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Eigen value and eigen function for electron in magnetic field using relativistic quantum mechanics

القيم الخاصة والدوال الخاصة لإلكترون في مجال مغناطيسي باستخدام ميكانيك الكم النسبي

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Abstract:

We have studied the motion of electron in electromagnetic field using relativistic quantum mechanics to find the eigen values of energy and eigen function for electron by solving Dirac-Equation for electron in a vector potential ($A_Z=A_X=0$) and $A_y=HX$ (the field H being along the Z-axis).
We used the second-order equation for the axially function Φ and assume that Φ is an eigen function of the operator Σ_Z with eigen value $\sigma =\pm 1$.

الخلاصة :

لقد درسنا حركة الالكترون في مجال كهرومغناطيسي باستعمال ميكانيك الكم النسبي لايجاد القيم الخاصة للطاقة للالكترون وايجاد الدوال الخاصة له.
وذلك بحل معادلة ديراك للالكترون في المجال الكهرومغناطيسي حيث إن اخذنا حالة خاصة وهي ($A_Z=A_X=0$) و $A_y=HX$ ويكون المجال المغناطيسي باتجاه المحور (Z).
المعادلة من الدرجة الثانية لايجاد الدالة المساعدة Φ وافترضنا إن Φ هي دالة خاصة للمؤثر Σ_Z والذي سياتي تعريفه لاحقا وبقيم خاصة $\sigma =\pm 1$.

Introduction:

The wave equation of free particles express essentially those properties which depends on the general requirements of space–time symmetry. Physical processes involving the particles however, depend on their interaction properties.
In relativistic theory it proves impossible to obtain by any simple generalization of the wave equation a description of particles that are capable of strong interactions i.e. a description going beyond the information contained in the equations for free particles.
The wave equation method however is applicable to the description of electromagnetic interaction of particles that are no capable of strong interaction these include electrons (and positrons) and the vary domain of electron quantum electrodynamics is there for accessible to the existing theory.
There are also unstable particles the Mesons which are not capable of strong interaction they are described by the same quantum electrodynamics as regards phenomena occurring in times short in comparison with tier lifetime.
In this research we shall discuss the problem of quantum electrodynamics which fall within the scope of a single-particle theory.
We shall apply this theory especially on electrons (or positrons) with spin equals to ($\sigma =\pm 1/2\ \hbar$) these are problems in which the number of particles is unchanged and the interaction can be represented in terms of an external electromagnetic field.
After Dirac put his equation in 1932 many scientist applied that equation in many field of relativistic quantum mechanics it had been applied to H-atom by Marshak 1933 also it applied to free electron [3], by Fyman 1935 [5] it applied to linear harmonic oscillator by Myazawa [7].
Here in this research we use relativistic quantum mechanics to find the Eigen values and Eigen function for electron (or positron) in constant magnetic field.

Theory:

The wave equation method is applicable to the description of electromagnetic interactions of particles that are not capable of strong interactions.
These electrons (and positron) and the very wide domain of electron quantum electrodynamics is therefore accessible to the existing theory. There are also unstable particles the Mesons which are not capable strong interaction they are described by the same quantum electrodynamics as regard phenomena occurring in times short in comparison with their lifetime with respect to weak interaction.
In this research we shall discuss problem of quantum electrodynamics which fall in to the scope of single particle theory. This problem in which the number of particles is unchanged. And the interaction can be represented in terms of an external electromagnetic field.
Besides the conditions which ensure that the ex. Field may be regarded as give there are condition arising from radioactive correction which also in limits on the validity of such theory.
The wave equations of an electron in a given ex. Field can be derived in the same way as in non relativistic theory [3].

Let $A^{\mu}=(\phi, \vec{A})$ be the 4-potential.

The external electromagnetic field ($\vec{A}$ being the potential and ϕ is the scalar potential).

