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Bifurcation Theory: A Review

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Abstract

Bifurcation theory is a field of mathematics that studies the qualitative changes in the behavior of a dynamical system as a parameter in the system is varied.
In this work, we review the history, the types of bifurcations and the relations of those concepts with the chaos phenomena and some other concepts such as sensitivity to the initial value. We also survey some applications of this important theory in the other fields
Like Physics, chemistry, biology, geography, … etc>

Keywords:
Bifurcation , Hopf bifurcation, Saddle- node bifurcation, Period doubling bifurcation, Chaos, SDIC

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1. INTRODUCTION

Bifurcation theory is a branch of mathematics that studies the qualitative changes in the behavior of a dynamical system as a parameter in the system is varied. These changes are often characterized by the sudden appearance or disappearance of certain types of solutions, such as equilibrium points or periodic orbits. Bifurcations can also lead to the creation of chaotic or strange attractor [6].

A bifurcation occurs when a small change in a system's parameters causes a sudden change in its behavior. This can happen, for example, when a fixed point of a dynamical system fails to be stable, leading to the appearance of a new equilibrium point, or when a periodic orbit is created or destroyed [2].

Bifurcation theory has important applications in many fields, including physics, chemistry, biology, engineering, and economics. In physics, it is used to study fluid flows, laser
dynamics, and electronic circuits. In chemistry, it is used to study chemical reactions, oscillations, and pattern formation. In biology, it is used to study the attitude of populations, neural networks, and genetic regulatory networks. In engineering, it studies the mechanical systems and control systems .In economics, it surveys the transactions in the market systems and other complex systems.

Bifurcation theory has also been used to research the behavior of complex systems, such as chaos theory and fractals, and it has been used to analyze and to predict the behavior of nonlinear systems in many fields [8].

The theory is usually applied to systems in a continuous mode, like differential equations, but also can be applied to discrete systems like maps [5].

The study of bifurcations is an active area of research, with ongoing work in both theoretical and applied areas. New techniques and computational tools are constantly being developed to aid in the analysis of bifurcations, including numerical methods, and computer simulations [25].

2.History of Bifurcation Theory:

The history of bifurcation theory goes back to the 18th and 19th centuries, with the work of mathematicians such as Jean-Baptiste Joseph Fourier, Henri Poincaré, and Andrey Markov.

Fourier, for example, in his work on the theory of heat conduction, studied the stability of periodic solutions of certain differential equations, which laid the foundations for later work in bifurcation theory.

In the early 20th century, the German mathematician David Hilbert and the French mathematician Émile Picard, studied the existence and uniqueness of solutions of certain differential equations, which also contributed to the foundations of bifurcation theory.

In the 1920s, the French mathematician Gaston Julia, while working on complex dynamics, introduced the concept of the bifurcation diagram, which is a graphical representation of the different solutions of a non-linear system as a parameter. which is varied. This was an important step in the development of bifurcation theory, as it provided a visual tool for understanding the behavior of non-linear systems.

Bifurcation theory is a field of mathematics which studies the changes in the qualitative or topological properties of a given family of dynamical systems as a parameter varies. The history of bifurcation theory can go back to the 18th and 19th centuries, with the work of mathematicians such as Jean-Baptiste Joseph Fourier, Henri Poincaré, and Andrey Markov. However, it wasn't until the 20th century that bifurcation theory began to be developed as a distinct field of study[6].

One of the key figures in the early development of bifurcation theory was Henri Poincaré, who studied the qualitative behavior of the dynamical systems, and the three-body problems. Another important figure was the Russian mathematician Andrey Markov, who studied the bifurcations of periodic orbits in dynamical systems.

In the 1950s and 1960s, the American mathematician Stephen Smale and the French mathematician René Thom independently developed a modern and abstract formulation of bifurcation theory. Smale's work focused on the study of dynamical systems using the theory of differentiable dynamical systems, whereas Thom's work used the theory of catastrophe theory.

In the 1970s and 1980s, bifurcation theory was further developed by mathematicians such as Jürgen Moser, Rufus Bowen and David Ruelle, who introduced new techniques and tools, such as the Melnikov method, the center manifold theorem, and the theory of strange attractors.

Today, bifurcation theory is an active field of research, with applications in diverse areas such as physics, engineering, biology, and economics.

One of the most famous articles in the field of bifurcation theory is "Period Three Implies Chaos," by American mathematician Mitchell Feigenbaum. The article was published in the journal "Physical Review Letters" in 1978.

In the article, Feigenbaum discovered a universal constant, now known as the Feigenbaum constant, that governs the behavior of certain types of non-linear systems.

He demonstrated that for certain types of non-linear systems, the ratio of consecutive bifurcation intervals converges to a universal value, now known as the Feigenbaum constant, regardless of the specific system.

The article was groundbreaking in the field of bifurcation theory and non-linear dynamics, as it provided a new way to understand and classify the behavior of non-linear systems, and it opened up new avenues for research in the field. It is widely considered to be a classic and a significant impact on the field of non-linear dynamics and chaos theory.

Another famous article in the field of bifurcation theory is Bifurcations and Catastrophes: Geometry of Solutions to Nonlinear Problems" by French mathematician René Thom. The article was published in the book "Structural Stability and Morphogenesis" in 1972..

