Summation of the Power Series

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Abstract

In this paper we development Gauss's idea which put mathematical structure for summation n from natural numbers $(\sum_{i=1}^{n} i)$. We introduce mathematical structures for summation of the power series from natural numbers $(\sum_{i=1}^{n} i^s)$ (where s natural numbers) and prove these structures by using mathematical induction. Finally introduce common structure for any s.

المستخلص

في هذا البحث قمنا بتطوير لما قام به كاوس من وضع صيغة رياضية لجمع n من الأعداد الطبيعية $(\sum_{i=1}^{n} i)$. حيث قمنا بوضع صيغ رياضية لجمع n من الأعداد الطبيعية مرفوعة لـ s $(\sum_{i=1}^{n} i^s)$ (حيث s عدد طبيعي) وقمنا ببرهنة هذه الصيغ بواسطة الاستقراء الرياضي. وأخيراً قدمنا صيغة عامة لأي s.

1. Introduction:

The Greek philosopher, Zeno of Elea (495_435 B.C.E) presented a number of paradox that perplexed many mathematicians of his day. Perhaps the most famous is the racecourse paradox, which may be stated as follows as a runner runs a racecourse, he must first run $1/2$ its length, then $1/2$ the distance that remains after that, and so on indefinitely. Since this is an infinite number of such half segments and the runner requires finite time to complete each one, he can never finish the rice. The sum of infinity many numbers may be finite this statement which may seem paradoxical at first, plays a central role in mathematics and has a variety of important application. (The purpose of this chapter is to explore its meaning and some of its consequences) you are already familiar with the phenomenon of a finite-valued infinite sum you know, for example, that the repeating decimal $0.333\ldots$ stand for the infinite sum $3/10 + 3/100 + 3/1000 + \cdots$ and that its value is the finite number $1/3$ the situation will be clarified in this introductory section. In 1784, Gauss put the basic formula for the nth summations.

2. Convergence and divergence of infinite series:

An expression of the form: $a_1 + a_2 + \cdots + a_n + \cdots$ is called an infinite series it customary to use summation notation to write series compactly as follows:

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} (a_n)$$

Roughly speaking, an infinite series:

$$a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} (a_k)$$

Is said to converge if it "add up" to a finite number and to diverge if it does not a more precise statement of the criterion for convergence involves the sum:

$$S_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} (a_k)$$

2.1 Geometric series [2]:

The series of the form**:**

$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} (ar^{n-1})$, where a is a constant $\neq 0$ and r is the ratio of successive term is called geometric series $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$

$$S_n = \begin{cases} \dfrac{a}{1-r} & \text{if } |r| < 1 \quad \text{the series is converge} \\ \text{can not exist} & \text{if } |r| \geq 1 \quad \text{the series is diverge} \end{cases}$$

Proposition [2]: A non geometric series as form $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is called Telescoping series. The telescoping series is convergent because:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$S_k = 1 - \frac{1}{k+1} = 1 \quad as \;\; k \to \infty$ (All other terms cancel)

3. Gausses idea

Gauss was imagine the first idea to find the nth summation when his teacher asked him to find of the sum of the numbers between 1 to 100, at once he already found a great method, he was know that is difficulty if he added step by step but in another way he think of addition the numbers such as this sets that define as follows:

$$\{0 + 100 = 100\} + \{1 + 99 = 100\} + \cdots + \{49 + 51 = 100\} + \{50 = 100/2\}$$

And count this sets by take:

$$\{100 + 100 + \cdots + 100\} + \{100/2\} = 50 * 100 + 50 = 5000 + 50 = 5050$$

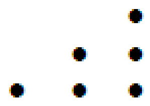
In general:

$$Sum(100) = \frac{100}{2} * 100 + \frac{100}{2} = \frac{100^2}{2} + \frac{100}{2} = \frac{100(100+1)}{2} = 5050$$

Now if we take $n = 100$ as any number such that $n \leq N$, we get $Sum(n) = \dfrac{n(n+1)}{2}$

At last he answered his teacher, but the teacher doesn't know how this pupil can count this numbers in another way we can express this rule by count a balls that arranged as the form [2]:

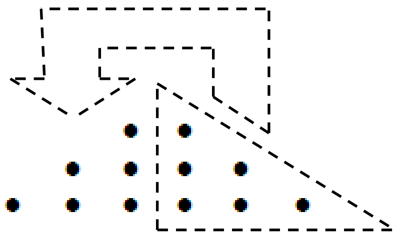
$$\sum_{i=1}^{3} i =$$



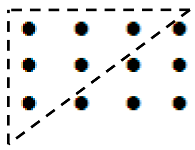
$$= 1 + 2 + 3 = 6$$

By multiply both sides by (2) to get:

$$2 * \sum_{i=1}^{3} i =$$



$$2 * \sum_{i=1}^{3} i =$$



$$= 3 * (3 + 1)$$

$$\sum_{i=1}^{3} i = \frac{3 * (3 + 1)}{2} = 6$$

We can take $n = 3$, n be any natural number such that $n \in \mathbb{N}$, we get:

To prove that by using mathematical induction [3].

1. let $n = 1$, then $\sum_{i=1}^{1} i = \frac{1 * (1 + 1)}{2}$

2. suppose $n = k$, then $\sum_{i=1}^{k} i = \frac{k(k + 1)}{2}$

3. let $n = k + 1$, T.P : $\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} (i + 1) + 1$

$$\sum_{i=1}^{k+1} i = \frac{(k + 1)(k + 2)}{2}$$

$$= \frac{k^2 + 3k + 2}{2}$$

$$= \frac{k^2 + k}{2} + \frac{2k + 2}{2}$$

$$= \frac{k(k + 1)}{2} + k + 1$$

$= \sum_{i=1}^{k} i + k + 1$, where $\sum_{i=1}^{k} 1 = k$ properties of summation

$$= \sum_{i=1}^{k} (i + 1) + 1$$

Therefore $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, $\forall\, n \in \mathbb{N}$

That was the first rule of nth summation that found where can calculate the sum of the natural numbers series (the series of natural numbers) and we will consider it the base to find the others nth summations.

4. Subtraction method of ratios:

During our studies to the properties of the nth summations, we were found relation between $\sum_{i=1}^{n} i$ and $\sum_{i=1}^{n} i^2$ and we will offer it as follows:

$$\sum_{i=1}^{2} i = 1 + 2 = 3 \quad , \quad \sum_{i=1}^{2} i^2 = 1^2 + 2^2 = 5$$

$$\sum_{i=1}^{3} i = 1 + 2 + 3 = 6 \quad , \quad \sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2 = 14$$

$$\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10 \quad , \quad \sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$$

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15 \quad , \quad \sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

$\vdots$

$$\left[\sum_{i=1}^{3} i^2 \div \sum_{i=1}^{3} i = \frac{14}{6} \; , \quad \sum_{i=1}^{2} i^2 \div \sum_{i=1}^{2} i = \frac{5}{3}\right] \to \frac{14}{6} - \frac{5}{3} = \frac{2}{3}$$

$$\left[\sum_{i=1}^{4} i^2 \div \sum_{i=1}^{4} i = \frac{30}{10} \; , \quad \sum_{i=1}^{3} i^2 \div \sum_{i=1}^{3} i = \frac{14}{6}\right] \to 3 - \frac{14}{6} = \frac{2}{3}$$

$$\left[\sum_{i=1}^{5} i^2 \div \sum_{i=1}^{5} i = \frac{55}{15} \; , \quad \sum_{i=1}^{4} i^2 \div \sum_{i=1}^{4} i = \frac{30}{10}\right] \to \frac{55}{15} - 3 = \frac{2}{3}$$

$\vdots$

Note that this ratio is continuous wherever we continued in this subtraction hence we can express about the last results by this relation:

$$\frac{\sum i^2}{\sum i} - \frac{\sum (i-1)^2}{\sum (i-1)} = \frac{2}{3}$$

And we can solve it to find $\sum_{i=1}^{n} i^2$ (we write $\sum_{i=1}^{n} i^2 = \sum i^2$ & $\sum_{i=1}^{n} i = \sum i$)

$$\frac{\sum i^2}{\sum i} - \frac{\sum (i-1)^2}{\sum (i-1)} = \frac{\sum i^2}{\sum i} - \frac{\sum i^2 + 2\sum i + n}{\sum (i-1)} = \frac{2}{3}$$

$$\frac{\sum i^2 \sum (i-1) - \sum i^2 \sum i + 2\sum i \sum i - n \sum i}{\sum i \sum (i-1)} = \frac{2}{3}$$

$$\sum i^2 = \frac{2\sum i \sum (i-1) - 6\sum i \sum i + 3n \sum i}{3\left[\sum (i-1) - \sum i\right]}$$

$$\sum i^2 = \left[\frac{n(n+1)}{2}\right] * \frac{\left[4 * \frac{n(n+1)}{2} - n\right]}{3n} = \frac{n(n+1)(2n+1)}{6}$$

