FINITE ELEMENT ANALYSIS OF THICK ORTHOTROPIC SQUARE PLATES ON ELASTIC FOUNDATIONS

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ABSTRACT

This paper deals with the linear elastic behavior of thick orthotropic square plates on Winkler type elastic foundations with both compressional and tangential resistances. The finite element method with different isoparametric thick plate and brick elements are used to solve problems, which were previously solved by the finite difference method. Good agreement is found between the different methods with percentage difference about 1%..

KEYWORDS

Finite element, Orthotropic material, Thick square plates, Winkler foundations.

NOTATIONS

Symbols	Description
A	Cross-sectional area of the plate.
[B]	Strain-displacement matrix.
c^2	Correction factor for transverse shear.
D_x, D_y	Flexural rigidities of orthotropic plates in x and y directions.
D_{xy}	Torsional rigidity of orthotropic plates in x and y directions.
E_x, E_y, E_z	Moduli of elasticity of orthotropic plates in x,y and z directions.
G_{xy}, G_{xz}, G_{yz}	Shearing modulus for xy, xz and yz planes.
h	Plate thickness.
I	Moment of inertia for plate section per unit width.
[J]	Jacobian matrix.
$[K_p]$	Stiffness matrix for the plate.
K_x, K_y, K_z	Moduli of subgrade reactions in x,y and z directions.
M_x, M_y	Bending moments in xz and yz planes (per unit width).
M_{xy}	Twisting moments (per unit width) in x and y direction.
[N]	Matrix containing the interpolation shape functions
$N_1, N_2 \ldots$	Shape functions.
P	Applied concentrated load.
P(x,y)	Soil reaction in Cartesian coordinates.
Q_x, Q_y	Transverse shearing force per unit width in x and y direction.

continued-NOTATIONS

Symbols	Description
q(x,y)	Transverse load per unit area in z direction
x,y,z	Cartesian coordinates.
u,v	Displacements in x and y directions
w	Displacement in z-direction.
$\{\delta\}$	Total displacements in the system.
$\varepsilon_x,\varepsilon_y,\varepsilon_z$	Normal strains in x, y and z directions.
ξ,ή,ζ	Local coordinates system.
ψ_x,ψ_y	Rotations of the transverse sections in xz or yz-planes.
$\gamma_{xy},\gamma_{yz},\gamma_{xz}$	Engineering shearing strains in xy, yz and xz-planes.
$\tau_{xy},\tau_{yz},\tau_{xz}$	Shearing stresses in xy, yz and xz planes.
$\sigma_x,\sigma_y,\sigma_z$	Normal stresses in x, y and z directions.

INTRODUCTION

Plates are plane structures of constant or variable thickness and bounded by two surfaces which are the top and bottom faces of the plate and by straight or curved transverse edges.

These are some reviews of early studies on thick plates.

Hinton et al. (1975) [1] used plate bending isoparametric finite elements with curved boundaries and variable thickness, allowing for the effect of transverse shearing deformations. The given examples show applications to thin, thick cellular and sandwich plates.

Rajapakse and Selvadurai (1986) [2] used the finite element analysis for the flexural interaction between an elastic thick plate and elastic half-space. It is found that the heterosis plate element is capable of modeling the plate-elastic medium interaction very efficiently. A square plate on elastic half space was considered**.**

Al-Jubori (1992) [3] solved the problem of isotropic thick rectangular plates on elastic foundations with both normal and frictional resistances by finite differences and finite elements. A four- node element was used. Results showed good agreements with the solution by finite differences especially for plate with large thicknesses and under distributed loadings.

Al-Mahdi (1994)[4] solved the problem of thick orthotropic rectangular plates on elastic foundations with both normal and frictional resistances by finite differences and finite elements. Results showed good agreement with the solution by finite differences especially for plates with large thickness and under distributed loadings.

Mishra and Chakrabarti (1996) [5] studied the behavior of flexible rectangular plates resting on tensionless elastic foundation. They analyzed the problem using the finite element method. Nine-node Mindlin elements has been adopted for modeling the plate to account for transverse shear effects with realistic design parameters being studied.

Buczkowski and Torbacki (2001) [6] analyzed rectangular and circular plates resting on two-parameter elastic foundation by using finite elements. The plate subjected to combined loading and permitting various types of boundary conditions. The formulation of the problem takes into account the shear deformation of the plate and the surrounding interaction effect outside the plate.

Liu and Riggs (2002) [7] derived a general formulation for a family of N-node, higher-order, displacement-compatible, triangular, Reissner/Mindlin shear-deformable plate elements. Many problems of isotropic and orthotropic thick rectangular and circular plates were solved using the finite element method with 3-nodes and 6- nodes triangular quadratic element

In this paper, Mindlin's thick plate theory is used to analyze thick orthotropic square plates on elastic foundations subjected to generalized loadings which are externally distributed shearing forces at top and bottom faces of the plate and distributed moments, in addition to the usually applied transverse loads. The transverse section has three degrees of freedom (the deflection w and the two rotations of the normal line to the middle plane, ψ_x and ψ_y, in case of plate bending element) or (the deflection w and the displacements u and v in case of brick element). The elastic foundation is represented by a Winkler model, which is assumed that the foundation is consisting of closely spaced independent linear springs normal and tangential to the plate as shown in figure (1).

FINITE ELEMENT MODEL

The finite element method is used to solve thick square plates by using 9 plate elements over a quarter of the plate. 8-node isoparametric plate bending elements are used. Also, 20-node isoparametric brick elements (two layers in thickness with 4 elements in each layer) are used. Different numbers of finite element mesh of brick element are used. The eight-element mesh gives accurate results. The mesh of the finite element is shown in figure (2).