In the article, Thom introduced the concept of catastrophe theory, which is a mathematical framework for understanding how smooth, non-linear systems can abruptly change their behavior as a parameter is varied. He used the theory of singularities from algebraic geometry and topology to describe the different types of bifurcations that can happen in dynamical systems and to classify them.

The article was highly influential in the field of bifurcation theory, as it provided a new way to understand and classify the behavior of non-linear systems, and it opened up new avenues for research in the field. It was one of the first articles to provide a geometric interpretation of bifurcations and it was a major step towards a more geometric and topological approach to non-linear dynamics. The catastrophe theory introduced by Thom was applied in many fields, like physics, biology, economics, and social sciences

3. Bifurcations and chaos phenomena :

Bifurcations and chaos are closely related concepts in nonlinear dynamics. A bifurcation happens when a small change in a parameter of a system leads to a qualitative transition in its behavior, such as a transition from a stable fixed point to a stable limit cycle.

Chaos is a type of dynamic behavior characterized by sensitive dependence on initial conditions and a lack of predictability.

One of the key ways that bifurcations and chaos are related is through the concept of strange attractors. A strange attractor is a geometric goal that represents the long-term conduct of a chaotic system, and it can be used to visualize the dynamics of the system. In many cases, the onset of chaos in a system is preceded by a series of bifurcations that lead to the formation of a strange attractor.

For example, period-doubling bifurcation is a common route to chaos in nonlinear systems. As the parameter of the system is increased, the system goes through a series of period-doubling bifurcations, which leads to the creation of a strange attractor that represents the chaotic behavior of the system.

In summary, bifurcations and chaos are closely related concepts in nonlinear dynamics. Bifurcations can be used to explain the qualitative changes in the behavior of a system, while chaos is a type of dynamic behavior characterized by sensitive dependence on initial conditions and a lack of predictability. The onset of chaos in many systems is preceded by a series of bifurcations, and the chaotic behavior can be represented by a strange attractor.

Bifurcation refers to a change in the qualitative behavior of a system as a control parameter is different In a bifurcation, a small change in a parameter can cause a drastic change in the long-term behavior of the system.

Chaos, on the other side, refers to the behavior of a system that appears random or unpredictable, but is actually deterministic. Chaos can occur in systems that are very sensitive to initial conditions, a property known as the butterfly effect.

Bifurcations can lead to chaos in a system. For example, as a control parameter is changed and a bifurcation occurs, the system may transition from a stable behavior to an unstable behavior. This instability can lead to chaos in the system's long-term behavior. Additionally, bifurcations can lead to the emergence of strange attractors, which are a characteristic feature of chaotic systems.

In summary, bifurcation is a change in the qualitative behavior of a system as a parameter is varied while chaos is the behavior of a system that appears random or unpredictable but is actually deterministic. Bifurcations can lead to chaos in a system as a small change in a parameter can cause a drastic change in the long-term behavior of the system, and it may transition from a stable to an unstable behavior, leading to chaos in behavior of the system.

Another way that chaos can arise is through the accumulation of period-doubling bifurcations. As a bifurcation parameter is increased, the period of a periodic orbit can double, leading to the creation of a new periodic orbit. If this process continues, it can lead to the creation of a fractal set of periodic orbits, known as a strange attractor.[4,6,19,25,27].

4. Some examples:

4**.1 The logistic map**: The logistic map is a simple mathematical sample of population growth. It is defined by the equation:

$$x_{n+1} = rx_n(1 - x_n),$$

Where x_n is the population at time n, and r is the growth rate. As the growth rate r is increased, the model exhibits a bifurcation, and eventually chaos. For example, as r is increased from 0 to 4, the population oscillates between two values. But as r is increased further, the population oscillates between 4 different values, then 8, and so on. Eventually, the population oscillates chaotically [6].

4.2 The Lorenz system: The Lorenz system is defined by the following three non-linear differential equations:

$$dx/dt = \delta * (y - x)$$
$$dy/dt = x * (\rho - z) - y$$
$$dz/dt = xy - \beta$$

where x, y, and z are the state variables, and δ, ρ and β are the parameters. The Lorenz system exhibits a bifurcation known as a "**Butterfly bifurcation**" in which a small perturbation in the parameter rho can results in chaotic weather patterns [6].

4.3 The Rössler system: The Rössler system is defined by the following three non-linear differential equations:

$$dx/dt = -y - z$$
$$dy/dt = x + ay$$
$$dz/dt = b + z(x - c)$$

where x, y, and z are the state variables, and a, b and c are the control parameters. The Rössler system is known for its "strange attractor", which is a geometric structure that characterizes the long-term conduct of the system. The strange attractor is formed as a result of a bifurcation, and it is a visual representation of the chaotic dynamics of the system[6].

4.4 The Belousov-Zhabotinsky reaction: The Belousov-Zhabotinsky reaction is a chemical reaction that exhibits oscillations, which can become chaotic under certain conditions. The reaction is defined by the following equation:

$$dx/dt = k_1Ay - k_2xy + k_3Ax - 2k_4x^2$$
$$dy/dt = -k_1Ay - k_2xy + \left(\frac{1}{2}\right) f\ k_5Bz$$
$$dz/dt = 2k_3Ax - k_5Bz$$

where k_i,i=1,2,...,5 are the rate constant, BZ, X and Y are the reactants. By varying the concentration of the reactants, a bifurcation can occur, and the system can transition from a stable oscillation to a chaotic one.

