (internal) loss factor of sub-system i, $\overline{E}_i$ represents the total stored energy in sub-system i, (according to SEA hypothesis, total energy equals to twice of kinetic energy as resonant vibration is assumed). The equation for the net power transmitted from sub-systems i to j is:-

$$\overline{P}_{trans,ij} = \omega\eta_{ij}\overline{E}_i - \omega\eta_{ji}\overline{E}_j \quad \text{.....(4)}$$

Where, η_{ij} , η_{ji} are the coupling loss factors from subsystems i to j and from subsystems j to i respectively and it depends on the type of connection between sub-systems, the material properties, the dimensions of the system and the center frequency of the band, ω (rad/sec). Coupling loss factor is a measure for how much energy is transmitted between coupled subsystems [5].

The equation for the power flows $\overline{P}_{in,i}$ between sub-systems 1 and 2 under typical conditions are expressed as follows:

$$\left(\overline{P}_{in}\right)_1 = \omega\eta_1\overline{E}_1 + \omega\eta_{12}n_1\left(\overline{E}_1 - \overline{E}_2\right)$$

$$\left(\overline{P}_{in}\right)_2 = \omega\eta_2\overline{E}_2 + \omega\eta_{21}n_2\left(\overline{E}_2 - \overline{E}_1\right) \quad \text{..(5)}$$

The coupling loss factors η_{12} and η_{21} are related by the following consistency relation,

$$n_1\eta_{12} = n_2\eta_{21} \quad \text{… (6)}$$

Where n_1 and n_2 are the modal densities of sub-systems 1 and 2 respectively. Modal density is one of the main parameters that describes a subsystem. It is defined as the average frequency spacing between modes or the number of modes per unit frequency (rad/sec in this case). The equation of power injected into sub-systems can now be written as:

$$\left(\overline{P}_{in}\right)_1 = \omega\eta_1\overline{E}_1 + \omega\eta_{12}n_1\left(\frac{\overline{E}_1}{n_1} - \frac{\overline{E}_2}{n_2}\right)$$

…. (7)

$$\left(\overline{P}_{in}\right)_2 = \omega\eta_2\overline{E}_2 + \omega\eta_{21}n_2\left(\frac{\overline{E}_2}{n_2} - \frac{\overline{E}_1}{n_1}\right)$$

A frequently encountered situation, (as in the present), is when one sub-system is directly driven by an external force and the other coupled sub-system is driven only through the coupling. Then, equations (7) reduces to, [3],

$$0 = \omega\eta_2\overline{E}_2 + \omega\eta_{21}n_2\left(\frac{\overline{E}_2}{n_2} - \frac{\overline{E}_1}{n_1}\right)$$

$$\left(\overline{P}_{in}\right)_1 = \omega\eta_1\overline{E}_1 + \omega\eta_{12}n_1\left(\frac{\overline{E}_1}{n_1} - \frac{\overline{E}_2}{n_2}\right)$$

…. (8)

Where, $\frac{\overline{E}_i}{n_i}$ is known as the energy density of sub-system i which has the units of Joules/mode. The modal density of flat plate in flexural vibration is given by [7]:

$$n(\omega) = \frac{S\sqrt{12}}{2\pi C_L t} \quad \text{. ..(9)}$$

.

Where, C_L is the longitudinal wave speed, S is the surface area of plate, and t is the plate thickness. The longitudinal wave speed in flexural motion is defined as,

$$C_L = \sqrt{\frac{Y}{\rho(1-\upsilon^2)}}$$

Where Y is the Young's modulus, υ is the Poisson's ratio and ρ is the material density. Although it has been derived for the prediction of modal density of a flat plate, equation (9) has wide range of structural applications. It is used to calculate the modal density of the conical shell having the same surface area as that of the flat plate (subsystem 2) [1]. The modal density of subsystem 1 (the cylindrical part) is obtained using the following relations [6]:

$$\frac{n(f)_{plate}}{n(f)_{cylinder}} = \sqrt{f / f_{ring}} \quad \text{for} \quad f/f_{ring} \le 0.5$$

$$= 1.4(f/f_{ring}) \quad \text{for} \quad 0.5 < f/f_{ring} < 0.8 \quad \ldots(10)$$

$$= \frac{0.08}{(F-1/F)}\left\{\begin{matrix} F\cos(1.75/F^2(f/f_{ring})^2) \\ -(1/F)\cos(1.75F^2(f/f_{ring})^2) \end{matrix}\right\} \quad \text{for} \quad f/f_{ring} > 0.8 .$$

Where $n(f) = 2\pi\, n(\omega)$, $f_{ring} = C_L/(2\pi r)$, is the ring frequency of cylinder where r is the mean radius of cylinder. F denotes the frequency band factor which has the value of 1.22 for 1/3rd octave frequency bands and the value of 1.414 for octave frequency bands. In the present work, the average modal density is calculated in different 1/3rd octave frequency bands of interest.