We obtain to desired equation on replace the 4-mometum operator $\hat{\vec{P}}$ in Dirac's equation by $\hat{\vec{P}}$ - e$\vec{A}$ where (e) is the charge on the

$$[\ \gamma(\hat{\vec{P}}-e\vec{A})-m\ ]\psi=0........................(1)$$

$$\gamma=\begin{pmatrix}\gamma^1\\ \gamma^2\\ \gamma^3\end{pmatrix}$$ are Dirac matrices

We put (C – the velocity of light equal=1) i.e. the unit of velocity in the research .
The corresponding Hamiltonian $\vec{H}$ is found to be [4]:

$$\hat{\vec{H}}=\alpha\ (\hat{\vec{P}}-e\vec{A})+\hat{\beta}m+e\phi.......................(2)$$

α and β are Pauli matrices.
The first-order equation (2) can be transformed to second-order equations by applying to them the operator $Y(\hat{\vec{P}}-e\vec{A})+m$ to them

$$\left[\gamma^{\mu}\gamma^{\upsilon}(P_{\mu}-eA_{\mu})(P_{\upsilon}-eA_{\upsilon})-m^2\right]\psi=0....................(3)$$

The existing of indices (μ ,υ) as sub and subscripts mean summation.
The product $Y^{\mu}Y^{\upsilon}$ may be written as following:

$$\gamma^{\mu}\gamma^{\upsilon}=1/2(\gamma^{\mu}\gamma^{\upsilon}+\gamma^{\upsilon}\gamma^{\mu})+1/2(\gamma^{\mu}\gamma^{\upsilon}-\gamma^{\upsilon}\gamma^{\mu})$$

$$=g^{\mu\upsilon}+\sigma^{\mu\upsilon}\ ....(\ 4\ )$$

Where $g^{\mu\upsilon}$, $\sigma^{\mu\upsilon}$ are symmetric and anti-symmetric matrix 4-tensor.

On multiplying by $\sigma^{\mu\upsilon}$ we can anti-symmetric by the substation :

$$(P_\mu - eA_\mu)(P_\upsilon - eA_\upsilon) \rightarrow 1/2\{(P_\mu - eA_\mu)(P_\upsilon - eA_\upsilon)\}$$
$$= 1/2(-A_\mu P_\upsilon + P_\upsilon A_\mu - P_\mu A_\upsilon + A_\upsilon P_\mu)$$
$$= 1/2\, i\, e(\partial_\upsilon A_\mu - \partial_\mu A_\upsilon)$$
$$= -1/2\, ie\, F_{\mu\upsilon} \text{.............................}(\ 5\ )$$

∂_υ mean differentiation with respect (X^Y), i=√-1, P_μ -is the operator of momentum in the direction (μ). We put here ħ=1 i.e. it the unit of action (Blank constant)

$F_{\mu\upsilon}$ is anti-symmetric electromagnetic field tensor defined from the bracket , the result is the second-order equation

$$\left[(\hat{\vec{P}} - e\hat{\vec{A}})^2 - m^2 - 1/2\, ieF_{\mu\upsilon}\sigma^{\mu\upsilon} \right] \psi = 0 \text{...................}(6)$$

The product of $F_{\mu\upsilon}\sigma^{\mu\upsilon}$ may be written three dimensional forms of the components:

$$\sigma^{\mu\upsilon} = (\ \hat{\vec{\alpha}}, i\hat{\sum}\ ), \qquad F_{\mu\upsilon} = (\ -\vec{E}, \vec{H}\ )$$

$\vec{E}$, and $\vec{H}$ is the electric and magnetic field then:

$$\left[(\hat{P} - e\vec{A})^2 - m^2 + e\vec{\sum}.\vec{H} - ie\,\vec{\alpha}.\vec{E} \right] \psi = 0 \text{........................}(7)$$

Or in ordinary units:

$$\left[(i\hbar\frac{\partial}{\partial t} - \frac{e}{c}\phi)^2 - (i\hbar\vec{\nabla} + \frac{e}{c}\vec{A})^2 - m^2C^2 + \frac{e\hbar}{c}\vec{\sum}.\vec{H} - i\frac{e\hbar}{c}\vec{\alpha}.\vec{E} \right] \psi = 0 \text{.....}(8)$$

The occurrence in these equations of term in the fields $\vec{E}$ *and* $\vec{H}$ is due to the spin of the particle. The customary procedure is that if ϕ is any solution of the second-order equation then solution of the correct first order equation

$$\Psi = [\ \gamma(\hat{\vec{P}} - e\vec{A}) + m\ ]\ \phi$$

For on multiplying this equation by $(\hat{\vec{P}} - e\vec{A}) - m$, we see the right hand vanishes if (ϕ) satisfy eq.6.

The stationary state solution of Dirac's equation in an external field may include of both continuous spectrum and the discrete spectrum. As in the non-relativistic theory states with continuous spectrum correspond to infinite motion in which particle can be at infinity. It may there be regarded as free particle .

