FINDING THE SEPARATION CONSTANT (λ) FOR LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

Our aim in this search is to find the values of separation constant (λ) for the complete solution without using any conditions for some classified of the linear second order partial differential equations for all cases which contains the case $A_3 = \frac{A_2^{\,2}}{4}$ or $A_2 = \frac{A_1^{\,2}}{4}$ or $B_2 = \frac{B_1^{\,2}}{4}$ since that is not easy to find the values (λ) without using any conditions for an application homogenous linear partial differential equations (L.P.D.Es) which is solving by separation of variable method.

المستخلص

هدفنا هو ايجاد قيمة ثابت الفصل (λ) للحل التام بدون أي شروط لبعض تصنيفات المعادلة التفاضلية الجزئية الخطية من الرتبة الثانية لكل الحالات التي يكون فيها

$$A_3 = \frac{A_2^{\,2}}{4} \text{ or } A_2 = \frac{A_1^{\,2}}{4} \text{ or } B_2 = \frac{B_1^{\,2}}{4}$$

حيث ليس من السهولة ايجاد قيمة (λ) بدون أي شروط لاي تطبيق حول المعادلات التفاضلية الجزئية من الرتبة الثانية .

1- INTRODUCTION

One of the power full methods of generating asset of solutions for a given homogeneous linear partial differential equations is the method of separation variables.

The dependent variable in separation variable is assumed to be a product of functions and by finding the partial derivatives in the original partial differential equations by this method we obtain ordinary differential equations are independent of each other, and each ordinary differential equation must be affixed to constant (separation of constant) which is supposed to be(λ) and by substituting initial and boundary conditions we obtain the values of separation constant (λ) .

We could find in the complete solution order linear partial differential equations by using the assumptions [2]:

$$z(x,y) = e^{\int \frac{u(x)}{x}dx + \int v(y)dy} \quad , \quad z(x,y) = e^{\int u(x)dx + \int \frac{v(y)}{y}dy} \quad \text{and}$$

$$z(x,y) = e^{\int \frac{u(x)}{x}dx + \int \frac{v(y)}{y}dy} \quad ,$$

which are converted linear partial differential equations of second order to linear ordinary differential equations of first order, and classified linear partial differential equations to many cases. In this search we take all cases which is concerned the case $A_3 = \frac{A_2^{\,2}}{4}$ or $A_2 = \frac{A_1^{\,2}}{4}$ or $B_2 = \frac{B_1^{\,2}}{4}$ for finding (λ) without using any conditions.

2- Basic Definitions [1]

2-1 Definition

A partial differential equation is an equation that contains partial derivatives.

2-2 Definition

The equation whose derivatives of the first degree are not multiplied together , is called linear partial differential equation.

2-3 Definition

The equation whose partial derivatives are equal in the order , is called with homogeneous terms linear partial differential equation.

3-Extension of Linear Second Order Partial Differential Equations

The linear second order partial differential equations, which have the general form

$$A(x,y)z_{xx} + B(x,y)z_{xy} + C(x,y)z_{yy} + D(x,y)z_x + E(x,y)z_y + F(x,y)z = 0 \ldots(1),$$

where some of $A(x,y), B(x,y), C(x,y), D(x,y), E(x,y)$ and $F(x,y)$ are functions of x or y or both x and y .

In order to find the complete solution of the equation(1), in[2] search function $u(x)$ and $v(y)$ such that the assumptions

$$z(x,y) = e^{\int \frac{u(x)}{x}dx + \int v(y)dy} \quad , \quad z(x,y) = e^{\int u(x)dx + \int \frac{v(y)}{y}dy} \quad \text{and}$$

$$z(x,y) = e^{\int \frac{u(x)}{x}dx + \int \frac{v(y)}{y}dy} \quad ,$$

represent the complete solution of the equation (1).

4- Classified and Complete Solution of Second Order Linear Partial Differential Equations

Case 1:

$$A_1 x^2 z_{xx} + D_1 x z_x + C z_{yy} + E z_y + F z = 0 \cdots (2),$$

$(i.e B = 0)$

$\ni A_1, D_1, C, E$ and F are constants and not identically zero.