The two-dimensional isoparametric thick plate element in local coordinates ξ and η has n nodes. Each node i has three degrees of freedom. They are (w_i, ψ_{xi}, ψ_{yi}) in Cartesian coordinates. Thus, the element degrees of freedom may be listed in the vector (or column matrix).

$$\{\delta^e\} = [w_1, \psi_{x1}, \psi_{y1}, \ldots\ldots\ldots\ldots w_n, \psi_{xn}, \psi_{yn}]$$

The degrees of freedom in Cartesian coordinate (w, ψ_x and ψ_y) can be defined in terms of shape function:

$$\left.\begin{aligned} w(\xi,\eta) &= \sum_{i=1}^{n} N_i . w_i \\ \psi_x(\xi,\eta) &= \sum_{i=1}^{n} N_i . \psi_{xi} \\ \psi_y(\xi,\eta) &= \sum_{i=1}^{n} N_i . \psi_{yi} \end{aligned}\right\} \tag{1}$$

The Jacobian matrix [J] in Cartesian coordinates is obtained from the following expression:

$$[J] = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} = \sum_{i=1}^{n} \begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} x_i & \dfrac{\partial Ni}{\partial \xi} y_i \\ \dfrac{\partial N_i}{\partial \eta} x_i & \dfrac{\partial N_i}{\partial \eta} y_i \end{bmatrix} \tag{2}$$

The strains are defined in terms of the nodal displacements and shape function derivatives, the expression in Cartesian coordinates is given:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \sum_{i=1}^{n} \begin{bmatrix} 0 & -\dfrac{\partial N_i}{\partial x} & 0 \\ 0 & 0 & -\dfrac{\partial Ni}{\partial y} \\ 0 & -\dfrac{\partial N_i}{\partial y} & -\dfrac{\partial Ni}{\partial x} \\ \dfrac{\partial N_i}{\partial x} & -N_i & 0 \\ \dfrac{\partial N_i}{\partial y} & 0 & -N_i \end{bmatrix} \begin{Bmatrix} w_i \\ \psi_{xi} \\ \psi_{y_i} \end{Bmatrix} \tag{3}$$

or

$$\{\varepsilon^e\} = \sum_{i=1}^{n} [B_i]\{.\delta^e\} \tag{4}$$

The generalized stress-strain relationship for a plate of orthotropic elastic materials in Cartesian coordinates is written as:

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} D_x & \nu_{yx}.D_x & 0 & 0 & 0 \\ \nu_{xy}.D_y & D_y & 0 & 0 & 0 \\ 0 & 0 & D_{xy} & 0 & 0 \\ 0 & 0 & 0 & c^2.G_{xz}h & 0 \\ 0 & 0 & 0 & 0 & c^2.G_{yz}h \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \quad (5)$$

or

$$\{\sigma^e\} = [D]_{oc}\{\varepsilon^e\} \quad (6)$$

where $[D]_{oc}$ is the matrix of elastic constant for the orthotropic elastic thick plate in Cartesian coordinates.

The element stiffness matrix for thick orthotropic plate in Cartesian coordinates is given as:

$$[K_P] = \sum_{i=1}^{n} \int_{-1}^{+1}\int_{-1}^{+1} [B_i]^T [D]_{oc} [B_i] \det J d\xi d\eta \quad (7)$$

The three-dimensional element in local coordinates (ξ, η, ζ) at node i with nodal displacements at (x,y,z) are u_i, v_i and w_i respectively. Thus, the element displacement may be listed in the vector (or column matrix).

$$\{\delta^e\} = [w_1, u_1, v_1, ..., w_n, u_n, v_n]$$

The isoparametric definition of the brick element is:

$$\left.\begin{aligned} u(\xi,\eta,\zeta) &= \sum_{i=1}^{n} N_i(\xi,\eta,\zeta).u_i \\ \nu(\xi,\eta,\zeta) &= \sum_{i=1}^{n} N_i(\xi,\eta,\zeta).\nu_i \\ w(\xi,\eta,\zeta) &= \sum_{i=1}^{n} N_i(\xi,\eta,\zeta).w_i \end{aligned}\right\} \quad (8)$$

where $N_i(\xi,\eta,\zeta)$ represents the shape functions for the global coordinates $x(\xi,\eta,\zeta)$,$y(\xi,\eta,\zeta)$, $z(\xi,\eta,\zeta)$ at node i.

The shape function N_i is a function of the local coordinates, while the derivatives of shape function should be expressed in terms of the global Cartesian coordinates:

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \\ \dfrac{\partial N_i}{\partial \zeta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} & \dfrac{\partial z}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} & \dfrac{\partial z}{\partial \eta} \\ \dfrac{\partial x}{\partial \zeta} & \dfrac{\partial y}{\partial \zeta} & \dfrac{\partial z}{\partial \zeta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \\ \dfrac{\partial N_i}{\partial z} \end{Bmatrix} \quad (9)$$

The strains are defined in terms of the nodal displacement and shape function derivatives in Cartesian coordinates by the expression:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \sum_{i=1}^{n} \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial x} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial z} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial x} \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} \qquad (10)$$

or

$$\{\varepsilon^e\} = \sum_{i=1}^{n} [B_i].\{\delta^e\} \qquad (11)$$

For the stress-strain relations:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{Bmatrix} = \begin{bmatrix} \frac{(1-\nu_{yz}.\nu_{zy})E_x}{\varphi} & \frac{(\nu_{yx}+\nu_{zx}.\nu_{yz})E_x}{\varphi} & \frac{(\nu_{zx}+\nu_{yx}.\nu_{zy})E_x}{\varphi} \\ \frac{(\nu_{yx}+\nu_{zx}.\nu_{yz})E_y}{\varphi} & \frac{(1-\nu_{xz}.\nu_{zx})E_y}{\varphi} & \frac{(\nu_{zy}+\nu_{xy}.\nu_{zx})E_y}{\varphi} \\ \frac{(\nu_{zx}+\nu_{yx}.\nu_{zy})E_z}{\varphi} & \frac{(\nu_{zy}+\nu_{xy}.\nu_{zx})E_z}{\varphi} & \frac{(1-\nu_{xy}.\nu_{yx})E_z}{\varphi} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{Bmatrix} \qquad (12a)$$

$$\begin{Bmatrix} \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} G_{xy} & 0 & 0 \\ 0 & G_{xz} & 0 \\ 0 & 0 & G_{yz} \end{bmatrix} \begin{Bmatrix} \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \qquad (12b)$$

where,

$$\phi = 1-\nu_{xy}\nu_{yx}-\nu_{yz}\nu_{zy}-\nu_{zx}\nu_{xz}-2\nu_{xy}\nu_{yz}\nu_{zx}$$

For certain orthotropic materials, an approximate relation exists for the shear modulus [8]:

$$G_{xy} = \frac{E_x . E_y}{E_x + E_y (1 + 2.\nu_{xy})} \qquad (13)$$

Similar expressions exist for G_{xz} and G_{yz}. These relations are used in the present study.

The stiffness matrix for orthotropic elastic brick element in Cartesian coordinates is given as:

$$[K_p] = \sum_{i=1}^{n} \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} [B_i]^T [D][B_i] \det J \, d\xi d\eta d\zeta \qquad (14)$$

APPLICATIONS

Two cases of thick orthotropic square plates on elastic foundations are considered. The cases are a simply supported and a fixed edge plate under uniformly distributed load as shown in figure (3).