In summary, these examples demonstrate how bifurcations can lead to chaos through small changes in a parameter and how it can transition from a stable behavior to an unstable behavior. The logistic map, Lorenz system, Rössler system and Belousov-Zhabotinsky reaction are mathematical models and chemical reactions that provide concrete examples of how bifurcations can lead to chaos.

4.5 An example for transition to chaos through some bifurcations:

One classic example of a transition to chaos through bifurcations is the logistic map, which is a simple nonlinear difference equation that describes the dynamics of a population. The logistic map is given (see 4.1).

As the control parameter r is increased, the logistic map passes a series of bifurcations that lead to the creation of chaotic behavior. At low values of r, the logistic map has a stable fixed point, which represents an equilibrium population. As r is increased, the fixed point loses stability and a periodic orbit is created through a period-doubling bifurcation.

As r is increased further, the period of the orbit doubles again, leading to the creation of a new periodic orbit. This process continues, leading to the accumulation of period-doubling bifurcations and the creation of a fractal set of periodic orbits, known as a strange attractor.

At a critical value of r, the logistic map exhibits chaotic behavior, characterized by sensitive dependence on initial conditions, aperiodic and unpredictable behavior, and the presence of a strange attractor.

The logistic map is a simple example but it illustrates how a simple nonlinear system can exhibit complex and seemingly random behavior through a series of bifurcations that lead to the creation of chaos. This type of transition to chaos is a common feature in many nonlinear systems and it's widely studied in the field of nonlinear dynamics and chaos theory.

5. Bifurcation and Sensitivity:

The bifurcation leads to sensitive dependence on initial conditions. This means that small changes in the initial conditions of a system can lead to vastly different outcomes as the system evolves. This behavior is often observed in nonlinear systems, such as those that exhibit bifurcations.

Examples:

5.1 An example of sensitive dependence on initial conditions is the logistic map, which is a simple mathematical model of population growth. As we have shown the logistic map exhibits a bifurcation as the parameter controlling population growth is increased. At a certain critical value of this parameter, the system transitions from stable behavior to chaotic behavior.

For example, if the initial population is slightly greater than 0.5 and the growth rate parameter is set at 3.8, the population will eventually stabilize at a steady state value. However, if the initial population is slightly less than 0.5 and the growth rate parameter is still set at 3.8, the population will oscillate chaotically between different values.

This shows how little changes in initial conditions may be results great different outcomes in a nonlinear system.

5.2 Example from complex dynamics:

An example of sensitive dependence on initial conditions from complex dynamics is the Lorenz system. The Lorenz system is a system of nonlinear differential equations that models the conduct of a simple weather system. It exhibits a range of behavior, including periodic, quasiperiodic, and chaotic solutions.

For example, if the initial conditions for the Lorenz system are (x,y,z) = (1,1,1) and the system's parameters are set to the default values, the system will exhibit chaotic behavior, with the solutions to the equations never repeating, and sensitive to initial conditions. However, if the initial conditions are changed slightly to (1.0001,1,1) the solutions will evolve into a stable limit cycle, showing how to small changes in initial conditions can result from huge different outcomes in the system.

5.3 The bifurcation and SDIC in the dynamic of an exponential family:[26,5]

The bifurcation and sensitive dependence on initial conditions (SDIC) can also occur in the dynamics of exponential family distributions.

For example, consider a dynamic process that generates data from an exponential family distribution, such as a mixture model.

The mixture model is a probabilistic model for representing the presence of subpopulations within an overall population. The parameters of the mixture model are the mixture coefficients and the parameters of the component distributions.

As the parameters of the mixture model are varied, the model can undergo bifurcations that cause the component distributions to merge or split. This can lead to sensitive dependence on initial conditions, as small changes in the parameters of the mixture model can lead to vastly different outcomes in the generated data.

Another example is the dynamic of the exponential family with a time dependent natural parameter, the solution of the ODE representing the dynamic can have a bifurcation point, where the solutions change from stable to chaotic.

In conclusion, exponential family distributions can also exhibit bifurcations and sensitive dependence on initial conditions in their dynamic, as the distributions of the parameters are varied over time.

6. Types of bifurcations : [2, 6, 13]

6.1 A saddle-node bifurcation :-A type of bifurcation appears in dynamical systems, where two fixed points of the system collide and disappear. In the real plane, saddle-node bifurcations can occur in systems dissolve by one-dimensional represented maps or differential equations.

6.1.1 Real Plane: A classic example of a saddle-node bifurcation in the real plane is the logistic map, which is a one-dimensional map defined by the function.

$$f(x) = rx(1-x)$$

For parameter r between 0 and 1, the map has a single fixed point at x = 0, which is a stable equilibrium. As the parameter r increases, a second fixed point appears at $x = 1 - I/_r$.

As the parameter r continues to increase, the two fixed points collide and disappear at a critical value of r = 1. This type is a saddle-node bifurcation.

6.1.2 Complex Plane: A saddle-node bifurcation can also occur in systems described by complex maps, such as polynomial maps. For example, consider the polynomial map $f(z) = z^2 - c$ where c is a complex parameter. For values of c that are inside the unit circle, the map has two fixed points at $z = \sqrt{c}$ and $z = -\sqrt{c}$.

As c moves towards the unit circle, the two fixed points collide and dissolve at the critical value c = 1.