The assumption $z(x,y) = e^{\int \frac{u(x)}{x}dx + \int v(y)dy}$ …(3) ,
represents the complete solution of equation (2),
this assumption will transform the equation(2), to first ordinary differential equation by finding z_x, z_y, z_{xx} and z_{yy} from the equation (3), and substitution the above partial derivatives in equation(2), we get

$$A_1(xu'(x)+u^2(x)-u(x))+D_1u(x)+C(v'(y)+v^2(y))+Ev(y)+F=0 \ldots(4)$$

the equation (4), is of the first order ordinary differential equation and contains two independent functions $u(x)$ and $v(y)$, then the complete solution of equation(2),is given by :

i) $z(x,y)=x^{\frac{-A_2}{2}}e^{\frac{-B_1}{2}y}\left(a_1x^{\sqrt{\frac{A_2^2}{4}-A_3}}+a_2x^{-\sqrt{\frac{A_2^2}{4}-A_3}}\right)\left(d_1\cos\sqrt{B_2-\frac{B_1^2}{4}}y+d_2\sin\sqrt{B_2-\frac{B_1^2}{4}}y\right)$ If

$\frac{A_2^2}{4} \neq A_3$ and $B_2 \neq \frac{B_1^2}{4}$

ii) $z(x,y)=x^{\frac{-A_2}{2}}e^{\frac{-B_1}{2}y}\left(a_1x^{\sqrt{\frac{A_2^2}{4}-A_3}}+a_2x^{-\sqrt{\frac{A_2^2}{4}-A_3}}\right)(y-c_2)$

If $\frac{A_2^2}{4} \neq A_3$ and $B_2 = \frac{B_1^2}{4}$

iii) $z(x,y)=x^{\frac{-A_2}{2}}e^{\frac{-B_1}{2}y}\left(d_1\cos\sqrt{B_2-\frac{B_1^2}{4}}y+d_2\sin\sqrt{B_2-\frac{B_1^2}{4}}y\right)(\ln(c_4x))$, $(c_4x)>0$

If $\frac{A_2^2}{4} = A_3$ and $B_2 \neq \frac{B_1^2}{4}$

iv $z(x,y)=k_1x^{\frac{-A_2}{2}}e^{\frac{-B_1}{2}y}(y-c_2)\ln(c_4x)$, $(c_4x)>0$

If $\frac{A_2^2}{4} = A_3$ and $B_2 = \frac{B_1^2}{4}$

where $A_2=\frac{D_1}{A_1}-1, A_3=\frac{-\lambda^2}{A_1}, B_2=\frac{F+\lambda^2}{C}$ and $B_1=\frac{E}{C}$

and $\lambda, a_i, d_i, (i=1,2), k_1=e^g, c_2$ and c_4 are arbitrary constants

***Case* 2:**

$Az_{xx}+Dz_x+C_1y^2z_{yy}+E_1yz_y+Fz=0\cdots(5),$
$(i.eB=0)$

∍ A, B, C_1, E_1 and F are constants and not identically zero.

The assumption $z(x,y)=e^{\int u(x)dx+\int \frac{v(y)}{y}dy}$ …(6) ,
represents the complete solution of equation (5),
this assumption will transform the equation(5), to first ordinary differential equation then the complete solution of equation (5) is given by :

i) $z(x,y)=y^{\frac{-B_1}{2}}e^{\frac{-A_1}{2}x}\left(a_1y^{\sqrt{\frac{B_1^2}{4}-B_2}}+a_2y^{-\sqrt{\frac{B_1^2}{4}-B_2}}\right)\left(d_1\cos\sqrt{A_2-\frac{A_1^2}{4}}x+d_2\sin\sqrt{A_2-\frac{A_1^2}{4}}x\right);$ If