1. For the simply supported edge plate, figures (4) and (5) show the deflection profile and bending moment diagram in x-direction by both the finite differences [Al-Mahdi (1994)[4]] and the present study. The results show good agreement by these two methods. The difference in central deflection is 3.8% and in central moment it is 2.8 % in case of using plate bending elements and the difference in central deflection is 1.9% and in central moment 1.4 % in

case of using brick elements. Table (1) shows the result of central deflection by different methods.

2. For the clamped edge plate, figures (6) and (7), show the deflection profile and bending moment diagram in x-direction by both the finite differences [Al-Mahdi (1994) [4]] and the present study. The difference in central deflection is 3.6% and in central moment 8.4 % in case of using plate bending elements and the difference in central deflection is 1.8% and in central moment 3.7 % in case of using brick elements.

PARAMETRIC STUDY

To study the effects of elastic foundations and thickness on the behavior of thick orthotropic square plates, a simply supported thick plate shown in figure (8) (K_x=K_y =20000 kN/m^3) is considered. The loading was taken to be uniformly distributed load (q=25 kN/m^2). The effects of variation of vertical and horizontal subgrade reactions on the results of central deflection and bending moments of the thick orthotropic plate are considered. The following points are concluded from the study of the variation of vertical and horizontal subgrade reactions.

- To show the effect of variation of the vertical subgrade reaction on the results, an orthotropic square plate with simply supported edges and resting on vertical subgrade reaction with various values (neglecting the effect of frictional restraints) are studied. Figures (9) and (10) show the variation of vertical subgrade reaction on the central deflection and bending moments. From these figures, the central deflections and moments will decrease as the vertical subgrade reaction is increased because of increasing resistance from the foundation. It was found that by increasing the vertical subgrade reaction from (0.0 to 30000 kN/m^3), the central deflection is decreased by 0.45% and the central moment by 0.50% [9].

- To show the effect of variation of horizontal subgrade reaction, a simply supported thick plate with vertical subgrade reaction (K_z=10000 kN/m^3) and horizontal subgrade reactions of various values of (K_x and K_y) are considered. Figures (11) and (12) show the variation of horizontal subgrade reaction (K_x and K_y) with central deflections and bending moments. From these figures, small reduction on central deflections and bending moments occurs as the horizontal subgrade reactions are increased because of slightly increasing of foundation resistances. It was found that by increasing the horizontal subgrade reaction from (0.0 to 30000 kN/m^3), the central deflection is decreased by 0.04% and the central moment is decreased by 0.08% [9].
- To study the effect of thickness (or stiffness) of plate on the results of central deflection and moments, simply supported plates with various thicknesses are considered. Figures (13) and (14) show the effect of variation of thickness of plate on

central deflection and bending moments of thick orthotropic square plate. From these figures, the central deflection will decrease as the thickness of plate is increased because of increasing plate stiffness. But, the central moment will increase as thickness of plate is increased. It was found that by increasing the thickness of the thick plate from (0.15 to 0.3 m), the central deflection is decreased by 82.90% and the central resisting moment is increased by 1.0% [9].

CONCLUSIONS

1. The results from the finite element method are plotted with the results of examples previously solved by using finite differences to check the accuracy of this explicitly different method. Good agreements are obtained between these methods with percentage difference about 1.0%.
2. The central deflection will decrease as the thickness of plate is increased because of increasing plate stiffness. But, the central moment will increase as thickness of plate is increased. It was found that by increasing the thickness of the thick plate from (0.15 to 0.3 m), the central deflection is decreased by 82.90% and the central resisting moment is increased by 1.0%
3. The central deflections and moments will decrease as the vertical subgrade reaction is increased because of increasing resistance from the foundation. It was found that by increasing the vertical subgrade reaction from (0.0 to 30000

kN/m^3), the central deflection is decreased by 0.45% and the central moment by 0.50%

4. Small reduction on central deflections and bending moments occurs as the horizontal subgrade reactions are increased because of slightly increasing of foundation resistances. It was found that by increasing the horizontal subgrade reaction from (0.0 to 30000 kN/m^3), the central deflection is decreased by 0.04% and the central moment is decreased by 0.08%

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Table (1): Central Deflection of Orthotropic Square Plate.

Boundary condition	Deflection (m)			
	Al-Mahdi (1994)[3] (Finite difference)	**Plate element (Present study)**	**Brick element (Present study)**	**Exact solutions (Timoshenko and Woinosky - Krieger, S. (1959) [10])**
Simply supported plate	12.4×10^{-4}	11.987×10^{-4}	12.256×10^{-6}	12.657×10^{-6}
Clamped edge plate	7.764×10^{-4}	7.557×10^{-6}	7.732×10^{-6}	7.785×10^{-6}

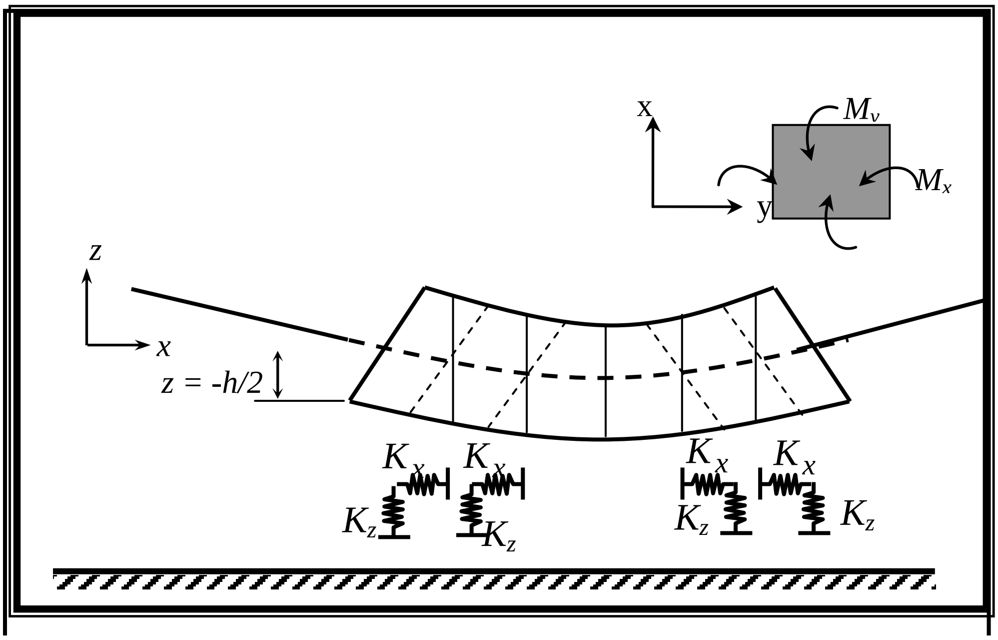

Figure (1) Winkler Compression and Friction Model.

