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The Equivalence Between Ultrasemiprime and Ultraprime Under Certain Conditions

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Abstract

Abstract—Mathieu proved that any prime algebra which is also C^*- algebra is an ultraprime. Brešar shows that C^*- algebra is an ultrasemiprime. This paper is give condtions to ultrasempirime algebras to get ultraprime. Mathieu defined ultraprime algebras by using four equivalent conditions. In his definition of ultraprime algebra, he used the ultrapower of algebra. In contrast, in other identical conditions, he once used sequences; in another occasion, he used a particular linear operator. Our proof adds a new condition by using the prime algebra to the ultrasemiprime algebra to get an equivalent condition to ultraprime algebras.

Keywords:
ultraprime algebra, ultrasemiprime algebra, ultrapower algebra.

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I. INTRODUCTION

In this paper, every algebra is an associative algebra. Ultraprime algebra was defined by Mathieu [1] when he discovered the norm for the algebra of quotients. For normed algebra, A is called ultraprime if it exists $c > 0$ such that $c\|a\|\|b\| \leq \|M_{a,b}\|$ for all $a, b \in A$, where $M_{a,b}: A \longrightarrow A$ is linear operator defined by $M_{a,b}(x) = axb$ for $x \in A$. If $b = a$ for all $a \in A$, then the definition of ultraprime algebra A has been generalized to the ultrasemiprime algebra, which is also mentioned in [1]. For more studies about ultraprime algebra, see [2] [3] [4]. Any ultraprime algebra is also an ultrasemiprime algebra, but The opposite is invalid.

Mohammed, in [5], proved that $\mathbb{R}^n$ and $\mathbb{C}^n$, $n \in \mathbb{N}$ are ultrasemiprime, but $\mathbb{R}^n$ and $\mathbb{C}^n$ are not ultraprime algebras which Al-Neima in [6], later generalized this to the direct product of any two ultrasemiprime algebras. In [7], we note in Lemma 3.1 that there are four equivalent conditions, one of which is used to define the ultraprime algebra. In this paper, we will add another requirement to obtain the ultraprime algebra using ultrasemiprime algebra.

II. ULTRASEMIPRIME ALGEBRAS

Mathieu provides an analytical adjective for quotient algebra in [1] when he studied ultraprime algebra. Also, an ultrasemiprime algebra is defined in [1].

The normed algebra A is called an ultrasemiprime when exists $c > 0$ such that $c\|a\|^2 \leq \|M_{a,a}\|$ for all $a \in A$. For more studies about the ultrasemiprime, see [5] [6] [8]. Every associative ultraprime algebra is ultrasemiprime, but the converse is invalid since every associative finite dimensional algebra is ultrasemiprime, as Mohammed proved in [5]. The following Theorem is needed in our main result, which has been proved in [6].

Theorem 2.1

For the normed algebra A. The requirements listed below are equivalent.

1. Any sequence (a_m) in A with $\|a_m\| = 1$, for all $m \in \mathbb{N}$, there exists bounded sequence (b_m), in A, such that the sequence $(a_m b_m a_m)$, does not tend to zero.
2. There is $c > 0$, satisfies $c\|a\|^2 \leq \|M_{a,a}\|$ for all a in A.

The four equivalent conditions for ultraprime algebra were proved by Mathieu in [7].

Lemma 2.2

Let A be normed algebra and u the free ultrafilter on $\mathbb{N}$. Then, the requirements listed below are equivalent.

1. For any pair $(x_k), (z_k)$, $k \in \mathbb{N}$ of sequence in A such that $\|x_k\| = \|z_k\| = 1$, for all $k \in \mathbb{N}$, there is bounded sequence, (y_k), $k \in \mathbb{N}$ such that $(x_k y_k z_k)$, $k \in \mathbb{N}$ does not tend to zero.
2. There is $c > 0$ such that $c\|a\|\|b\| \leq \|M_{a,b}\|$ for all $a, b \in A$.
3. For any ultrapower $\hat{A}_u$ of A is prime
4. There exists an ultrapower $\hat{A}_u$ of A, which is prime.

Mathieu used the fourth condition of Lemma 2.2 to define ultraprime algebras. Still, later, the second condition became the most famous condition used in defining associative ultraprime algebras, while the fourth condition remained used when the ultraprime algebras were non-associative see [9].

Mathieu proved that the C^*- algebra is prime if and only if it is ultraprime algebra see [1; proposition 3.1(a)]. In [8], Brešar proved that C^*- algebra is ultrasemiprime algebra. The proving of Brešar and Mathieu mentioned above allows us to generalize proposition 3.1(a) in [1], using ultrasemiprime algebra instead of C^*- algebra. Our new generalize the new equivalent condition to get an ultraprime algebra where the other conditions are mentioned in Lemma 2.2.

The condition *, which we use it in the Theorem 2.3 for every sequence $(x_k y_k x_k)$, $k \in \mathbb{N}$ does not tend to zero and is convergent. For any pair $(x_k), (y_k)$, $k \in \mathbb{N}$ of sequence in A such that $\|x_k\| = 1$, for all $k \in \mathbb{N}$, and the bounded sequence, (y_k),

Theorem 2.3

Let A be normed algebra with condition *. Then A is ultraprime algebra if and only if A is ultrasemiprime and prime algebra.

Proof:

Let A be an ultraprime normed algebra; it is clear that A is ultrasemiprime and prime algebra.

Conversely, let A be an ultrasemiprime and prime algebra, and $(a_k), (b_k)$, be sequences in A such that $\|a_k\| = \|b_k\| = 1$ for all $k \in \mathbb{N}$.