$(\frac{B_1^2}{4}-B_2)\geq 0$

$A_2\neq\frac{A_1^2}{4}$ and $\frac{B_1^2}{4}\neq B_2$

ii) $z(x,y)=y^{\frac{-B_1}{2}}e^{\frac{-A_1}{2}x}\left(d_1\cos\sqrt{A_2-\frac{A_1^2}{4}}x+d_2\sin\sqrt{A_2-\frac{A_1^2}{4}}x\right)(\ln cy);(cy)>0$

If $A_2\neq\frac{A_1^2}{4}$ and $\frac{B_1^2}{4}=B_2$

iii) $z(x,y)=y^{\frac{-B_1}{2}}e^{\frac{-A_1}{2}x}(x-c_3)\left(a_1y^{\sqrt{\frac{B_1^2}{4}-B_2}}+a_2y^{-\sqrt{\frac{B_1^2}{4}-B_2}}\right);(\frac{B_1^2}{4}-B_2)\geq 0$

If $A_2=\frac{A_1^2}{4}$ and $\frac{B_1^2}{4}\neq B_2$

iv) $z(x,y)=k_1y^{\frac{-B_1}{2}}e^{\frac{-A_1}{2}x}(x-c_3)(\ln cy);(cy)>0$

If $A_2=\frac{A_1^2}{4}$ and $\frac{B_1^2}{4}=B_2$

where $A_1 = \frac{D}{A}, A_2 = \frac{\lambda^2}{A}, B_2 = \frac{F - \lambda^2}{C_1}$ and $B_1 = \frac{E_1}{C_1} - 1$

and $\lambda, a_i, d_i, (i = 1,2), k_1 = e^g, c_3$ and c are arbitrary constants

***Case* 3:**

$$A_1 x^2 z_{xx} + D_1 x z_x + C_1 y^2 z_{yy} + E_1 y z_y + Fz = 0 \cdots (7),$$

$(i.e B = 0)$

$\ni A_1, B_1, C_1, E_1, and F$ are constants and not identically zero.

The assumption $z(x,y) = e^{\int \frac{u(x)}{x} dx + \int \frac{v(y)}{y} dy}$ …(8) ,
represents the complete solution of equation (7),
this assumption will transform the equation(7), to first order ordinary differential equation,
then the complete solution of equation (7) is given by:

i)

$$z(x,y) = x^{\frac{-A_2}{2}} y^{\frac{-B_1}{2}} \left(a_1 x^{\sqrt{\frac{A_2^2}{4} - A_3}} + a_2 x^{-\sqrt{\frac{A_2^2}{4} - A_3}} \right) \left(d_1 y^{\sqrt{\frac{B_1^2}{4} - B_2}} + d_2 y^{-\sqrt{\frac{B_1^2}{4} - B_2}} \right);$$

$(\frac{A_2^2}{4} - A_3) \geq 0 and (\frac{B_1^2}{4} - B_2) \geq 0$

If $A_3 \neq \frac{A_2^2}{4}$ and $\frac{B_1^2}{4} \neq B_2$

ii) $z(x,y) = x^{\frac{-A_2}{2}} y^{\frac{-B_1}{2}} \left(a_1 x^{\sqrt{\frac{A_2^2}{4} - A_3}} + a_2 x^{-\sqrt{\frac{A_2^2}{4} - A_3}} \right) (\ln cy); (\frac{A_2^2}{4} - A_3) \geq 0 and (cy) > 0$

If $A_3 \neq \frac{A_2^2}{4}$ and $\frac{B_1^2}{4} = B_2$

iii) $z(x,y) = x^{\frac{-A_2}{2}} y^{\frac{-B_1}{2}} \left(a_1 y^{\sqrt{\frac{B_1^2}{4} - B_2}} + a_2 y^{-\sqrt{\frac{B_1^2}{4} - B_2}} \right) (\ln(c_2 x)); (\frac{B_1^2}{4} - B_2) \geq 0; (c_2 x) > 0$

If $A_3 = \frac{A_2^2}{4}$ and $\frac{B_1^2}{4} \neq B_2$,

iv) $z(x,y) = kx^{\frac{-A_2}{2}} y^{\frac{-B_1}{2}} (\ln(c_2 x))(\ln(cy)); (c_2 x) > 0 and (cy) > 0$

If $A_3 = \frac{A_2^2}{4}$ and $\frac{B_1^2}{4} = B_2$.