The time average input power to a general multi-modal system can be written as [6],

$$\left(\overline{P_{in}}\right) = \frac{1}{2}\left|\tilde{\ell}\right|^2 Re\left\{\tilde{Z}_m^{-1}\right\} \quad \ldots..(11)$$

Where $\left|\tilde{\ell}\right|^2$ denotes the mean square value of the excitation. The real part of drive–point mechanical impedance $\tilde{Z}_m$ of a cylinder of thickness t and mass per unit area ρ_a in flexural vibration is [7]:-

$$Re\left\{\tilde{Z}_m\right\} = 8\sqrt{\frac{Yt^3\rho_a}{12\left(1-\upsilon^2\right)}} \quad \ldots(12)$$

The coupling loss factor η_{12} between flexural mode groups for line junction between two flat plates is evaluated by Lyon [1] and Cremer *et al* [7]. It is conveniently given in terms of the wave transmission coefficient, τ_{12}, (the ratio of transmitted power to the incident power). For a line junction of length L, it can be written as,

$$\eta_{12} = \frac{2C_B L \tau_{12}}{\pi\omega S_1} \quad \ldots(13)$$

Equation (13) reveals that the coupling loss factor depends on the surface area of the driven subsystem, S_1 and the bending wave speed C_B. Where;

$$C_B = \sqrt{1.8 C_L t f} \qquad \ldots(14)$$

f is the center frequency of the 1/3rd octave frequency band in Hz. The normal incidence transmission coefficient for two coupled flat plates is given by [8]:-

$$\tau_{12}(0) = 2\left\{\psi^{1/2} + \psi^{-1/2}\right\}^{-2} \qquad \ldots(15)$$

Where;

$$\psi = \frac{\rho_1 C_{L1}^{3/2} t_1^{5/2}}{\rho_2 C_{L2}^{3/2} t_2^{5/2}} \qquad \ldots(16)$$

ρ_1 and ρ_2 are the mass densities of plates1 and 2 respectively, t_1 and t_2 are the thickness of plates 1 and 2 respectively. The random incidence transmission coefficient τ_{12} is approximated by:

$$\tau_{12} = \tau_{12}(0)\frac{2.754\ X}{1 + 3.24\ X} \qquad \ldots(17)$$

Where;

$$X = \frac{t_1}{t_2} \qquad \ldots\ldots(18)$$

Equation (17) is used, with some approximation, to evaluate the coupling loss factor between cylindrical and conical shells having the same surface areas as the two coupled plates [9]. When all the steps are followed correctly and all variables are defined carefully and then fed to the SEA matrix equation (2), the output will be the total vibrational energies (E_i) that are stored in the coupled subsystems. The vibrational displacement estimates (D_i) are then obtained from energy estimates using the relationship:

$$\overline{D_i} = \sqrt{\frac{E_i}{m_i \omega^2}}\ . \qquad \ldots(19)$$

Where, m_i is the total mass of subsystem i, and ω is the centre frequency of the band.

It should be noted that the dissipation loss factors η_1 and η_2 are obtained from a previous work [10] under academic supervision by the author. A journal paper containing the work has been submitted and it should be published in due time. The experiments are done by exciting each subsystem individually by an impact hammer and inferring the average modal damping ratio, using the conventional half power bandwidth method [2], on the experimentally constructed frequency response functions. The set-up is a composite cylinder-cone coupled system made of chopped E-Glass reinforced polyester resin. The measured value of average modal damping ratio of the coupled subsystems is found to be about 2%. Note that, at resonance, the value of dissipation loss factor is calculated as twice that of damping ratio [2]. Material properties of the composite structures, such as Young's modulus, density and Poisson's ratio are experimentally obtained using samples of such a structure and found to be 3.964 Gpa, 1369 kg/m^3 and 0.45 respectively [10]. All experimental data described above are used in SEA model..

A computer program for a two-subsystem Statistical Energy Analysis model is constructed according to the theory described above. Results of the displacement response of the coupled system due to the randomly exciting system 1 (cylinder) are obtained in different 1/3rd octave frequency bands centered at 315 Hz to 20000 Hz. A constant value of 10000 N^2 mean square value is selected for the excitation in all the frequency bands. The results are shown in figures (7), (9) and (10). It should be noted that each data point on those curves results from averaging of many values over the frequency band of interest and also it is a spatial average of many response, and excitation locations on the coupled system.