This is a saddle-node bifurcation in the complex plane.

In this example, the two fixed points of the map are complex numbers, and they move in the complex plane as the parameter c changes. The two fixed points start out as stable equilibria, but as c moves towards the unit circle, the stability of the fixed points changes and they both become unstable. At the critical value c = 1, the two fixed points collide and disappear.

In summary, a saddle-node bifurcation is a kind of bifurcation that is born in dynamical systems, where two fixed points of the system collide and disappear. It can occur in systems represented by one-dimensional maps or differential equations in the real plane and in systems described by complex maps, such as polynomial maps in the complex plane.

6.2 Transcritical bifurcation:[20,27]

A transcritical bifurcation is a type of bifurcation that happen in dynamical systems, where two fixed points of the system exchange their stability properties as a parameter of the system is varied. In the real plane, transcritical bifurcations can occur in systems represented by one-dimensional maps or differential equations.

6.2.1 Real Plane: A classic example of a transcritical bifurcation in the real plane is the logistic map, which is a one-dimensional map defined by the function.

$$f(x) = rx(1-x)$$

For values of the parameter r less than 1, the map has a single fixed point at x = 0, which is a stable equilibrium. As the parameter r increases, a second fixed point appears at x = r-1. As the parameter r continues to increase, the two fixed points exchange their stability properties at a critical value of r = 1. This type is a transcritical bifurcation.

6 .2.2 Complex Plane: A transcritical bifurcation can also occur in systems described by complex maps, such as polynomial maps. For example, consider the polynomial map: $f(z) = z^2 - z - c$ where c is a complex parameter. For values of c that are inside a certain region, the map has two fixed points at $z_1 = (1 + \sqrt{1+4c})/2$ and $z_2 = (1 - \sqrt{1+4c})/2$.

As c changes, the stability of the fixed points change and they exchange their stability properties at a certain critical value of c. This type is a transcritical bifurcation.

For example, for values of c that are negative, the fixed point z1 is stable but the fixed point z2 is unstable. As c increases and approaches the critical value of c = 0, the stability of the fixed points change, and z1 becomes unstable and z2 becomes stable.

6.3. A pitchfork bifurcation: is a type of bifurcation that occurs in dynamical systems, where a single fixed point splits into three as a parameter of the system is varied. In the real plane, pitchfork bifurcations can occur in system represented by one-dimensional maps or differential equations.

6.3.1 Real Plane: A classic example of a pitchfork bifurcation in the real plane is the cubic map, which is a one-dimensional map defined by the function:-

$$f(x) = rx - x^2$$

for values of the parameter r less than 0, the map has a single stable fixed point at x = 0. As the parameter r increases, a pair of stable fixed points appear symmetrically on both sides of x = 0 at $x = \sqrt{r}$ and $x = -\sqrt{r}$. These new stable fixed points are created from the original one and the bifurcation happens at the critical value r = 0. This is a pitchfork bifurcation.

6.3.2 Complex Plane: A pitchfork bifurcation can also occur in systems described by complex maps, such as polynomial maps. For example, consider the polynomial map $f(z) = z^3 + c$ where c is a complex parameter. For values of c that are inside a certain region, the map has a single fixed point at z = 0. As c increases and approaches the critical value c = 0, the fixed point splits into three symmetric fixed points at $z = \sqrt{-c}$,
$z = -\sqrt{-c}$ and z = 0. This is a pitchfork bifurcation.

6.4. Hopf bifurcation: A Hopf bifurcation is a kind of bifurcation that appear in dynamical systems, where a stable fixed point misses stability and switches on to a stable limit cycle as a parameter of the system is varied. In the real plane, Hopf bifurcations can occur in systems described by two-dimensional maps or differential equations.

6.4.1 Real Plane: A classic example of a Hopf bifurcation in the real plane is the van der Pol oscillator, which is a two-dimensional system described by the differential equations:

$$dx/dt = x$$
$$dy/dt = \mu(1-x^2)y - x$$

Where mu is a parameter. For values of mu less than 1, the system has a stable fixed point at the origin (x=0, y=0). As mu increases, a stable limit cycle appears in a neighborhood of the origin. This is a Hopf bifurcation.

6.4.2 Complex Plane: A Hopf bifurcation can also occur in systems described by complex maps, such as polynomial maps. For example, consider the polynomial map: $f(z) = z^2 + c$ where c is a complex parameter. For values of c that are inside a certain region, the map has a stable fixed point at z = 0. As c increases and approaches the critical value c = -0.25, the fixed point loses stability and gives rise to a stable limit cycle in a neighborhood of z = 0. This is a Hopf bifurcation.

Therefore, this is a type of bifurcation that appears in dynamical systems, where a stable fixed point misses out on stability and leads to a stable limit cycle as a parameter of the system is varied. It can occur in systems described by two-dimensional maps or differential equations in real planes and in systems described by complex maps, such as polynomial maps in complex planes. The model of the van der Pol oscillator and polynomial map illustrates how the stable fixed point can lose stability and give rise to a stable limit cycle when the parameter of the system changes

6.5 Supercritical and subcritical:[20,25]

6.5.1 Supercritical Pitchfork Bifurcation: In as supercritical pitchfork bifurcation, the stability of the new fixed points created by the bifurcation is different from that of the original fixed point. For example, in the cubic map example provided earlier, the original fixed point at x = 0 is stable, whereas the new fixed points at $x = \sqrt{r}$ and $x = -\sqrt{r}$ are unstable for r > 0.