Where $A_2 = \frac{D_1}{A_1} - 1, A_3 = \frac{\lambda^2}{A_1}, B_2 = \frac{F - \lambda^2}{C_1} and B_1 = \frac{E_1}{C_1} - 1$

and $\lambda, a_i, d_i, (i = 1,2), k = e^g, c_2$ and c are arbitrary constants

***Case* 4:**

$$A_1 x^2 z_{xx} + D_1 x z_x + C_1 y^2 z_{yy} + E_1 y z_y + B_1 xyz_{xy} + Fz = 0, (B \neq 0) \quad \ldots(9)$$

∍ A_1, D_1, C_1, E_1, B_1 and F are constants and not identically zero.

The assumption $z(x, y) = e^{\int \frac{u(x)}{x} dx + \int \frac{v(y)}{y} dy}$ …(8) ,
represents the complete solution of equation (9),
this assumption will transform the equation(9), to first order ordinary differential equation,then the complete solution of equation (9) is given by:

i) $z(x, y) = x^{\frac{-A_2}{2}} y^{\lambda} \left(a_1 x^{\sqrt{\frac{A_2^2}{4} - A_3}} + a_2 x^{\sqrt{\frac{A_2^2}{4} - A_3}} \right); (\frac{A_2^2}{4} - A_3) \geq 0$

If $\frac{A_2^2}{4} \neq A_3$

ii) $z(x, y) = kx^{\frac{-A_2}{2}} y^{\lambda} (\ln c_3 x); (c_3 x) > 0$

If $\frac{A_2^2}{4} = A_3$

Where $A_2 = \frac{D_1 + B_1 \lambda}{A_1} - 1 and A_3 = \frac{C_1(\lambda^2 - \lambda) + E_1 \lambda + F}{A_1}$

and $\lambda, a_i, (i = 1,2), k = e^g$ and c_3 are arbitrary constants .

5- How to Find The Separation of Constant(λ) For Linear Partial Differential Equation

Case 1:

$$A_1 x^2 z_{xx} + D_1 x z_x + C z_{yy} + E z_y + Fz = 0, (i.e B = 0)$$

ii) If $\frac{A_2^2}{4} \neq A_3$ and $B_2 = \frac{B_1^2}{4}$

When $B_2 = \frac{F + \lambda^2}{C} and B_1 = \frac{E}{C}$

Proof :

Since $B_2 = \frac{B_1^2}{4}$ hence

$$\frac{F+\lambda^2}{C}=\frac{E^2}{4C^2}\Rightarrow$$

$$\lambda^2=\frac{E^2}{4C}-F$$

then the complete solution becomes

$$z(x,y)=x^{\frac{-1}{2}(\frac{D_1}{A_1}-1)}e^{\frac{-E}{2C}y}\left(a_1x^{\sqrt{\frac{1}{4}(\frac{D_1}{A_1}-1)^2+\frac{E^2-4CF}{4CA_1}}}+a_2x^{-\sqrt{\frac{1}{4}(\frac{D_1}{A_1}-1)^2+\frac{E^2-4CF}{4CA_1}}}\right)(y-c_2)$$

iii) If $\frac{A_2{}^2}{4}=A_3$ and $B_2\neq\frac{B_1{}^2}{4}$

When $A_2=\frac{D_1}{A_1}-1 and A_3=\frac{-\lambda^2}{A_1}$

Proof :