Finite Element model

A Finite Element random vibration analysis is performed on the coupled system under consideration, (shown in figure (1)), using the PSD module of *Ansys* finite element package. The cylinder (subsystem1) is excited by a random, white noise, loading having a constant spectral density value and a mean square value of 10000 N^2. An 8 node, six degrees of freedom shell63 element is chosen for the analysis after performing a convergence test. Random vibration analysis is chosen rather than any other types of analyses because the basic hypothesis of SEA assumes that the excitations of all modes of the coupled system are equally probable. A typical modal picture for the coupled structure under investigation is shown in figure (2). The picture is obtained by performing harmonic analysis with an arbitrary harmonic force of 1000 N applied at subsystem 1 and an overall average modal damping of 0.2%. The modal picture reveals low modal overlap factor at low frequencies and high modal overlap factor as one travels further in frequency. The Modal overlap factor is a measure of the number overlapped modes in the frequency bands of interest. It is a function of dissipation loss factor, modal density and centre frequency of the band. Figures (3)-(6) show the number of interacting resonant modes in selected 1/3rd octave bands at low and high frequencies. Random response (displacement) estimates are predicted numerically in different 1/3rd octave frequency bands centered at 315 Hz to 20000 Hz and the results are shown in Figure (7), (8) and (9).

Results and discussions

As stated earlier, the purpose of the present study is to indicate the effectiveness of Statistical Energy Analysis as a technique for the prediction of vibration response estimate of coupled multi-modal structures. This is done by the comparison of SEA outcome with the numerical results obtained from Finite Element method. SEA response estimates (displacement) of the coupled cone-cylinder structure are shown in figure (7). The figure displays the vibrational displacement of each subsystem in the form of frequency and spatial averages in different 1/3rd octave frequency bands. It is shown clearly that the energy of the directly excited subsystem 1 is always greater than that of subsystem 2 which is receiving an indirect excitation. A difference of 4-6 dB is noted between the displacement estimates of the coupled subsystems in all the

frequency bands. The corresponding results obtained from FEM are shown in figure (8). Similar observation is noted for the difference between the energies of the coupled systems. Finite Element Method is considered by the researchers as one of the most trusted, certain approximate methods. It performs the analysis in the basis of mode by mode calculations. On the other hand, SEA predicts the response estimate in the form of vibrational energy of a coupled subsystem in an overall manner, without going into great technical details, in a quick and smart procedure that is favorable to Engineers especially in the early stages of design. Although SEA is attractive approach it hides some degree of uncertainty. An attempt has been made by the author of the present work [11] to explore and quantify the uncertainty in applications of SEA to different one dimensional and two dimensional multi-modal systems. However, reasonable findings have been obtained concerning the uncertainty in applications of SEA, but lots of research works yet to be done in this area to encourage researchers that are using SEA.

The comparison between FEM and SEA displacement response estimates of subsystem 1 is shown in figure (9). The agreement between results obtained from the two methods is good at the bands whose centre frequencies are high. SEA response estimate approaches that of FEM as the frequency increases. The reason behind this behavior is attributed to the increase in modal densities of the coupled systems which leads to an increase in the number of the interacting, coupled resonant modes in the frequency band of interest as shown in the typical high modal overlap picture of figure (6). The large discrepancies between the results shown at low frequencies are due to the lack in the number of the coupled resonant modes. Examples of the modal pictures at low order frequency bands are shown in figures (3), (4), and (5) where only one or two modes (low modal overlap factor) are exist in the frequency bands of interest. The small number of modes is obviously not sufficient for the energy sharing process between coupled subsystems. This is considered as one of the drawbacks in application of SEA.

Similar trends, as above, are noted in figure (10) when comparing the displacement estimates that are obtained from SEA and FEM for subsystem 2.

To support the statement described above, percentage errors between SEA and FEM is obtained. It is calculated as,

$$\%\,Error = \left| \frac{SEA\,estimate - FEM\,estimate}{FEM\,estimate} \right| * 100 \qquad \ldots(20)$$

The percentage errors of displacement estimate in the different 1/3rd octave frequency bands are displayed in figure (11). The figure shows that low order frequency bands results exhibit high percentage errors compared with those obtained in the high order frequency bands for both coupled subsystems. This is attributed to the number of interacting resonant frequency modes involved in the analysis.

Concluding remarks

The main concluding remarks obtained from the present work can be summarized as follows:

1. Statistical Energy Analysis is a powerful tool for the prediction of random response, in mid and high frequencies, of coupled multi-mode systems in simple explicit formulae without going into great details as other methods do. The penalty of the apparent simplicity of this approach is that the accuracy of response prediction for any individual practical system is subject to uncertainty. One of the important sources of uncertainty relates to the center frequency and analysis band, i.e. the number of coupled resonant modes involved in analysis (modal overlap factor)
2. The comparison of response estimates obtained from SEA calculations agree well with those obtained from FEM analysis at mid and high frequencies.
3. The percentage error obtained from the comparison of SEA and FEM displacement drops sharply as the frequency increases.