6.5.2 Subcritical Pitchfork Bifurcation: In a subcritical pitchfork bifurcation, the stability of the new fixed points created by the bifurcation is the same as that of the original fixed point. For example, look at the following polynomial map: $f(x) = rx - x^3 \quad for\, r < 0$
in this case the original fixed point at x = 0 is unstable, and the new fixed points at $x = \sqrt{-r}$ and $x = -\sqrt{-r}$ are also unstable for r < 0.

Hence, in a pitchfork bifurcation, a single fixed point splits into three as a parameter of the system is varied. There are two types of pitchfork bifurcations: supercritical and subcritical. In a supercritical pitchfork bifurcation, the stability of the new fixed points created by the bifurcation is different from that of the original fixed point. In a subcritical pitchfork bifurcation, the stability of the new fixed points created by the bifurcation is the same as that of the original fixed point.

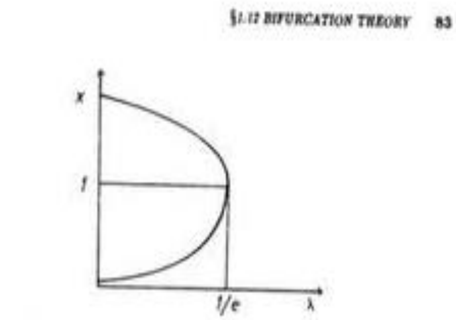
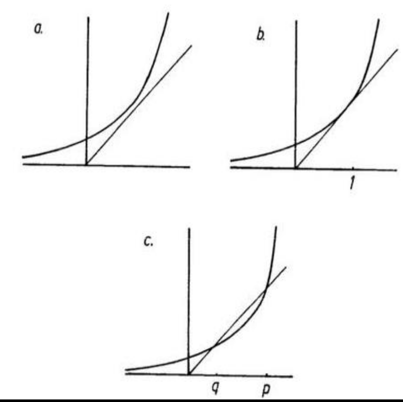


Fig. 12.3. The bifurcation diagram for $E_\lambda(x) = \lambda e^x$; x is plotted versus λ.



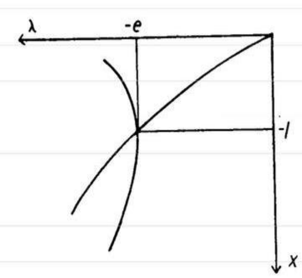


Fig. 12.6. The bifurcation diagram for $E_\lambda(x) = \lambda e^x$; x is plotted versus λ.

6.6 Period-doubling bifurcation:
A period-doubling bifurcation is a type of bifurcation that subsist in dynamical systems, where a stable fixed point loses stability and gives rise to a stable periodic orbit of twice the period as a parameter of the system is varied. In the real plane

6.6.1 Real Plane: A classic example of a period-doubling bifurcation in the real plane is the logistic map, which is a one-dimensional map defined by the function:

$$f(x) = rx(1-x)$$

For values of the parameter r less than 3.57, the map has a single stable fixed point at x = 0. As the parameter r increases, a stable periodic orbit of period 2 appears in a neighborhood of the origin. This periodic orbit becomes more stable as r increases, and at certain critical values of r, it gives rise to periodic orbits of periods 4, 8, 16, and so on. This type is a period-doubling bifurcation cascade.

6.6.2 Complex Plane: A period-doubling bifurcation can also occur in systems described by complex maps, such as polynomial maps. For example, consider the polynomial map: $f(z) = z^2 + c$ .where c is a complex parameter. For values of c that are inside a certain region, the map has a stable fixed point at z = 0. As c increases, a stable periodic orbit of period 2 appears in a neighborhood of z = 0. This periodic orbit becomes more stable as c increases, and at certain critical values of c, it gives rise to periodic orbits of periods 4, 8, 16, and so on. This is a period-doubling bifurcation cascade.

So we can say that , a period-doubling bifurcation is a type of bifurcation that happen in dynamical systems, where a stable fixed point loses stability and gives rise to a stable periodic orbit of twice the period as a parameter of the system is varied. It can occur in systems represented by one-dimensional maps or differential equations in real planes and in systems described by complex maps, such as polynomial maps in complex planes.

The example of the logistic map and polynomial map illustrates how a stable fixed point can lose stability and change to a stable periodic orbit with twice the period as the parameter of the system changes, and at certain critical values of the parameter, it gives rise to periodic orbits of period 4, 8,16, and so on, in a sequence known as a period-doubling bifurcation cascade. As the parameter value increases, the period of the periodic orbit doubles at each bifurcation.

This process of period-doubling bifurcations is a well-studied phenomenon in nonlinear dynamics and it has been observed in many physical systems such as electronic circuits, fluid dynamics, and chemical reactions. The period-doubling bifurcation cascade is believed to be a route to chaos in many systems, and it is also related to the Feigenbaum constants, which are mathematical constants that describe the scaling of the bifurcations in the cascade.

In general, a period-doubling bifurcation is a type of bifurcation that is encountered in dynamical systems, where a stable fixed point loses stability and switch to a stable periodic orbit of twice the period as a parameter of the system is varied. The examples of logistic maps and polynomial maps illustrate how this process can occur, and it is also a well-studied phenomenon in nonlinear dynamics and it has been observed in many physical systems. The period-doubling bifurcation cascade is believed to be a route to chaos in many systems.