Therefore:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

To prove that by using mathematical induction:

1. Let $n = 1$, then : $\sum_{i=1}^{n} i^2 = \frac{(1+1)(2+1)}{6} = 1$

2. Suppose that $n = k$, then : $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$

3. Let $n = k+1$, T.P : $\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} (i+1)^2 + 1$

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{2k^3 + 9k^2 + 13k + 6}{6}$$

$$= \frac{2k^3 + 3k^2 + k}{6} + 2\left(\frac{k^2 + k}{2}\right) + k + 1$$

$$= \frac{k(k+1)(2k+1)}{6} + 2\left(\frac{k(k+1)}{2}\right) + k + 1$$

$$= \sum_{i=1}^{k} i^2 + 2\sum_{i=1}^{k} i + \sum_{i=1}^{k} 1 + 1$$

$$= \sum_{i=1}^{k} (i^2 + 2i + 1) + 1$$

$$= \sum_{i=1}^{k} (i+1)^2 + 1$$

To find $\sum_{i=1}^{n} i^3$, such a last method in numerical calculation we can see that:

$\sum_{i=1}^{2} i = 1 + 2 = 3$, $\sum_{i=1}^{2} i^3 = 1^3 + 2^3 = 9$

$\sum_{i=1}^{3} i = 1 + 2 + 3 = 6$, $\sum_{i=1}^{3} i^3 = 1^3 + 2^3 + 3^3 = 36$

$\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10$, $\sum_{i=1}^{4} i^3 = 1^3 + 2^3 + 3^3 + 4^3 = 100$

$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15$, $\sum_{i=1}^{3} i^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225$

$\vdots$

$$\left[\sum_{i=1}^{3} i^3 \div \sum_{i=1}^{3} i = \frac{36}{6} \quad , \quad \sum_{i=1}^{2} i^3 \div \sum_{i=1}^{2} i = \frac{9}{3}\right] \rightarrow 6 - 3 = 3$$

$$\left[\sum_{i=1}^{4} i^3 \div \sum_{i=1}^{4} i = \frac{100}{10} \quad , \quad \sum_{i=1}^{3} i^3 \div \sum_{i=1}^{3} i = \frac{36}{6}\right] \rightarrow 10 - 6 = 4$$

$$\left[\sum_{i=1}^{5} i^3 \div \sum_{i=1}^{5} i = \frac{225}{15} \quad , \quad \sum_{i=1}^{4} i^3 \div \sum_{i=1}^{4} i = \frac{100}{10}\right] \rightarrow 15 - 10 = 5$$

$\vdots$

Note that this ratio is continuous wherever we continued in this subtraction hence we can express about the last results by this relation:

$\frac{\sum i^3}{\sum i} - \frac{\sum (i-1)^3}{\sum (i-1)} = n$, and such the last method if we solve it we can;

Show that:

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

To prove that by using mathematical induction:

1. Let $n = 1$, then : $\sum_{i=1}^{n} i^3 = \left[\frac{(1+1)}{2}\right]^2 = 1$

2. Suppose that $n = k$, then : $\sum_{i=1}^{k} i^3 = \left[\frac{k(k+1)}{2}\right]^2$

3. Let $n = k+1$, T.P : $\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} (i+1)^3 + 1$

$$\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2 * (k+2)^2}{4} + \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4}$$

$$= \frac{k^4 + 2k^3 + k^2}{4} + 3\frac{(2k^3 + 3k^2 + k)}{6} + 3\frac{(k^2 + k)}{2} + k + 1$$

$$= \frac{\left(k(k+1)\right)^2}{4} + 3\frac{k(k+1)(2k+1)}{6} + 3\frac{k(k+1)}{2} + k + 1$$

$$= \sum_{i=1}^{k} i^3 + 3\sum_{i=1}^{k} i^2 + 3\sum_{i=1}^{k} i + \sum_{i=1}^{k} 1 + 1$$

$$= \sum_{i=1}^{k} (i^3 + 3i^2 + 3i + 1) + 1$$

$$= \sum_{i=1}^{k} (i+1)^3 + 1$$

And by the same way we can find $\sum_{i=1}^{n} i^4$ by find a relation between $\sum_{i=1}^{n} i^4$ and $\sum_{i=1}^{n} i^2$ such that $\frac{\sum i^4}{\sum i^2} - \frac{\sum (i-1)^4}{\sum (i-1)^2} = \frac{6n}{5}$, if we solve this equation we will found:

$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$ note that we used $\sum_{i=1}^{n} i^2$ instead of use $\sum_{i=1}^{n} i$ avoid the big and complicated ratios to prove:

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

To prove that by using mathematical induction;

$$1.\ \text{Let } n=1, \text{then: } \sum_{i=1}^{n} i^4 = \frac{1(1+1)(2*1+1)(3*1^2+3*1-1)}{30} = 1$$

$$2.\ \text{Suppose that } n=k, \text{then: } \sum_{i=1}^{k} i^4 = \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}$$

$$3.\ \text{Let } n=k+1\ , \qquad \text{T.P: } \sum_{i=1}^{k+1} i^4 = \sum_{i=1}^{k} (i+1)^4 + 1$$

$$\sum_{i=1}^{k+1} i^4 = \frac{(k+1)(k+2)(2k+3)(3(k+1)^2+3k+2)}{30}$$

$$= \frac{6k^5+45k^4+130k^3+180k^2+119k+30}{30}$$

$$= \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} + 4\left[\frac{k(k+1)}{2}\right]^2 + 6\frac{k(k+1)(2k+1)}{6} + 4\frac{k(k+1)}{2} + k + 1$$

$$= \sum_{i=1}^{k} i^4 + 4\sum_{i=1}^{k} i^3 + 6\sum_{i=1}^{k} i^2 + 4\sum_{i=1}^{k} i + \sum_{i=1}^{k} 1 + 1, \text{we know that: } \sum_{i=1}^{k} 1 = k$$

$$= \sum_{i=1}^{k} (i^4+4i^3+6i^2+4i+1) + 1$$

$$= \sum_{i=1}^{k} (i+1)^4 + 1$$

Note that:

$$\frac{\sum i^5}{\sum i^3} - \frac{\sum (i-1)^5}{\sum (i-1)^3} = \frac{4n}{3}$$

Such as the last methods if we solve it, we will obtain this result:

$$\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

To prove that by using mathematical induction:

$$1.\ \text{Let } n=1\ , \qquad \text{then: } \sum_{i=1}^{n} i^5 = \frac{1^2(1+1)^2(2*1^2+2*1-1)}{12} = 1$$

$$2.\ \text{Suppose that } n=k\ , \qquad \text{then: } \sum_{i=1}^{k} i^5 = \frac{k^2(k+1)^2(2k^2+2k-1)}{12}$$

3. Let $n = k + 1$, T.P : $\sum_{i=1}^{k+1} i^5 = \sum_{i=1}^{k} (i+1)^5 + 1$

$$\sum_{i=1}^{k+1} i^5 = \frac{(k+1)^2(k+2)^2(2(k+1)^2 + 2(k+1) - 1)}{12}$$

$$= \frac{2k^6 + 18k^5 + 65k^4 + 120k^3 + 119k^2 + 60k + 12}{12}$$

$$= \frac{2k^6 + 6k^5 + 5k^4 - k^2}{12} + \frac{5(6k^5 + 15k^4 + 10k^3 - k)}{30} + \frac{10(k^4 + k^3 + k^2)}{4}$$

$$+ \frac{10(2k^3 + 3k^2 + k)}{6} + \frac{5(k^2 + k)}{2} + k + 1$$

$$= \sum_{i=1}^{k} i^5 + 5\sum_{i=1}^{k} i^4 + 10\sum_{i=1}^{k} i^3 + 10\sum_{i=1}^{k} i^2 + 5\sum_{i=1}^{k} i + \sum_{i=1}^{k} 1 + 1$$

$$= \sum_{i=1}^{k} (i^5 + 5i^4 + 10i^3 + 10i^2 + 5i + 1) + 1$$

$$= \sum_{i=1}^{k} (i+1)^5 + 1$$

4.1 The relations between the summations and the ratios:

The last method is confined to creating some relations between the summations and as same as ratios, so it may be growing to complicated ratios whenever we continuous to increase of power to i so we can found another way to found other summations by conversion to linear system, and we will be taken this motif in this section notice:

$$\sum_{i=1}^{n} i = \frac{n^2}{2} + \frac{n}{2} , \qquad \sum_{i=1}^{n} i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} , \qquad \sum_{i=1}^{n} i^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

$$\sum_{i=1}^{n} i^5 = \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12}$$