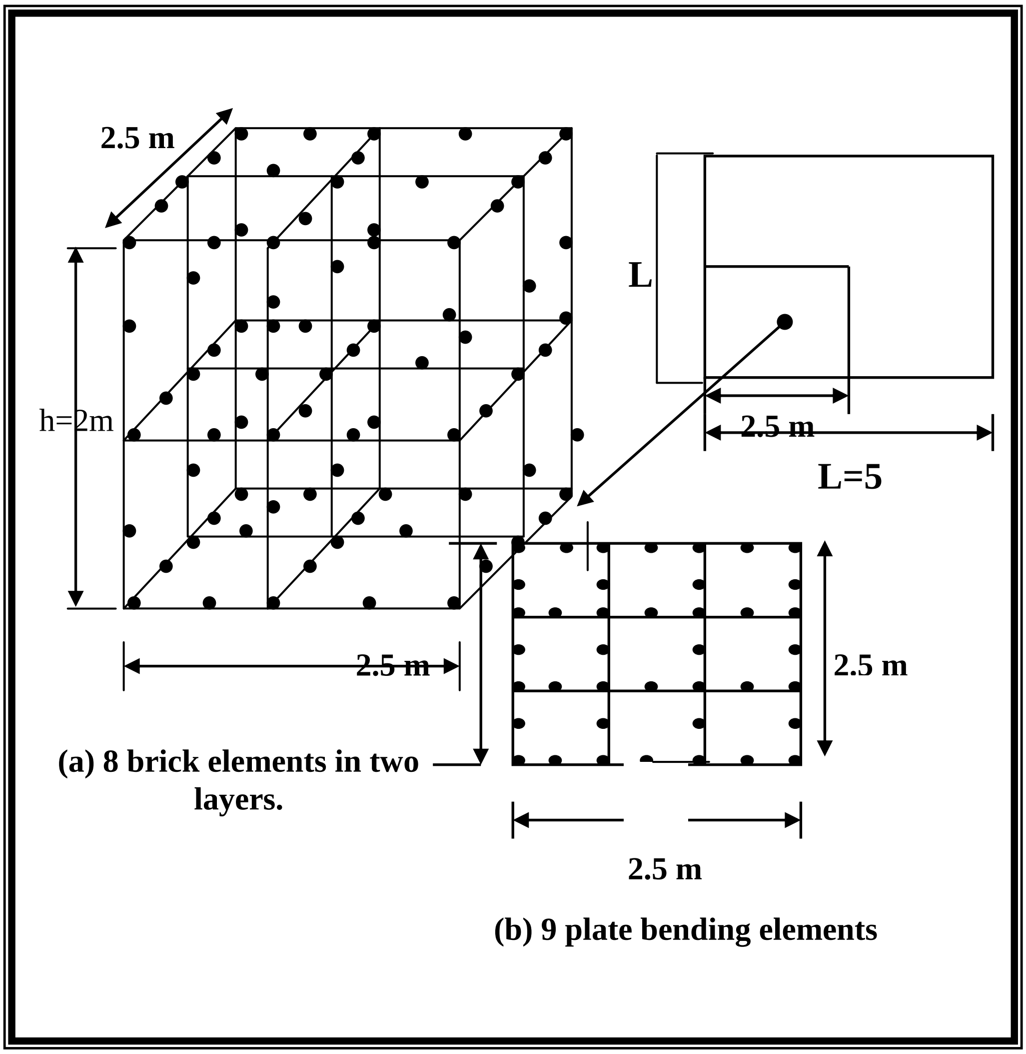

Figure (2) Finite Element Mesh.

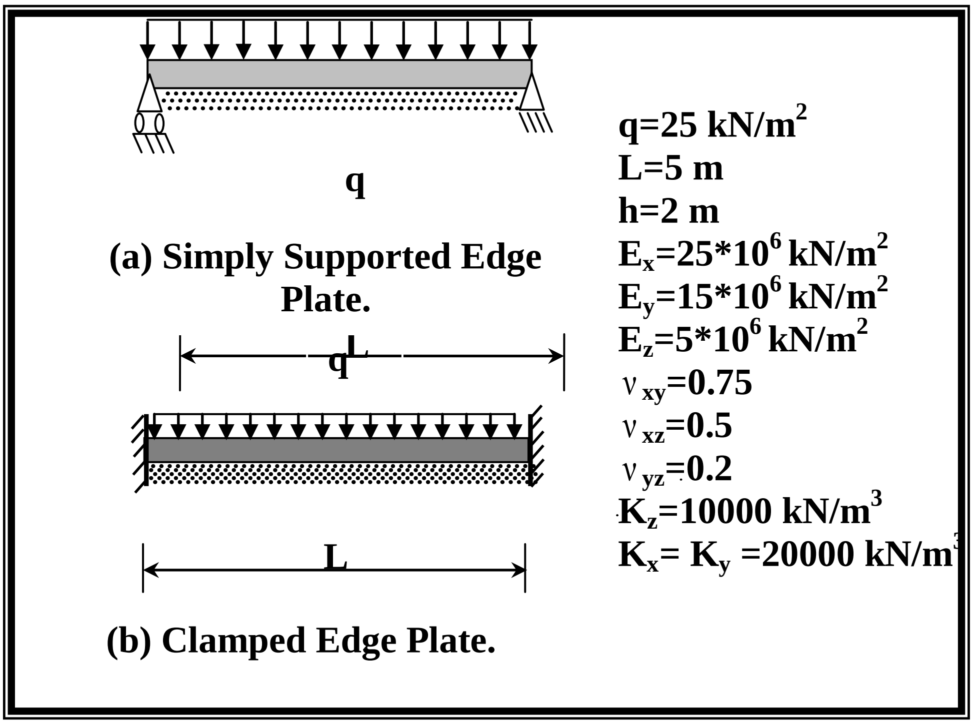

Figure (3) Orthotropic Square Plate Geometry and Loading.