7. A torus bifurcation[10]:-

is a type of bifurcation that happens in dynamical systems, where a stable fixed point misses out stability and moves rise to a stable torus (a two-dimensional torus is a donut-shaped surface) as a parameter of the system is varied. In the real plane, torus bifurcations can occur in systems described by three-dimensional maps or differential equations.

7.1 Real Plane: A classic example of a torus bifurcation in the real plane is the standard map, which is a two-dimensional map defined by the function:

$$f(x,y) = (x + y + ksin(x), y + \sin(x))$$

where k is a parameter. For values of k less than a critical value, the system has a stable fixed point. As k increases, a stable torus exhibit in a neighborhood of the fixed point. This torus becomes more stable as k increases, and at certain critical values of k, it gives rise to a sequence of tori of increasing dimension.

7.2 Complex Plane: Torus bifurcations can also occur in systems described by complex maps, such as polynomial maps. For example, consider the polynomial map:

$$f(z) = (z^2 + c, z^3 + c)$$

“ where c is a complex parameter. For values of c that are inside a certain region, the map has a stable fixed point. As c increases, a stable torus appears in a neighborhood of the fixed point. This torus becomes more stable as c increases, and at certain critical values of c, it gives rise to a sequence of tori of increasing dimension”.

So, a torus bifurcation is a type of bifurcation that occurs in dynamical systems, where a stable fixed point loses stability and gives rise to a stable torus as a parameter of the system is varied. It can occur in systems described by three-dimensional maps or differential equations in real planes and in systems described by complex maps, such as polynomial maps in complex planes. The example of the standard map and polynomial map illustrates how a stable fixed point can lose stability and give rise to a stable torus as the parameter of the system changes. The torus is a two-dimensional donut-shaped surface,and it can exist in higher dimensions as well.

Torus bifurcations are relatively rare, and they are considered a rare phenomenon in nonlinear dynamics. They are not as well-studied as other types of bifurcations, such as saddle-node, transcritical, pitchfork and Hopf bifurcations, but some recent studies have focused on understanding the properties and behavior of systems undergoing torus bifurcations.

So, a torus bifurcation is a type of bifurcation that happen in dynamical systems, where a stable fixed point loses stability and a stab changed letorus whenever of the system is varied. The examples of standard maps and polynomial maps illustrate how this process can occur, but it is considered a rare phenomenon in nonlinear dynamics and it's not as well-studied as other types of bifurcations.

8. A crisis bifurcation is a type of bifurcation that occurs in dynamical systems: [13,16,21]

where a stable periodic orbit loses stability and switch to a chaotic attractor as a parameter of the system is varied. In the real plane, crisis bifurcations can take place in systems described by one-dimensional maps or differential equations.

8.1 Real Plane: A classic example of a crisis bifurcation in the real plane is the logistic map, which is a one-dimensional map defined by the function:

$$f(x) = rx(1 - x)$$

For values of the parameter r less than 3.57, the map has a single stable fixed point at x = 0. As the parameter r increases, a stable periodic orbit of period 2 appears in a neighborhood of the origin. This periodic orbit becomes more stable as r increases, and at certain critical values of r, it switch on to periodic orbits of period 4, 8, 16, and so on. This is a period-doubling bifurcation cascade. However, for values of r greater than 3.57, the system becomes chaotic. The transition from a periodic orbit to chaos is called a crisis.

8.2 Complex Plane: A crisis bifurcation can also occur in systems described by complex maps, such as polynomial maps. For example, consider the polynomial map:

$$f(z) = z^2 + c$$

where c is a complex parameter. For values of c that are inside a certain region, the map has a stable fixed point at z = 0. As c increases, a stable periodic orbit of period 2 appears in a neighborhood of z = 0. This periodic orbit becomes more stable as c increases, and at certain critical values of c, it gives rise to periodic orbits of periods 4, 8, 16, and so on. However, for values of c greater than a certain critical value, the system becomes chaotic. The transition from a periodic orbit to chaos is called a crisis.

9. SOME APPLICATIONS FOR BIFURCATION THEORY

9.1 Applications of bifurcation in physics: **[1,3,10,11,15]**
bifurcations have many applications in physics, including laser dynamics, chaos in electronic circuits, fluid dynamics, chemical reactions, nonlinear dynamics in mechanical systems, synchronization in coupled systems, Complex systems and quantum mechanics. Understanding bifurcations and their properties is important for understanding the behavior of nonlinear systems in physics.

9.1. 1 Laser dynamics: Bifurcations have an important role in the dynamics of laser systems. For example, the bifurcation which occurs from a stable fixed point to a stable limit cycle is used to explain the phenomenon of self-pulsing in lasers.

9.1.2 Chaos in electronic circuits: Bifurcations are used to explain the emergence of chaos in electronic circuits, such as the period-doubling bifurcation cascade, which is a route to chaos in the logistic map.

9.1.3 Fluid dynamics: Bifurcations are used to explain the behavior of fluid systems, such as the bifurcation which from a stable fixed point to a stable limit cycle, which is used to explain the behavior of laminar flow in pipe systems.