Notice all this summations results in form:

$\sum_{i=1}^{n} i^s = \frac{n^{s+1}}{s+1} + \frac{n^s}{2} + \cdots$, such that (s) is any natural number $\neq 1$

$i.e: s \in \mathbb{N}; s > 1$, so we can offer $\sum_{i=1}^{n} i^6$ as follows:

$$\sum_{i=1}^{n} i^6 = \frac{n^7}{7} + \frac{n^6}{2} + X_1 n^5 + X_2 n^4 + X_3 n^3 + X_4 n^2 + X_5 n \qquad \dots (*)$$

Such that: X_1, X_2, X_3, X_4, X_5 is ratios will be corrected idiom of $\sum_{i=1}^{n} i^6$ now if we insert $n = 1$ as follows:

$$\sum_{i=1}^{1} i^6 = \frac{1}{7} + \frac{1}{2} + X_1 + X_2 + X_3 + X_4 + X_5 = 1$$

$$= X_1 + X_2 + X_3 + X_4 + X_5 = 1 - \frac{1}{2} - \frac{1}{7} = \frac{5}{14} \qquad \ldots(1)$$

$$\text{if } n = 2 : \sum_{i=1}^{2} i^6 = \frac{2^7}{7} + \frac{2^6}{2} + 2^5 * X_1 + 2^4 * X_2 + 2^3 * X_3 + 2^2 * X_4 + 2 * X_5$$

$$= 32X_1 + 16X_2 + 8X_3 + 4X_4 + 2X_5 = \frac{103}{7} \qquad \ldots(2)$$

$$\text{if } n = 3 : \sum_{i=1}^{3} i^6 = \frac{3^7}{7} + \frac{3^6}{2} + 3^5 * X_1 + 3^4 * X_2 + 3^3 * X_3 + 3^2 * X_4 + 3 * X_5$$

$$= 243X_1 + 81X_2 + 27X_3 + 9X_4 + 3X_5 = \frac{4555}{14} \qquad \ldots(3)$$

$$\text{if } n = 4 : 1024X_1 + 256X_2 + 64X_3 + 16X_4 + 4X_5 = \frac{3510}{7} \qquad \ldots(4)$$

$$\text{if } n = 5 : 3125X_1 + 625X_2 + 125X_3 + 25X_4 + 5X_5 = \frac{21585}{14} \qquad \ldots(5)$$

By: (1) , (2) , (3) , (4) and (5) we have:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 32 & 16 & 8 & 4 & 2 \\ 243 & 81 & 27 & 9 & 3 \\ 1024 & 256 & 64 & 16 & 4 \\ 3125 & 625 & 125 & 25 & 5 \end{bmatrix} = \begin{bmatrix} 5/14 \\ 103/7 \\ 4555/14 \\ 3510/7 \\ 21585/14 \end{bmatrix}$$

We can solve this linear system by (Gauss method [1]) or (MATLAB program)

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ -1/6 \\ 0 \\ 1/42 \end{bmatrix} \quad \ldots(**), \qquad \text{by } (*) \text{ and } (**) \text{ we get:}$$

$$\sum_{i=1}^{n} i^6 = \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42}$$

In the same method we can found:-

$$\sum_{i=1}^{n} i^7 = \frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} - \frac{7n^4}{24} + \frac{n^2}{12}$$

$$\sum_{i=1}^{n} i^8 = \frac{n^9}{9} + \frac{n^8}{2} + \frac{2n^7}{3} - \frac{7n^5}{15} + \frac{2n^3}{9} - \frac{n}{30}$$

$$\sum_{i=1}^{n} i^9 = \frac{n^{10}}{10} + \frac{n^9}{2} + \frac{3n^8}{4} - \frac{7n^6}{10} + \frac{n^4}{2} - \frac{3n^2}{20}$$

$$\sum_{i=1}^{n} i^{10} = \frac{n^{11}}{11} + \frac{n^{10}}{2} + \frac{5n^9}{6} - n^7 + n^5 - \frac{n^3}{2} + \frac{5n}{66}$$