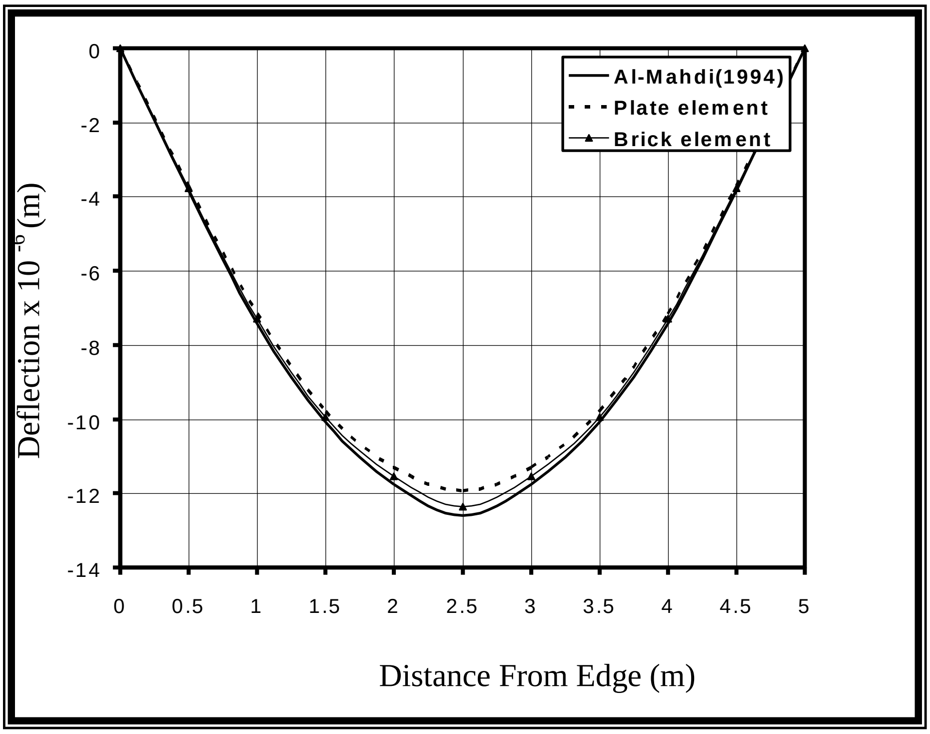

Figure (4) Deflection Profile in x-Direction for Simply Supported Thick Orthotropic Square Plate.

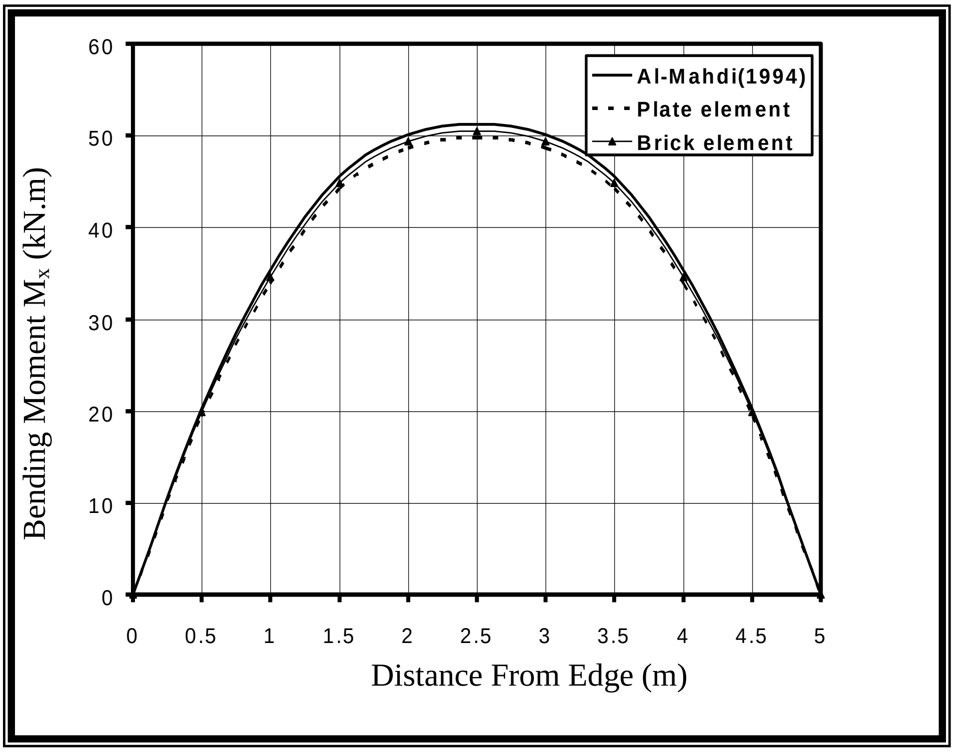

Figure (5) Bending Moment (M_x) Diagram for Simply Supported Thick Orthotropic Square Plate.

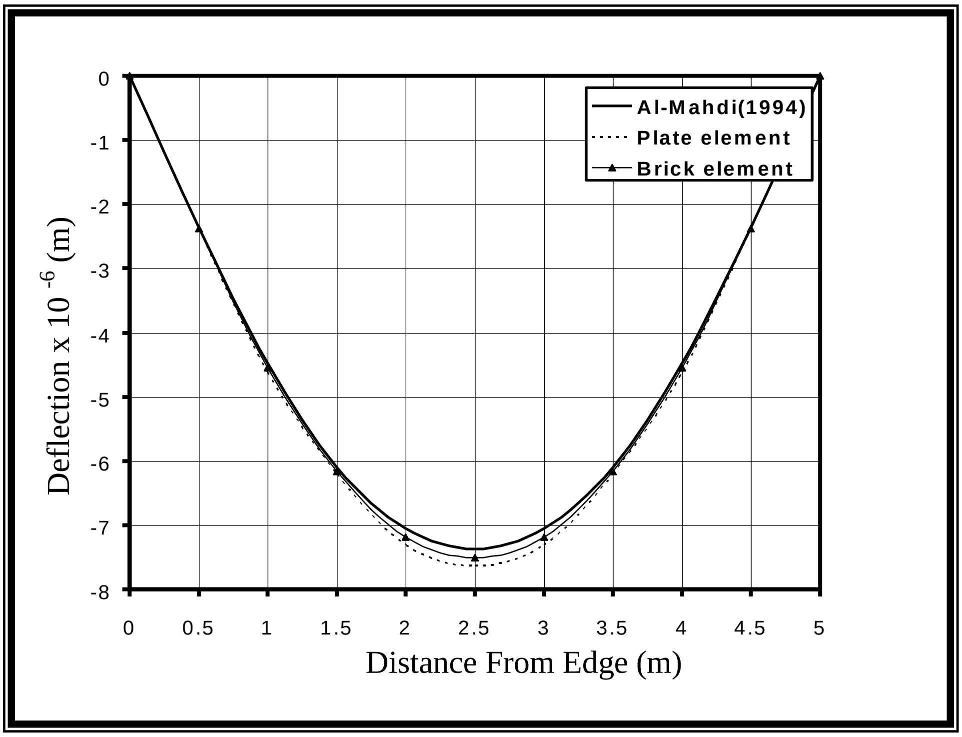

Figure (6) Deflection Profile in x- Direction for Clamped Thick Orthotropic Square Plate.