Since the Eigin-values of the Hamiltonian of a free particle are (±√ $\hat{\vec{P}} + m^2$) it is clear that the continuous spectrum of energy eigin-values is in arranges (Σ ≥ m) and (Σ ≤ - m), if -m< Σ < m the particles cannot be at infinity and the motion is therefore finite and the state belongs to the discrete spectrum.

The wave equation of electron in constant magnetic field:

The vector potential is A_z=A_x=0, A_y=H_x (the field $\vec{H}$ being along the Z-axis and $\vec{H} = \vec{\nabla} \times \vec{A}$. The component P_y, P_z of generalize momentum (as well as the energy) are conserved.

We use the second-order equation for axillaries (ϕ) and assume that ϕ is an eigin-function of the operator (Σ) with eigin-value σ =±1, and the operator P_y, P_z . the equation of (ϕ) is:

$$\left[-\frac{d^2}{dx^2}+(eH_x-P_y)^2-eH\sigma\right]\varphi=(\varepsilon^2-m^2-P_z^2)\varphi.................(9)$$

To solve eq.9 we put: eH_x-P_y=√eH θ, eH = φ $\vec{\varphi}$, dx=(1/√eH) dθ. Where θ-is independent variable and $\vec{\phi}$ -is axillary's wave function.

Eq.9 become:

$$-\frac{d^2}{d\theta^2}\vec{\phi}+\theta^2\vec{\phi}=\frac{\varepsilon^2-m^2-P_z^2+\sigma eH}{eH}................(10)$$

Put: $$\bar{\varepsilon}=\frac{\varepsilon-m^2-P_z^2+\sigma eH}{eH}...................(11)$$

$$-\frac{d^2}{d\theta^2}\overline{\varphi}+\theta^2\overline{\varphi}=\bar{\varepsilon}\varphi...................(12)$$

Put: $\overline{\varphi}=g(\theta)e^{-\frac{\theta^2}{2}}$, $\frac{d\overline{\varphi}}{dt}=-\theta\, g(\theta)e^{-\frac{\theta^2}{2}}+e^{-\frac{\theta^2}{2}}\frac{dg(\theta)}{dt}$

$$\frac{d^2\overline{\varphi}}{d\theta^2}=\theta^2e^{-\frac{\theta^2}{2}}g(\theta)-2\theta e^{-\frac{\theta^2}{2}}\frac{dg(\theta)}{dt}+e^{-\frac{\theta^2}{2}}\frac{d^2g(\theta)}{d\theta^2}...............(13)$$

Substitute eq.13 in eq.10 we obtain:

$$-\frac{d^2g(\theta)}{dt^2}+2\theta\frac{dg(\theta)}{d\theta}=(\bar{\varepsilon}-1)\,g(\theta).....................(14)$$

We can integrate eq.14 by power-series :

$$g(\theta)=g_0+g_1\theta+g_2\theta^2+.........=\sum_{n=0}^{\infty}g_n\theta^n..................(15)$$

Where g_n are undetermined factors.

$$\frac{dg}{dt}=g_1+2g_2\theta+3g_3\theta^2+..........=\sum_{n=1}^{\infty}ng_n\theta^{n-1}................(16)$$

$$2\theta\frac{dg}{d\theta}=2\theta\,g(\theta)+4g_2\theta^2+6g_3\theta^3+.....=\sum_{n=1}^{\infty}2ng_n\theta^n.....(17)$$

$$\frac{d^2g}{d\theta^2}=2g_2+6g_3\theta+..................=\sum_{k=2}^{\infty}(k-1)kg_k\theta^{k-1}......(18)$$

We change the suffix of summation to (k), if we put (k= -2=n) we have:

$$\frac{d^2g}{dt^2}=\sum_{n=0}^{\infty}(n+2)(n+1)g_{n+2}\theta^n......................(19)$$

Substitute eq.19 and eq.17 in eq.14 we obtain:

$$\sum_{n=0}^{\infty}\theta^n\left[-(n+2)(n+1)g_{n+2}+2ng_n-(\bar{\varepsilon}-1)g_n\right]=0\ldots\ldots\ldots\ldots\ldots\ldots(20)$$

We knew that any series to be equal zero the factors must equal zero :

$$\Rightarrow g_{n+2}=g_n\frac{2n+1-\bar{\varepsilon}}{(n+2)(n+1)}\ldots\ldots\ldots\ldots(21)$$

Studying of the series let us suppose that $g_0\neq0$ we obtain from eq.19 the values of ($g_2,g_4,g_6\ldots..$) and don't appear in series and odd factor if $g_0=0$ and $g_1\neq0$ we obtain ($g_3,g_5,\ldots..$) and don’t appear any even factor in the series.