Since $\frac{A_2{}^2}{4}=A_3$ hence

$$\frac{1}{4}(\frac{D_1}{A_1}-1)^2=\frac{-\lambda^2}{A_1}\Rightarrow$$

$$\lambda^2=\frac{-A_1}{4}(\frac{D_1}{A_1}-1)^2$$

then the complete solution becomes

$$z(x,y)=x^{\frac{-1}{2}(\frac{D_1}{A_1}-1)}e^{\frac{-E}{2C}y}\left(d_1\cos\sqrt{\frac{4A_1F-(D_1-A_1)^2}{4A_1C}-\frac{E^2}{4C^2}}\,y+d_2\sin\sqrt{\frac{4A_1F-(D_1-A_1)^2}{4A_1C}-\frac{E^2}{4C^2}}\,y\right)(\ln(c_4x))$$

Case 2:

$$Az_{xx}+Dz_x+C_1y^2z_{yy}+E_1yz_y+Fz=0,$$

$$(i.eB=0)$$

ii) If $A_2\neq\frac{A_1{}^2}{4}$ and $\frac{B_1{}^2}{4}=B_2$

When $B_2=\frac{F-\lambda^2}{C_1}$ and $B_1=\frac{E_1}{C_1}-1$

Proof:

Since $\frac{B_1{}^2}{4}=B_2$ hence

$$\frac{F-\lambda^2}{C_1}=\frac{1}{4}(\frac{E_1}{C_1}-1)^2\Rightarrow$$

$$\lambda^2 = F - \frac{(E_1 - C_1)^2}{4C_1}$$

then the complete solution becomes

$$z(x,y) = y^{\frac{-1}{2}(\frac{E_1}{C_1}-1)} e^{\frac{-D}{2A}x} \left(d_1 \cos\sqrt{\frac{4FC_1-(E_1-C_1)^2}{4C_1A} - \frac{D^2}{4A^2}}x + d_2 \sin\sqrt{\frac{4FC_1-(E_1-C_1)^2}{4C_1A} - \frac{D^2}{4A^2}}x \right)(\ln cy)$$

iii) If $A_2 = \frac{A_1^2}{4}$ and $\frac{B_1^2}{4} \neq B_2$

When $A_1 = \frac{D}{A}$ and $A_2 = \frac{\lambda^2}{A}$

Proof:

Since $A_2 = \frac{A_1^2}{4}$ hence

$$\frac{D^2}{4A^2} = \frac{\lambda^2}{A} \Rightarrow$$

$$\lambda^2 = \frac{D^2}{4A}$$

then the complete solution becomes

$$z(x,y) = y^{\frac{-1}{2}(\frac{E_1}{C_1}-1)} e^{\frac{-D}{2A}x} (x-c_3) \left(a_1 y^{\sqrt{\frac{1}{4}(\frac{E_1}{C_1}-1)^2 - \frac{4AF-D^2}{4AC_1}}} + a_2 y^{-\sqrt{\frac{1}{4}(\frac{E_1}{C_1}-1)^2 - \frac{4AF-D^2}{4AC_1}}} \right)$$

Case 3:

$$A_1x^2z_{xx} + D_1xz_x + C_1y^2z_{yy} + E_1yz_y + Fz = 0, (i.e B = 0)$$

ii) If $A_3 \neq \frac{A_2^2}{4}$ and $B_2 = \frac{B_1^2}{4}$

When $B_2 = \frac{F-\lambda^2}{C_1}$ and $B_1 = \frac{E_1}{C_1} - 1$

Proof:

Since $\frac{B_1^2}{4} = B_2$ hence

$$\frac{F-\lambda^2}{C_1} = \frac{1}{4}(\frac{E_1}{C_1}-1)^2 \Rightarrow$$

$$\lambda^2 = \frac{-C_1}{4}(\frac{E_1}{C_1}-1)^2 + F$$

then the complete solution becomes

$$z(x,y)=x^{\frac{-1}{2}(\frac{D_1}{A_1}-1)}\,y^{\frac{-1}{2}(\frac{E_1}{C_1}-1)}\left(a_1x^{\sqrt{\frac{1}{4}(\frac{D_1}{A_1}-1)^2-\frac{4C_1F-(E_1-C_1)^2}{4C_1A_1}}}+a_2x^{-\sqrt{\frac{1}{4}(\frac{D_1}{A_1}-1)^2-\frac{4C_1F-(E_1-C_1)^2}{4C_1A_1}}}\right)(\ln cy)$$

iii) If $A_3=\frac{A_2^{\,2}}{4}$ and $B_2\neq\frac{B_1^{\,2}}{4}$

When $A_2=\frac{D_1}{A_1}-1$ and $A_3=\frac{\lambda^2}{A_1}$

Proof:

Since $A_3=\frac{A_2^{\,2}}{4}$ hence

$\frac{1}{4}(\frac{D_1}{A_1}-1)^2=\frac{\lambda^2}{A_1}\Rightarrow$

$\lambda^2=\frac{A_1}{4}(\frac{D_1}{A_1}-1)^2$

then the complete solution becomes

$$z(x,y)=x^{\frac{-1}{2}(\frac{D_1}{A_1}-1)}\,y^{\frac{-1}{2}(\frac{E_1}{C_1}-1)}\left(a_1y^{\sqrt{\frac{1}{4}(\frac{E_1}{C}-1)^2-\frac{4A_1F-(D_1-A_1)^2}{4A_1C_1}}}+a_2y^{-\sqrt{\frac{1}{4}(\frac{E_1}{C_1}-1)^2-\frac{4A_1F-(D_1-A_1)^2}{4A_1C_1}}}\right)(\ln(c_2x))$$

Case 4:

$A_1x^2z_{xx}+D_1xz_x+C_1y^2z_{yy}+E_1yz_y+B_1xyz_{xy}+Fz=0,(B\neq0)$

ii) If $\frac{A_2^{\,2}}{4}=A_3$

When $A_2=\frac{D_1+B_1\lambda}{A_1}-1$ and $A_3=\frac{C_1(\lambda^2-\lambda)+E_1\lambda+F}{A_1}$

Proof:

Since $\frac{A_2^{\,2}}{4}=A_3$ hence

$\frac{1}{4}(\frac{D_1+B_1\lambda}{A_1}-1)^2-\frac{C_1(\lambda^2-\lambda)+E_1\lambda+F}{A_1}=0\Rightarrow$

$\lambda^2+\beta\lambda+\gamma=0 \quad \ldots(11)$

where $\beta=\frac{(2D_1B_1-2A_1B_1+4A_1C_1-4A_1E_1)}{B_1^{\,2}-4A_1C_1}$ and

$\gamma=\frac{D_1^{\,2}-2D_1A_1+A_1^{\,2}-4A_1F}{B_1^{\,2}-4C_1A_1}$, $B_1^{\,2}-4C_1A_1\neq0$

and by solving the equation(11) we can know the value of (λ).

6- Examples :

Examples 1: To solve the partial differential equation

$$x^2 z_{xx} + xz_x + z_{yy} + 4z_y + 3z = 0,$$

by using the (case(1),ii) then $\lambda^2 = 1$, thus the complete solution has the form

$$z(x,y) = e^{-2y}(a_1 x + a_2 x^{-1})(y - c_2)$$

Examples 2: To solve the partial differential equation

$$x^2 z_{xx} + 3xz_x + y^2 z_{yy} + 3yz_y + z = 0$$

by using the (case(3),iii) then $\lambda^2 = 1$, thus the complete solution has the form

$$z(x,y) = x^{-1} y^{-1}(a_1 y + a_2 y^{-1})\ln(c_2 x)$$

Examples 3: To solve the partial differential equation

$$x^2 z_{xx} + 5xz_x + 2y^2 z_{yy} + 5yz_y + 2xyz_{xy} + 4z = 0$$

by using the (case(4),ii) , we get $(\lambda^2 - \lambda = 0)$, thus either $\lambda = 0$ or $\lambda = 1$.

If $\lambda = 0$,then the complete solution has the form

$$z(x,y) = kx^{-2}\ln(c_3 x)$$

If $\lambda = 1$, then the complete solution has the form

$$z(x,y) = kx^{-3} y \ln(c_3 x).$$

References

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