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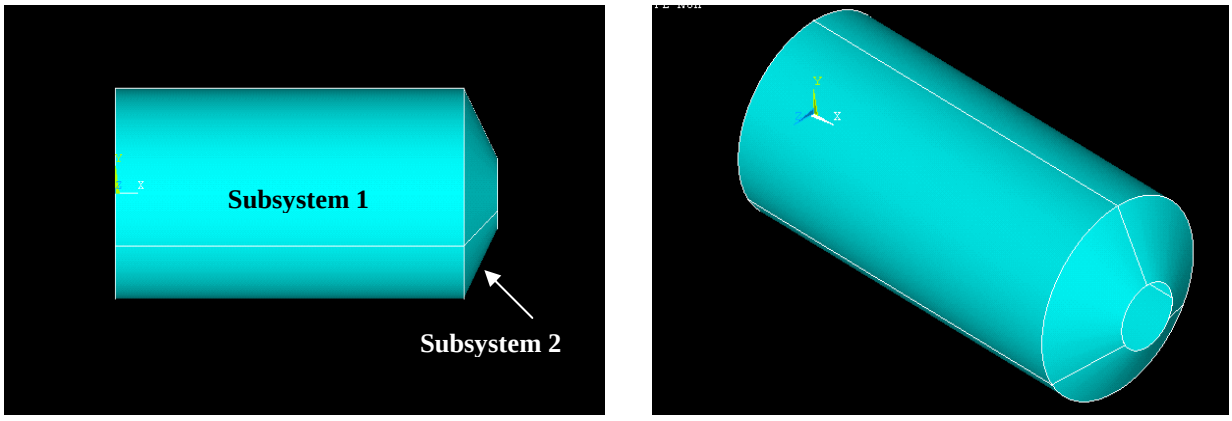

Figure (1): Coupled con-cylinder structure

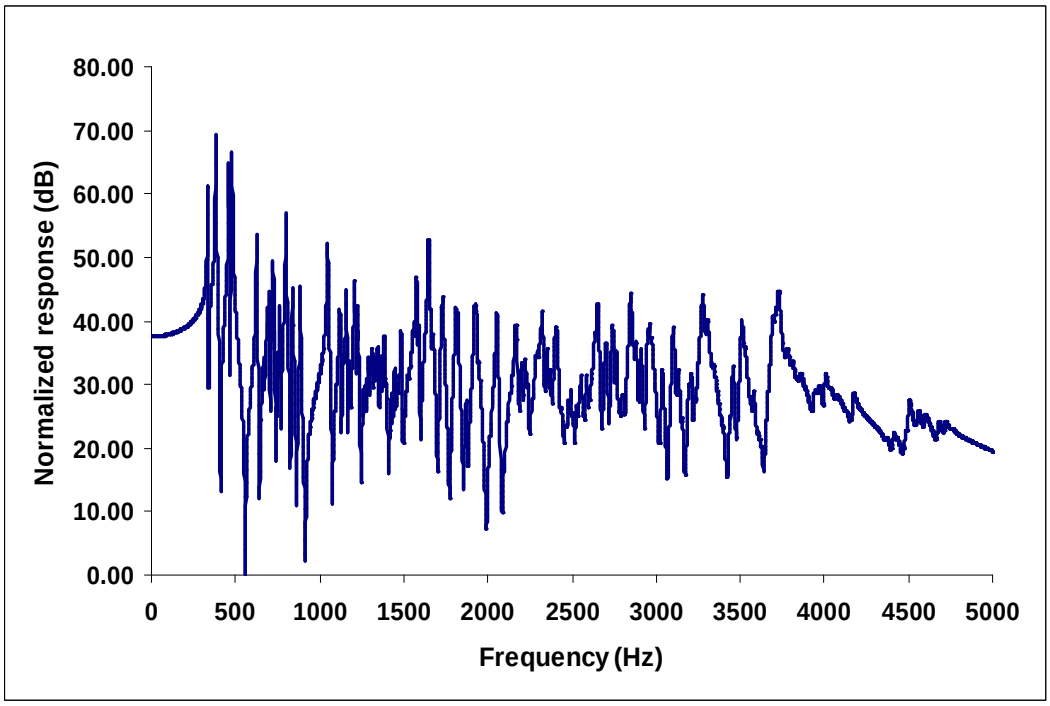

Figure (2): Harmonic response of co-cylinder structure. F=1000 N, damping ratio, ζ=0.2%