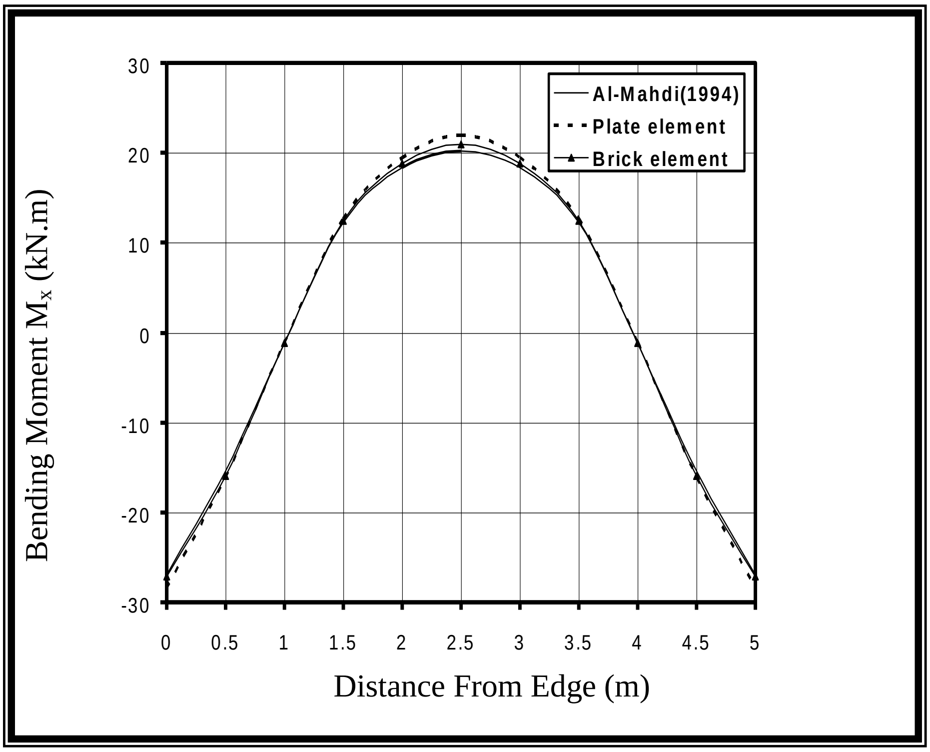

Figure (7) Bending Moment (M_x) Diagram for Clamped Thick Orthotropic Square Plate.

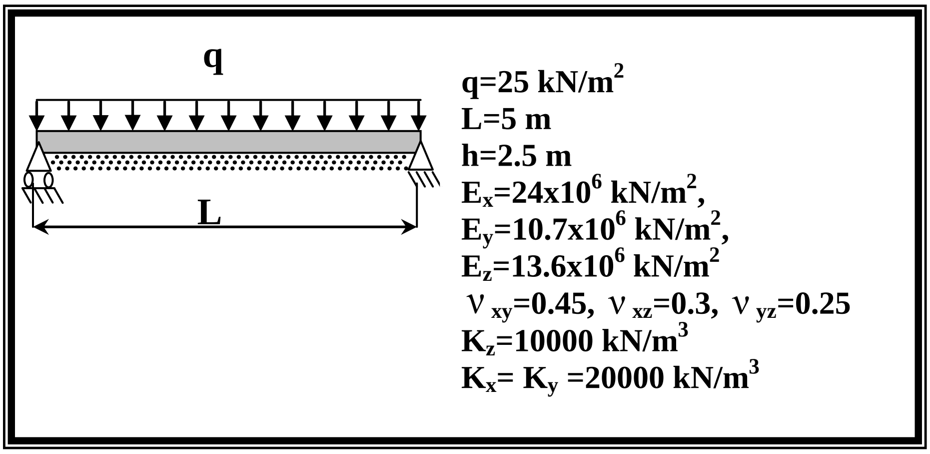

Figure (8) Orthotropic Square Plate Geometry and Loading.

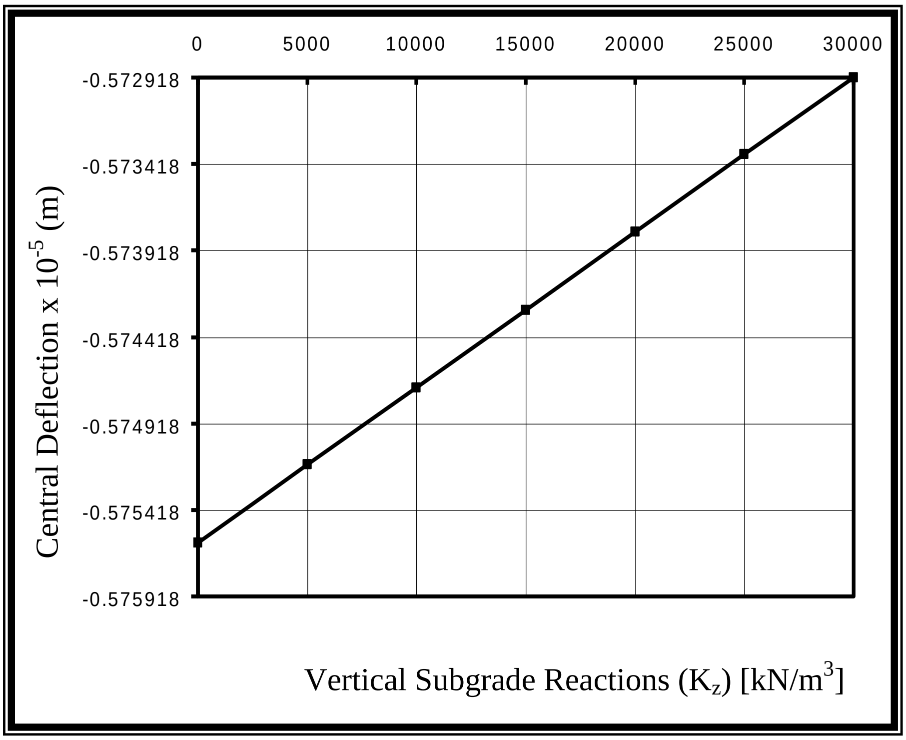

Figure (9) Effect of Vertical Subgrade Reactions on Central Deflection for Simply Supported Orthotropic Square Plate.