9.1.4 Chemical reactions: Bifurcations are used to explain the behavior of chemical reactions, This is the transition from a stable fixed point to a stable limit cycle, which is used to explain the behavior of the Belousov-Zhabotinsky reaction.

9.1.5 Nonlinear dynamics in mechanical systems:
Bifurcations can studies of nonlinear mechanical systems, such as the bifurcation which happen at a stable fixed point to a stable limit cycle, which is used to explain the behavior of nonlinear oscillators, such as the Duffing oscillator.

9.1.6 Synchronization in coupled systems: Bifurcations can be applicated to study of coupled systems, such as the bifurcation which occurs to trance a stable fixed point to a stable limit cycle, which is used to explain the phenomenon of synchronization in coupled oscillators. For example, the Kuramoto model, which describes the dynamics of a network of coupled oscillators, exhibits a supercritical Hopf bifurcation, which results in the emergence of synchronized behavior in the system.

9.1.7 Complex systems: Bifurcations can be surreyd the complex systems, such as the bifurcation from a stable fixed point to a chaotic attractor, which is used to explain the emergence of complex behavior in systems such as weather patterns, stock market fluctuations, and population dynamics.

9.1.8 Quantum mechanics: Bifurcations also appear in quantum mechanics, for example, in the study of quantum chaos. In quantum chaos, the quantum version of classical chaotic systems are studied. And for that, the study of bifurcations plays an important role.

9.2 Applications of bifurcation in Biology: **[5,24]**
Bifurcations are also important in studying of biological systems and have abundant applications in biology. Here are a few examples of how bifurcations are used in biology:

9.2.1. Ecology: Bifurcations can be used to research of ecological systems, such as the bifurcation which happens from a stable fixed point to a stable limit cycle, which is used to explain the behavior of population dynamics that exhibit oscillations in their activity, such as predator-prey interactions.

9.2.2. Genetics: Bifurcations can be used to etude the behavior of genetic systems this transition from fixed point to limit cycle, which is used to explain the behavior of gene expression that exhibit oscillations in their activity.

9.2.3. Neural systems: Bifurcations can be used to study the behavior of neural systems, such as the bifurcation which occurs from a stable fixed point to a stable limit cycle, which is used to explain the behavior of neural activity that exhibit oscillations in their activity.

9. 2.4 Immunology: Bifurcations can be used to study the behavior of immunological systems, which is used to explain the behavior of immune responses that exhibit oscillations in their activity.

9.2.5.Developmental biology: Bifurcations also to study of developmental systems, such as the bifurcation which happens from a stable fixed point to a stable limit cycle, which is used to explain the behavior of developmental pathways that exhibit oscillations in their activity.

9.2.6 Bifurcations in biochemical systems: Bifurcations can also be used to study the behavior of biochemical systems, such as the bifurcation which occurs from a stable fixed point to a stable limit cycle, which is used to explain the behavior of biochemical systems that exhibit oscillations in their concentrations.

9.2.7 Biological rhythms: Bifurcations can be used to study the behavior of biological rhythms, such as the bifurcation which occurs from a stable fixed point to a stable limit cycle, which is used to explain the behavior of circadian rhythms and other physiological processes that exhibit oscillations in their activity.

9.2.8 Systems biology: Bifurcations can be used in systems biology to study the behavior of complex biological systems, such as the bifurcation which happens from a stable fixed point to a stable limit cycle, which is used to explain the behavior of systems that exhibit oscillations in their activity

9.3 Applications of bifurcation in Chimestry: **[5,7,14,9]**
Bifurcations are also important in the study of chemical systems and have many applications in chemistry. Here are a few examples of how bifurcations are used in chemistry:

9.3. 1Chemical reactions: Bifurcations are used to explain the behavior of chemical reactions, such as the bifurcation which occurs from a stable fixed point to a stable limit cycle, which is used to explain the behavior of the Belousov-Zhabotinsky reaction. This reaction is an example of a chemical oscillator that exhibits a period-doubling bifurcation cascade, leading to the emergence of complex behavior in the system.

9.3.2 Catalytic reactions: Bifurcations can be used to study the behavior of catalytic reactions, which is used to explain the behavior of catalytic systems that exhibit oscillations in their reaction rates.

9.3.3 Enzyme kinetics: Bifurcations can be used to study the behavior of enzyme kinetics, such as the bifurcation which happens from a stable fixed point to a stable limit cycle, which is used to explain the behavior of enzymes that exhibit oscillations in their activity.

9.3.4 Chemical oscillators: Bifurcations can be used to study the behavior of chemical oscillators, such as the bifurcation which occurs from a stable fixed point to a stable limit cycle, which is used to explain the behavior of chemical systems that exhibit oscillations in their concentrations.

9.3.5 Phase transition: Bifurcations can also be used to study phase transitions in chemical systems. For example, the bifurcation which happen from a stable fixed point to a stable limit cycle is used to explain the behavior of systems that exhibit oscillations in their concentrations, such as the transition from liquid to gas.

9.3.6 Bifurcations in biochemical systems: Bifurcations can also be in studying of biochemical systems, such as the bifurcation which happens from a stable fixed point to a stable limit cycle, which is used to explain the behavior of biochemical systems that exhibit oscillations in their concentrations.