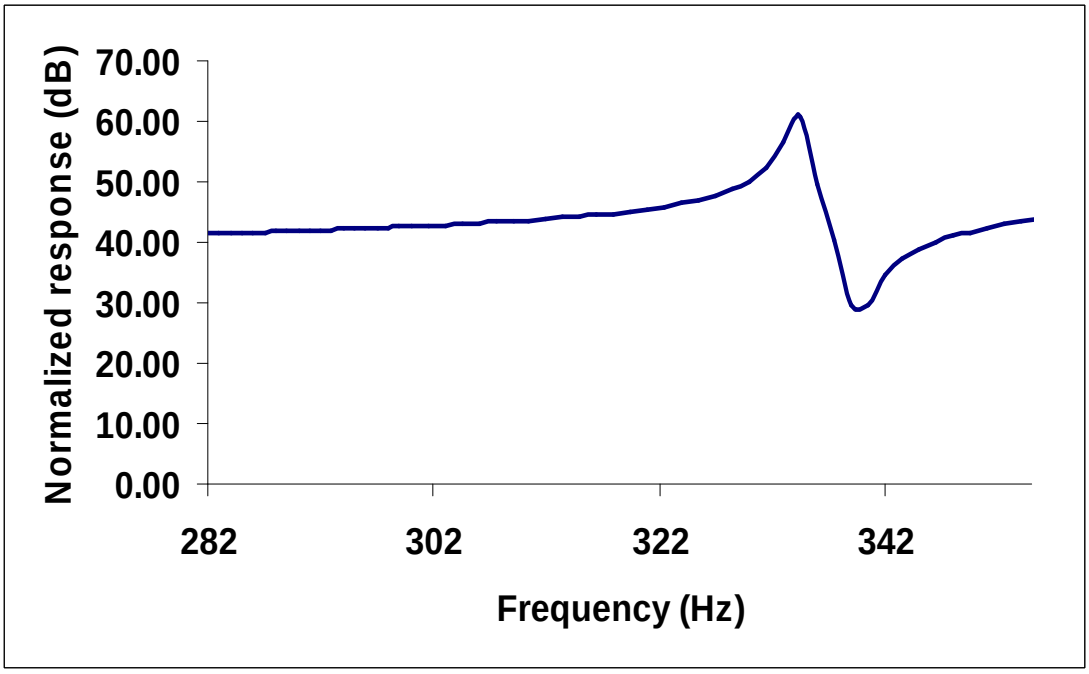

Figure (3): Single resonant mode in the 1/3rd octave band centered at 315 Hz, (low Modal Overlap Factor)

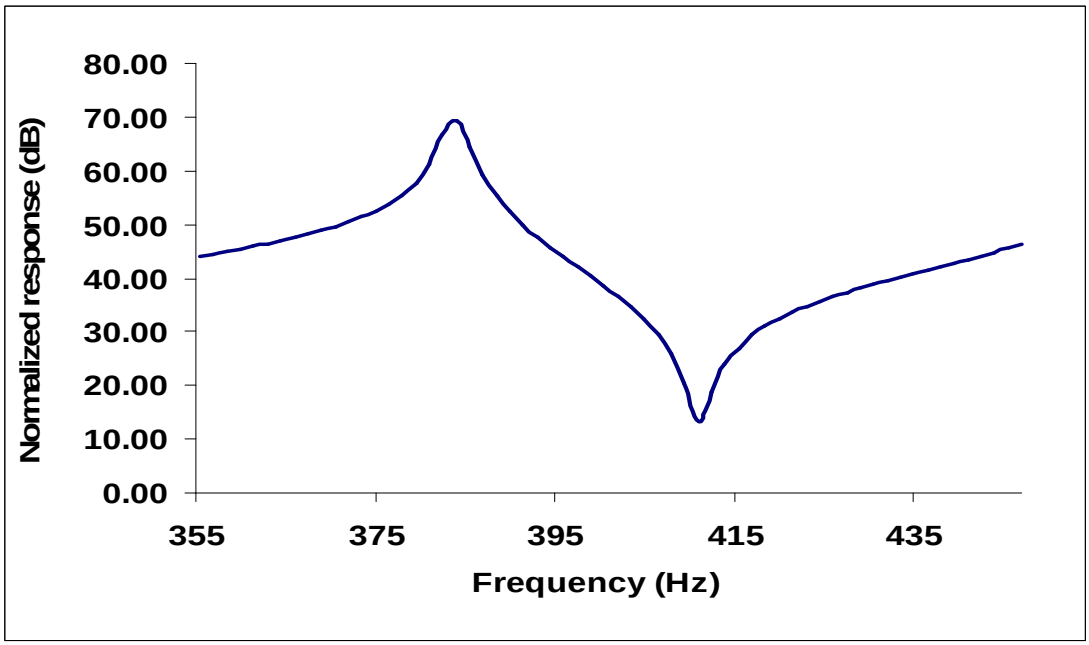

Figure (4): Single resonant mode in the 1/3rd octave band cantered at 400 Hz, (low modal overlap factor).