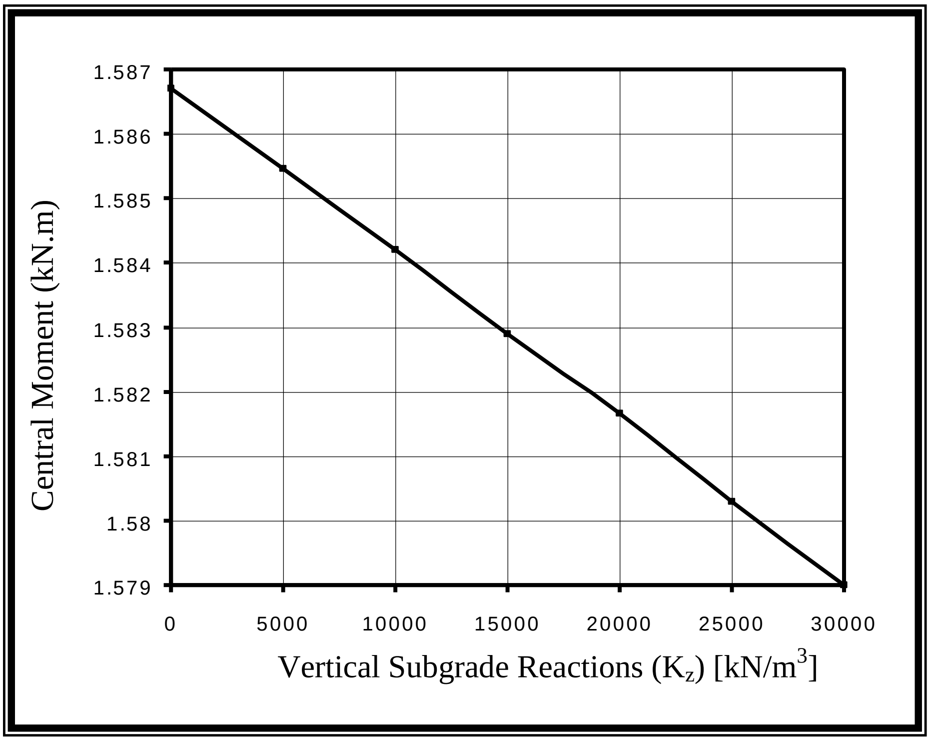

Figure (10) Effects of Vertical Subgrade Reactions on Central Moment for Simply Supported Orthotropic Square Plate.

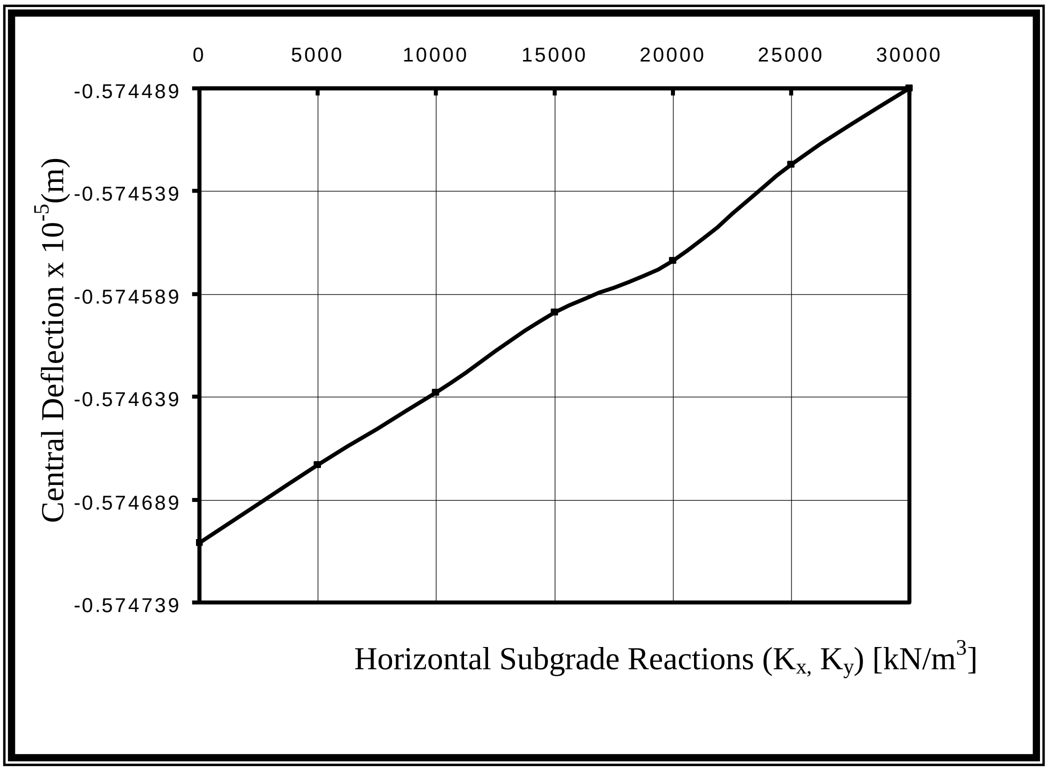

Figure (11) Effect of Horizontal Subgrade Reactions on Central Deflection for Simply Supported Orthotropic Square Plate.