If (n) is large number we can neglect the numbers which appear with (n) and we obtain:

$$g_{n+2}=\frac{2}{n}g_n\ldots\ldots\ldots\ldots(22)$$

If we take function have only odd (n) we obtain a series like that of even (n) because the number (1) can be neglect.

Therefore when (n) become large number the factors will be considers for even and odd series.

If we put ($g_{n'}=g_{2n}$) we obtain:

$$g_{n'+1}=\frac{g_0}{n'(n'-1)(n'-2)\ldots\ldots\ldots}=\frac{g_0}{n'!}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(23)$$

Therefore

$$g_{(\theta)}=\sum_{n=0}^{\infty}g_{n'}(\theta^2)=\sum\frac{g_{n'}(\theta^2)^{n'}}{n'!}=g_{n'}e^{\theta^2}\ldots\ldots\ldots\ldots\ldots\ldots(24)$$

And therefore g(θ) will be the exponential function (e^{θ^2}), but (e^{θ^2}) is un acceptable because the wave function must be limit at infinity because its square value is the probability of the existence.

There is one way to obtain determine value for ϕ at infinity. Its necessary that the series must vanish for value of (n) and the (g_n) when (n) is large than that value must equal to zero.

We see the eq.21 that if (g_{n+2}……..=zero) and (2n+1=ε) or:

$$\bar{\varepsilon}^2-m^2-P_2^2=|e||H|(2n+1)-eH\sigma\ldots\ldots\ldots\ldots\ldots\ldots(25)\ ,n=1,2\ldots\ldots$$

The product of g(θ) with ($e^{-\theta^2/2}$) tends to zero when $\theta\to\infty$ and this motion will be limit motion.

The Eigin Function For The Electron:

If we take n=1 we obtain:

$$\bar{\phi}_1=g_1\theta e^{-\frac{\theta^2}{2}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26)$$

And this function equals zero when (θ=0)

$$\bar{\phi}_2=g_0(1-2\theta^2)e^{-\frac{\theta^2}{2}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27)$$

And this function equals zero when (θ = ±1/√2).

If (e) is negative and (H) in the same direction of (σ), for (σ =1):

$$\bar{\varepsilon} = \sqrt{2|e||H|(n+1) + m^2 + P_z^2}$$

$$= \sqrt{2|e||H|(n+1) + (mc^2)^2 + (mv_z c)^2} \text{ ....(28)}$$

For (σ = -1):

$$\bar{\varepsilon} = \sqrt{2|e||H|(n) + m^2 + P_z^2}$$

$$= \sqrt{2|e||H|(n) + (mc^2)^2 + (mv_z c)^2} \text{ ....(29)}$$

If (e) is positive and H in the same direction of σ, for (σ =1):

$$\bar{\varepsilon} = \sqrt{2|e||H|(n) + m^2 + P_z^2}$$

$$= \sqrt{2|e||H|(n) + (mc^2)^2 + (mv_z c)^2} \text{ ....(30)}$$

For (σ = -1):

$$\bar{\varepsilon} = \sqrt{2|e||H|(n+1) + m^2 + P_z^2}$$

$$= \sqrt{2|e||H|(n+1) + (mc^2)^2 + (mv_z c)^2} \text{ ....(31)}$$

If (e) is negative and H in the opposite direction of σ, if H and σ have opposite direction:

$$\bar{\varepsilon} = \sqrt{2|e||H|(n) + m^2 + P_z^2}$$

$$= \sqrt{2|e||H|(n) + (mc^2)^2 + (mv_z c)^2} \text{ ....(32)}$$

If H and σ have the same direction:

$$\bar{\varepsilon} = \sqrt{2|e||H|(n+1) + m^2 + P_z^2}$$

$$= \sqrt{2|e||H|(n+1) + (mc^2)^2 + (mv_z c)^2} \text{ ....(33)}$$

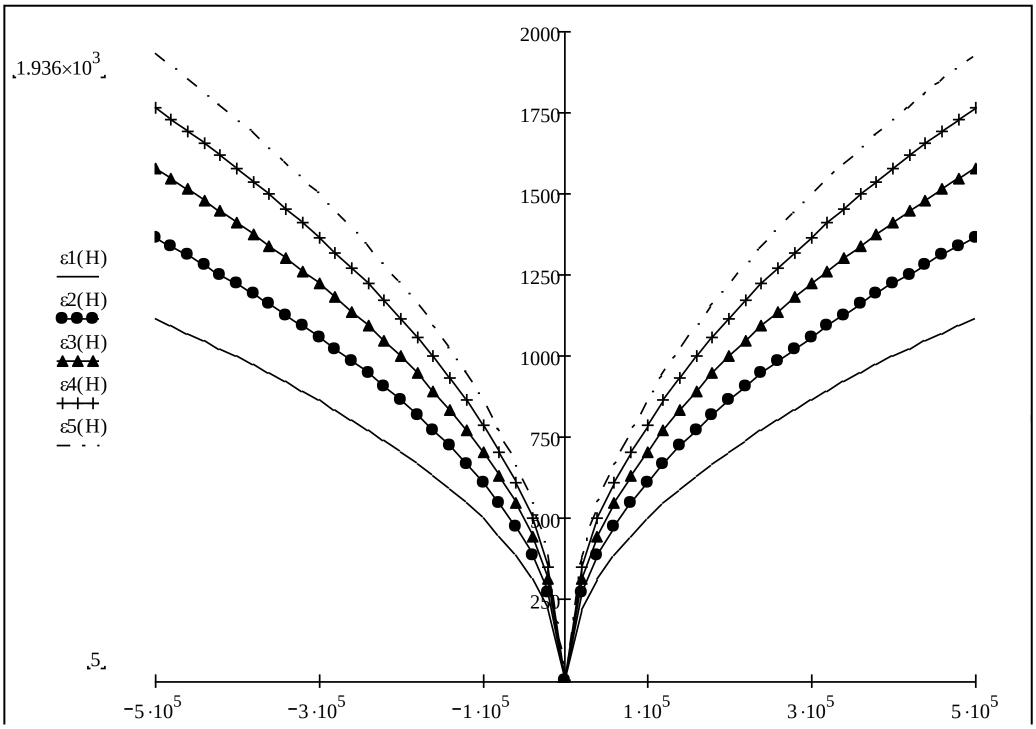
ξ

	n	$\bar{\varepsilon}$ *(eV)*
For H= 10^5 Guas And σ = 1	1	14.3786
	2	14.6607
	3	14.9375
	4	15.2092
	5	15.4762

	n	$\bar{\varepsilon}$ *x10^5 (eV)*
For H= 2x10^5 Guas And σ = 1	1	14.9375
	2	15.4762
	3	15.9967
	4	16.5009
	5	16.9901

	n	$\bar{\varepsilon}$ x10^5 (eV)
For H= 3x10^5 Guas And σ = 1	1	15.4762
	2	15.9967
	3	16.5009
	4	16.9901
	5	17.6456

	n	$\bar{\varepsilon}$ x10^5 (eV)
For H= 4x10^5 Guas And σ = 1	1	15.9967
	2	16.5009
	3	16.9901
	4	17.6456
	5	17.9285



Conclusion :

We see from the relation between the energy of the electron and the magnetic field, that the energy is proportional to the square root of (H) . energy is composite from 4-factors:

1. (mc^2) which it's the rest mass energy of the electron and this is the lower level of energy for the electron if there is no field .
2. $(P_Z C)^2$ the kinetic energy of the electron in relativistic mechanics.
3. (eHσ) the interaction energy of the spin of the electron with magnetic field.
4. { │e│ │H│(2n+1) } or { │e│ │H│(n) } it's the energy of the electron with magnetic field.

We see that the energy of electron is quantized and the relation between $\sum$ and (n) or (H) is conic section.

The gape between positive energy levels an negative energy levels is a zone that there are no electrons can be exist and its call Dirac sea of energy.

If the magnetic field is zero we see that the energy of electron is ($\bar{\varepsilon} = \sqrt{(mc^2)^2 + (P_z c)^2}$) and that is agree with the relativist mechanics. This proofs that the relation we have reach to its right.

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