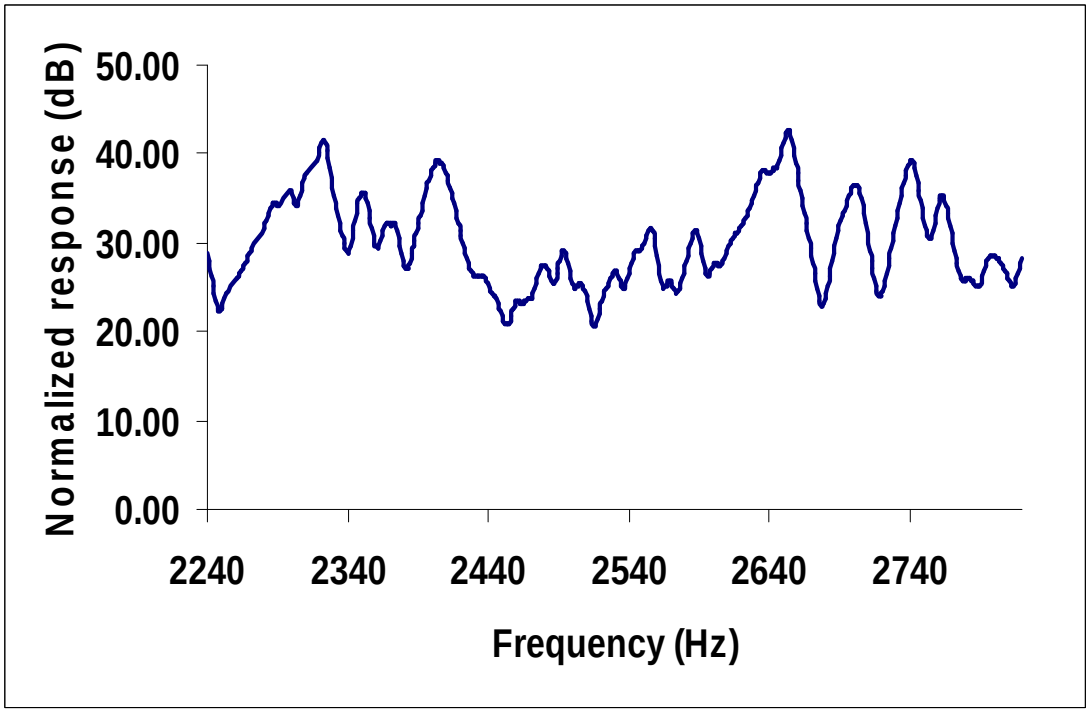

Figure (5): Many resonant modes in the 1/3rd octave band centered at 2000Hz, (high modal overlap factor)

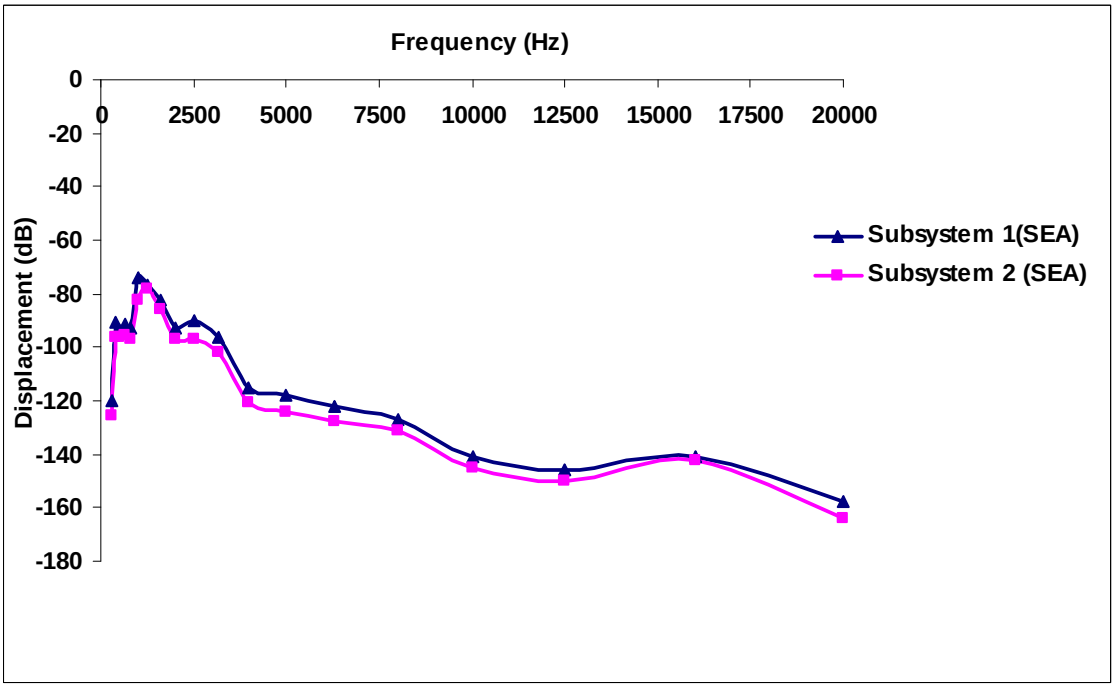

Figure (6): Frequency and special averaged SEA estimates of displacement of the coupled cone-cylinder structure in different 1/3rd octave frequency bands.