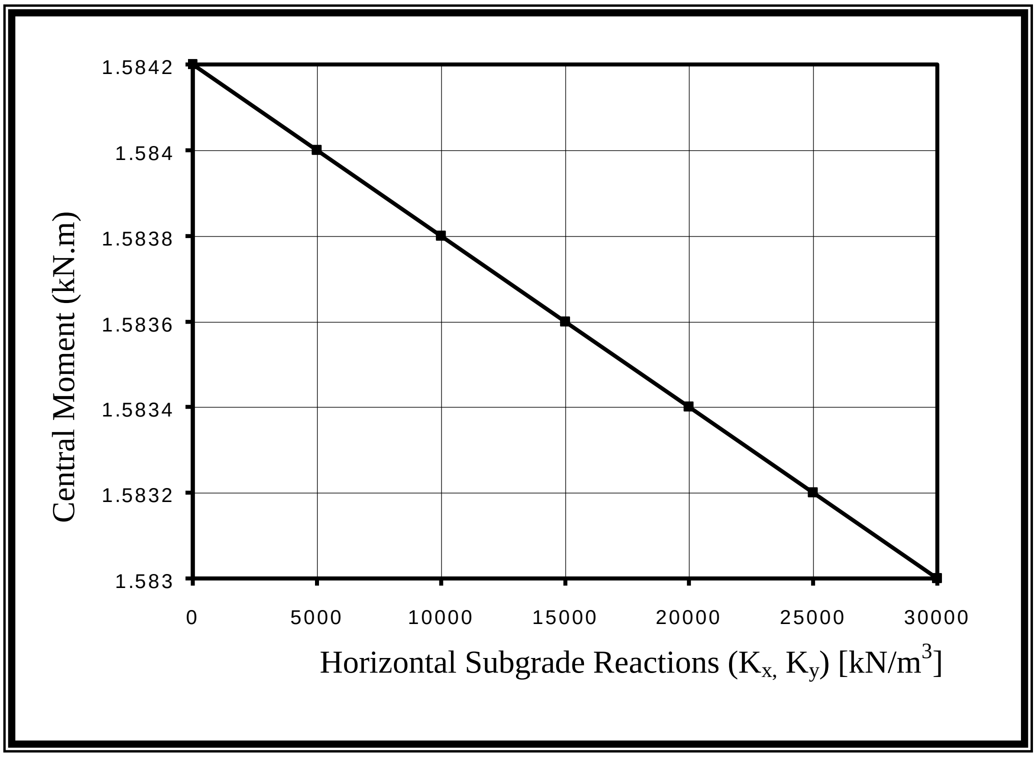

Figure (12) Effect of Horizontal Subgrade Reactions on Central Moment for Simply Supported Orthotropic Square plate.

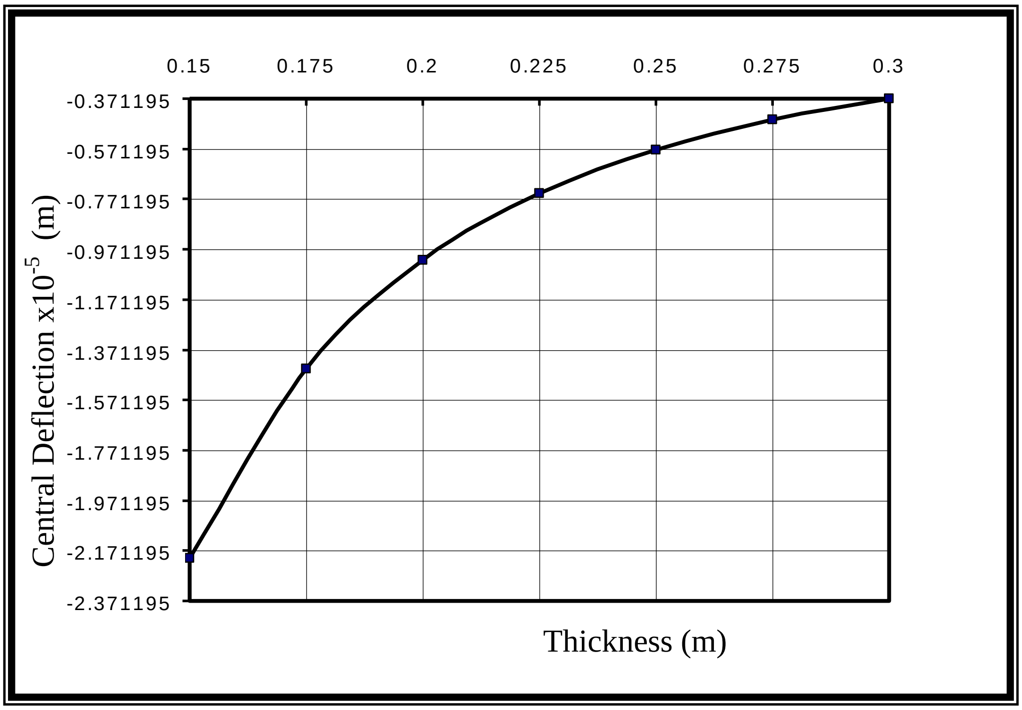

Figure (13) Effect of Thickness of Plate on Central Deflection of Simply Supported Orthotropic Square Plate.

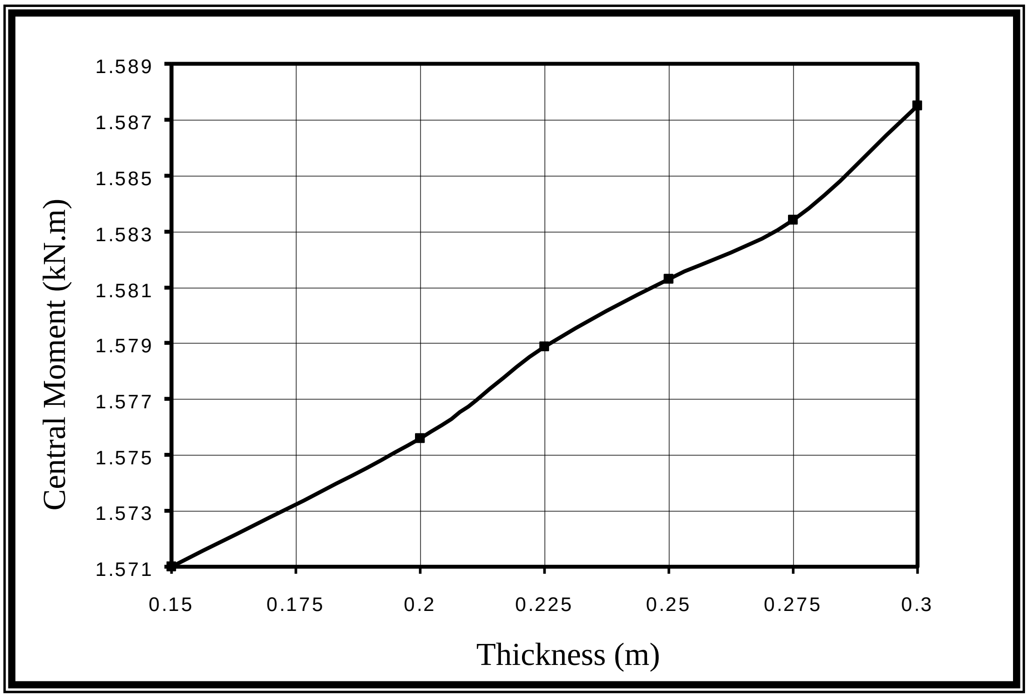

Figure (14) Effect of Thickness of Plate on Central Moment of Simply Supported Orthotropic Square Plate.