9.4 Applications of Bifurcation in Medicine:-[22,28,29]
Bifurcations are also important to the study of medical systems and have many applications in medicine. Here are a few examples of how bifurcations are used in medicine:

9.4.1 Cardiology: Bifurcations are used to explain the behavior of cardiovascular systems, such as the bifurcation which happens from a stable fixed point to a stable limit cycle, which is used to explain the behavior of heart rhythms that exhibit oscillations in their activity, such as arrhythmias.

9.4.2 Neurology: Bifurcations can be used also in the study of the behavior of neurological systems, such as the bifurcation which happens from a stable fixed point to a stable limit cycle, which is used to explain the behavior of seizures and other neurological disorders that exhibit oscillations in their activity.

9.4.3 Respiratory systems: Bifurcations can be studies the behavior of respiratory systems, such as the bifurcation which happens from a stable fixed point to a stable limit cycle, which is used to explain the behavior of breathing patterns that exhibit oscillations in their activity.

9.4.4 Endocrine systems: Bifurcations can study endocrine systems, such as the bifurcation which occurs from a stable fixed point to a stable limit cycle, which is used to explain the behavior of hormone levels that exhibit oscillations in their activity.

9.4.5 Cancer: Bifurcations can also be used to study the behavior of cancer systems, such as bifurcates the from a stable fixed point to a stable limit cycle which is used to explain the behavior of cancer cells that exhibit oscillations in their activity.

9.4.6 Medical imaging: such as the bifurcation which occurs from a stable fixed point to a stable limit cycle which is used to explain the behavior of medical imaging systems that exhibit oscillations in their activity.

9.5 Application of Bifurcation in Geography [7,12]:
Bifurcations have been used in the study of geography in several ways, including the study of physical systems and human systems. Here are a few examples of how bifurcations have been used in geography:

9.5.1 Hydrology: Bifurcations can be used to study the behavior of hydrological systems, such as the bifurcation which occurs from a stable fixed point to a stable limit cycle, which is used to explain the behavior of river systems and drainage basins that exhibit oscillations in their activity.

9.5.2 Climate: Bifurcations can be used to study the behavior of climate systems, such as the bifurcation that happen from a stable fixed point to a stable limit cycle, which is used to explain the behavior of climate systems that exhibit oscillations in their activity.

9.5.3 Land use change: Bifurcations can be used to study the behavior of land use change, such as the bifurcation which happens from a stable fixed point to a stable limit cycle, the behavior of medical imaging systems can be studied also using the bifurcation , which is used to explain the behavior of land use change that exhibit oscillations in their activity.

9.5.4 Urban systems: urban systems can be studied by Bifurcation, such as the bifurcation from a stable fixed point to a stable limit cycle, which is used to explain the behavior of urban systems that exhibit oscillations in their activity.

9.5.5 Geographical information systems: "Bifurcations can be used to study the behavior of geographical information systems, such as the bifurcation which occurs from a stable fixed point to a stable limit cycle, which is used to explain the behavior of geographical information systems that exhibits oscillations in their activity.
Perhaps the strangest application of bifurcation is that which occurs in linguistics."

However Bifurcations have been applied in several fields, but their application in linguistics is not as common as in other fields such as physics, chemistry, biology, engineering, and medicine. However, there are some studies that suggest the use of bifurcations in linguistic systems.

1. Phonology: Phonological systems can be studied by Bifurcation, such as the bifurcation which happens on a stable fixed point to a stable limit cycle, which is used to explain the behavior of phonological systems that exhibit oscillations in their activity.

2. Syntax: Bifurcations can be used to study the behavior of syntactic systems, such as the bifurcation from a stable fixed point to a stable limit cycle, which is used to explain the behavior of syntactic systems that exhibits oscillations in their activity.

3. Morphology: Bifurcations can be used to study the behavior of morphological systems, such as the bifurcation which occure transes a stable fixed point to a stable limit cycle, which is used to explain the behavior of morphological systems that exhibit oscillations in their activity.

4.Language change: Bifurcations can be used to study the behavior of language change, such as the bifurcation which happens from a stable fixed point to a stable limit cycle, which is used to explain behavior of language change that exhibit oscillations in their activity.

5. Language Acquisition: Bifurcations can be used to study the behavior of language acquisition, such as the bifurcation which occur trances a stable fixed point to a stable limit cycle, which is used to explain the behavior of language acquisition that exhibit oscillations in their activity.
bifurcations have some applications in linguistics but their application is not as common as in other fields. They can be used to study the behavior of phonological systems, syntactic systems, morphological systems, language change and language acquisition, but more research is needed to establish their application in linguistics.

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الملخص

نظرية التشعب هي فرع من فروع الرياضيات التي تدرس التغيرات النوعية في سلوك نظام ديناميكي تبعاً للتغير الحاصل على معلمات ذلك النظام **.** في هذا العمل ، نقدم مراجعة تاريخية للموضوع بالاضافة الى استعراض الانواع الرئيسية للتشعبات وترابطات هذا المفهوم مع ظاهرة الفوضى وبعض المفاهيم الأخرى مثل الحساسية للقيم الأولية. كما نقوم بتقديم بعض تطبيقات هذه النظرية الهامة في مجالات أخرى مثل الفيزياء والكيمياء والأحياء والجغرافيا وغيرها.

الكلمات المفتاحية : التشعب ،تشعب هويف،تشعب العقدة السرجية ،تشعب مضاعف الرتبة ،الفوضى ،الحساسية للقيم الابتدائية .