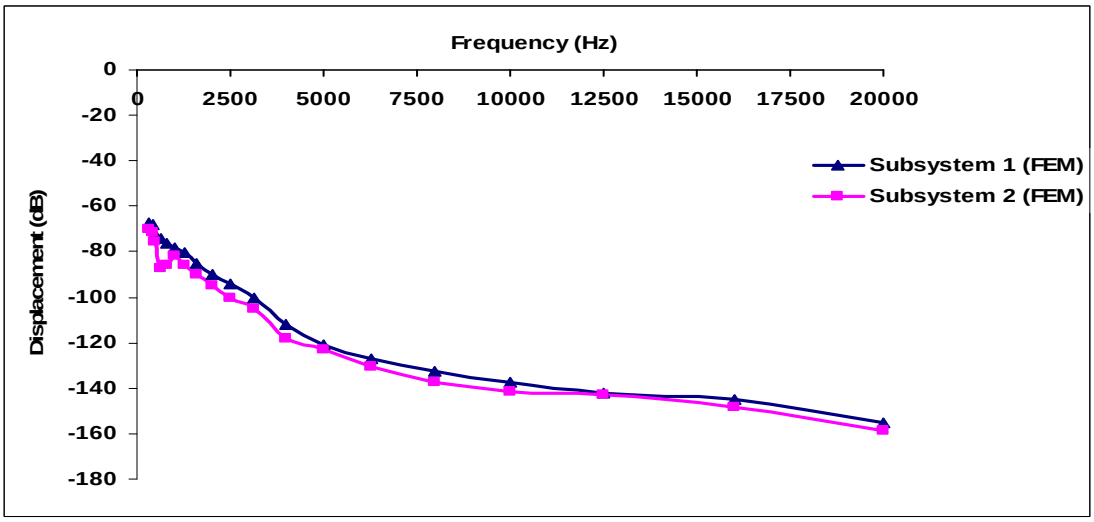

Figure (7): Frequency and specially averaged FEM estimates of displacement of the coupled cone-cylinder structure in different 1/3rd octave frequency bands.

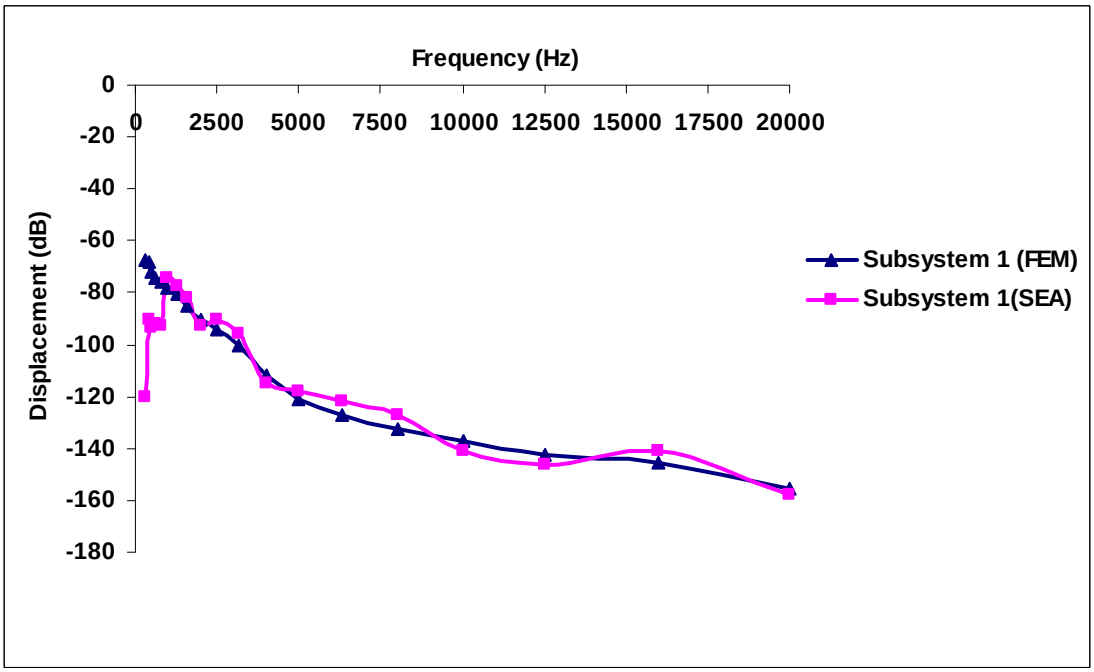

Figure (8): Comparison between FEM and SEA estimates of vibrational displacement of the excited subsystem 1 in different 1/3rd octave frequency bands.