$$\sum_{i=1}^{n} i^{11} = \frac{n^{12}}{12} + \frac{n^{11}}{2} + \frac{11n^{10}}{12} - \frac{11n^{8}}{8} + \frac{11n^{6}}{6} - \frac{11n^{4}}{8} + \frac{5n^{2}}{12}$$

$$\sum_{i=1}^{n} i^{12} = \frac{n^{13}}{13} + \frac{n^{12}}{2} + n^{11} - \frac{11n^{9}}{6} + \frac{22n^{7}}{7} - \frac{33n^{5}}{10} + \frac{5n^{3}}{3} - \frac{691n}{2730}$$

$$\sum_{i=1}^{n} i^{13} = \frac{n^{14}}{14} + \frac{n^{13}}{2} + \frac{13n^{12}}{12} - \frac{143n^{10}}{60} + \frac{143n^{8}}{28} - \frac{143n^{6}}{20} + \frac{65n^{4}}{12} - \frac{691n^{2}}{420}$$

$$\sum_{i=1}^{n} i^{14} = \frac{n^{15}}{15} + \frac{n^{14}}{2} + \frac{7n^{13}}{6} - \frac{91n^{11}}{30} + \frac{143n^{9}}{18} - \frac{143n^{7}}{10} + \frac{91n^{5}}{6} - \frac{691n^{3}}{90} + \frac{7n}{6}$$

$$\sum_{i=1}^{n} i^{15} = \frac{n^{16}}{16} + \frac{n^{15}}{2} + \frac{5n^{14}}{4} - \frac{91n^{12}}{24} + \frac{143n^{10}}{12} - \frac{429n^{8}}{16} + \frac{455n^{6}}{12} - \frac{691n^{4}}{24} + \frac{35n^{2}}{4}$$

$\vdots$

This method is considering easy way to find more than of the summations.

4.2 Great chain of the summations:

As we have said earlier or before there is note or rule that said $\sum_{i=1}^{n} i^{s} = \frac{n^{s+1}}{s+1} + \frac{n^{s}}{2} + \cdots$ such that (s) is any natural number $\neq 1$, $i.e: s \in \mathbb{N}; s > 1$.

We can develop this rule as big chain to comprise more than numbers of summation as follows:

$$\sum_{i=1}^{n} i^{s} = \frac{n^{s+1}}{s+1} + \frac{n^{s}}{2} + \frac{f'(n^{s})}{2*3!} - \frac{f'''(n^{s})}{6!} + \frac{f^{(5)}(n^{s})}{6*7!} - \frac{3*f^{(7)}(n^{s})}{10!} + \frac{5*f^{(9)}(n^{s})}{6*11!}$$
$$- \frac{691*f^{(11)}(n^{s})}{15!} + \frac{35*f^{(13)}(n^{s})}{2*15!} - \frac{10851*f^{(15)}(n^{s})}{5*18!}$$
$$+ \frac{43867*f^{(17)}(n^{s})}{42*19!} - \frac{1222277*f^{(19)}(n^{s})}{5*22!} + \cdots$$

Where: $f'(n^{s})$ is the first, $f'''(n^{s})$ is the second, …, $f^{(s)}(n^{s})$ is the (s) derivative of n.

We stop when:
If s is odd then $\rightarrow f^{(s-1)}(n^{s}) = 0$, conversely: if s is even then $\rightarrow f^{(s)}(n^{s}) = 0$

For example to found $\sum_{i=1}^{n} i^{4}, s = 4, f'(n^{4}) = 4n^{3}, f'''(n^{4}) = 24n, f^{(4)}(n^{4}) = 24$ but s is even;

$\therefore f^{(4)}(n^{4}) = 0$, by Great chain we have:

$$\sum_{i=1}^{n} i^{4} = \frac{n^{4+1}}{4+1} + \frac{n^{4}}{2} + \frac{4n^{3}}{12} - \frac{24n}{6!} = \frac{n^{5}}{5} + \frac{n^{4}}{2} + \frac{n^{3}}{3} - \frac{n}{30}$$

Reference:

[1] F. G. Curtis and O. W. Patrick, "Applied numerical analysis", Addison-Wesley, 1999.

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[3] K. H. Rosen, "Discrete mathematics and its applications", 5-Edition, McGraw-Hill, USA, 2005.