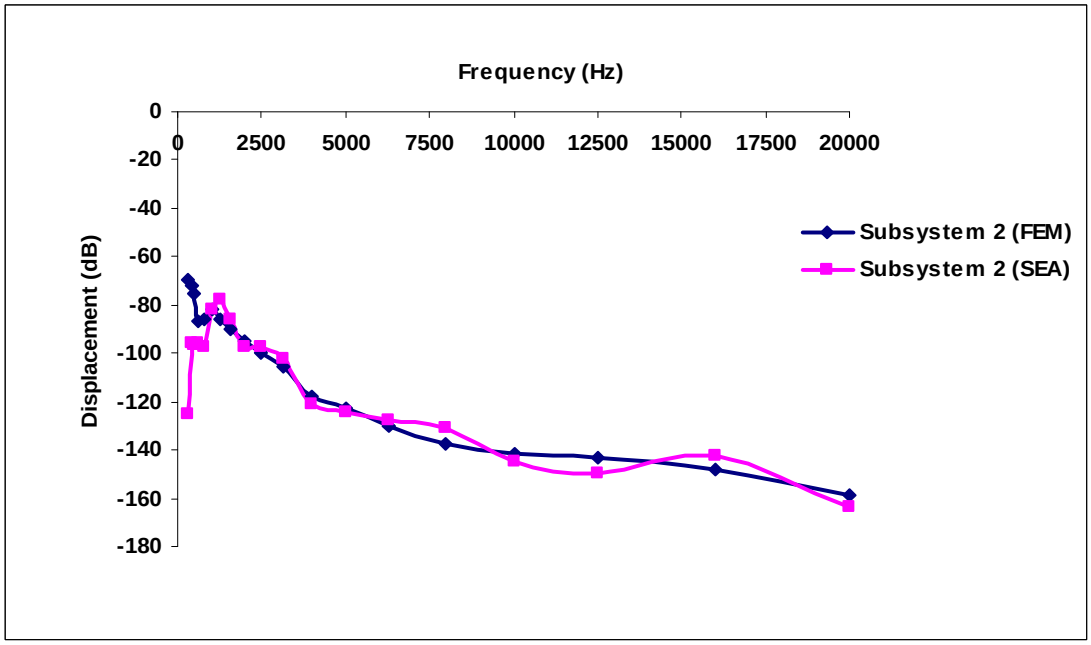

Figure (9): Comparison between FEM and SEA estimates of vibrational displacement of the indirectly excited subsystem 2 in different 1/3rd octave frequency bands.

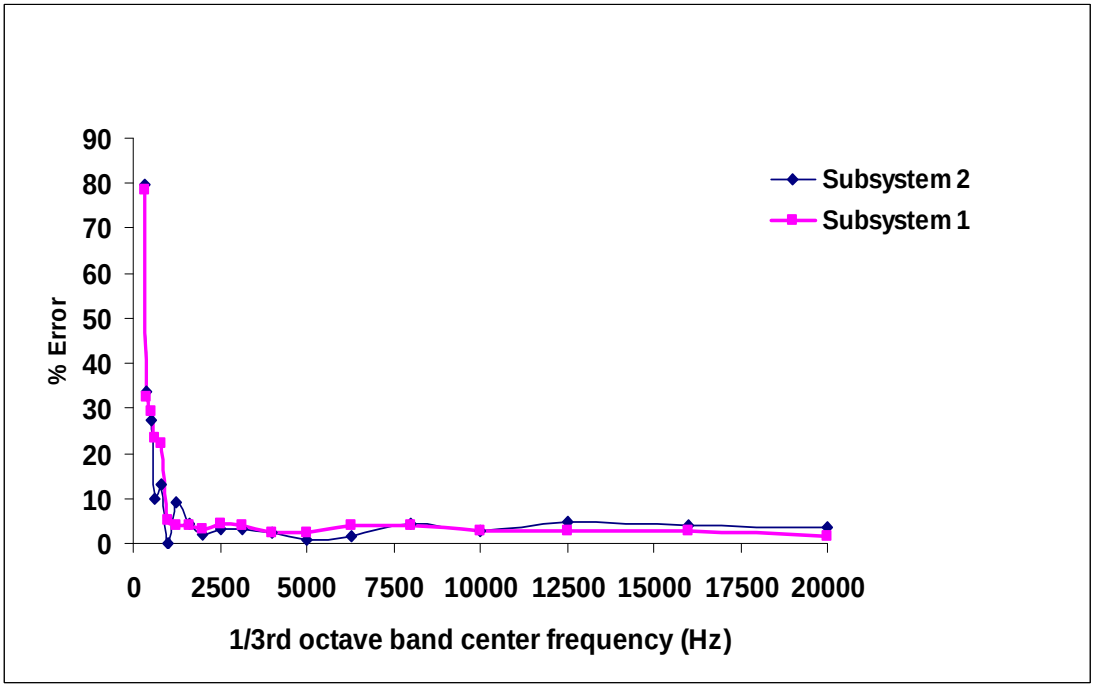

Figure (10): The Estimated percentage errors of displacement response of coupled subsystems obtained from the comparison of SEA and FEM results.