Since (a_k) is a sequence in A, $\|a_k\| = 1$, and A is ultrasemiprime by using Theorem 2.1, there is a sequence (x_k), $k \in \mathbb{N}$ in A which is bounded with bound c_x such that $(a_k x_k a_k)$ does not converge to zero. Let it converge to $x \neq 0$ by condition *.

Since (b_k) is a sequence in A, $\|b_k\| = 1$, and A is ultrasemiprime by using Theorem 2.1, there is a sequence (y_k), $k \in \mathbb{N}$ in A which is a bounded with bound c_y such that $(b_k y_k b_k)$ does not converge to zero. Similar to sequence $(a_k x_k a_k)$. Let it converge to $y \neq 0$.

Assume that A is not ultraprime algebra. From Lemma 2.2(1), get that for every bounded sequence (z_k), in A; the sequence $(a_k z_k b_k)$ tends to zero. We define (z_k) by $z_k = x_k a_k z b_k y_k$, where $z \in A$, (z_k) is a bounded sequence, because

$$\begin{aligned}\|z_k\| &= \|x_k a_k z b_k y_k\| \\ &\leq \|x_k\|\|a_k\|\|z\|\|b_k\|\|y_k\| \\ &\leq c_x c_y \|z\|\end{aligned}$$

Now $(a_k z_k b_k)$ is converge sequence to zero for any $z \in A$, so that for any $\epsilon > 0$, there exist $k_0 \in \mathbb{N}$ such that $\|a_k z_k b_k - 0\| < \epsilon$ for $k > k_0$

$$\begin{aligned}\|M_{a_k x_k a_k, b_k y_k b_k} - O\| &= \|M_{a_k x_k a_k, b_k y_k b_k}\| \\ &= \sup_{z \in A}\{\|a_k x_k a_k z b_k y_k b_k\|, \|z\| = 1\} \\ &= \sup_{z \in A}\{\|a_k z_k b_k\|, \|z\| = 1\} \\ &< \sup_{z \in A}\{\epsilon, \|z\| = 1\} = \epsilon \\ &\text{for all } k > k_0\end{aligned}$$

We get that $\|M_{a_k x_k a_k, b_k y_k b_k}\| \to 0$, when $k \to \infty$ we have $\|M_{a_k x_k a_k, b_k y_k b_k}\| \to \|M_{x,y}\|$, so $\|M_{x,y}\| = 0$ from the uniqueness of the convergent point. So $M_{x,y} = O$, that means for any $w \in A$, $0 = M_{x,y}(w) = xwy$, so $xAy = 0$. Since A is prime, then either $x = 0$ or $y = 0$. Therefore, either $(a_k x_k a_k)$ converges to zero or $(b_k y_k b_k)$ tends to zero; it is a contradiction, then must $(a_k z_k b_k) \nrightarrow 0$, by using Lemma 2.1(1), A is ultraprime normed algebra.■

The unitization of a normed algebra defined in [10] for algebra A over $\mathbb{F}$, represented by $A + \mathbb{F}$, the normed algebra containing the set $A \times \mathbb{F}$ with standard addition, scalar multiplication defined in $A \times \mathbb{F}$. The product in $A \times \mathbb{F}$ defined as $(x, \alpha) \cdot (y, \beta) = (xy + \beta x + \alpha y, \alpha\beta)$ and have the norm defined by $\|(x, \alpha)\| = \|x\| + |\alpha|$. So $A \times \mathbb{F}$ is normed algebra with identity element (0,1) where $x, y \in A, \alpha, \beta, 1 \in \mathbb{F}$.

Mathieu proved in [7] that if the ultraprime algebra A is without identity, then the unitization of A is ultraprime algebra. We demonstrate in the following proposition the same result for ultrasemiprime algebra.

Proposition 3.1

Let A be any ultrasemiprime algebra without identity with condition *. Then the unitization of A is ultrasemiprime.

Proof:

Let A be ultrasemiprime normed algebras, then $A \times \mathbb{F}$ is normed algebra with identity. take $(x_n, \alpha_n), n \in \mathbb{N}$ is a sequence in normed algebra $A \times \mathbb{F}$ with $\|(x_n, \alpha_n)\| = 1$ for all $n \in \mathbb{N}$, $\|(x_n, \alpha_n)\| = \|x_n\| + |\alpha_n| = 1$, so $\|x_n\| = 1 - |\alpha_n|$.

If the sequence (α_n) is convergent to zero, then we have $\lim_{n\to\infty} \|(x_n, \alpha_n)\| = \lim_{n\to\infty} \|x_n\|$

Now, the sequence $(z_n) = \frac{1}{1-|\alpha_n|}(x_n)$ is a sequence in A with $\|z_n\| = 1$. By using Theorem 2.1 and A is ultrasemiprime algebra, there is a bounded sequence (y_n), $n \in \mathbb{N}$ in A with $\|y_n\| \leq k$, a result of which the sequence $(z_n y_n z_n), n \in \mathbb{N}$ does not tend to zero.

There is a sequence $(y_n, 0)$ in $A \times \mathbb{F}$, $(y_n, 0)$ is bounded because

$\|(y_n, 0)\| = \|y_n\| \leq k$. We get the following

$$
\begin{aligned}
&(x_n, \alpha_n)(y_n, 0)(x_n, \alpha_n) = \\
&= (x_n y_n + 0x_n + \alpha_n y_n, 0)(x_n, \alpha_n) \\
&= (x_n y_n + \alpha_n y_n)x_n + \alpha_n(x_n y_n + \alpha_n y_n) + 0x_n, 0) \\
&= (x_n y_n x_n + \alpha_n y_n x_n + \alpha_n x_n y_n + {\alpha_n}^2 y_n, 0) \\
&= (x_n y_n x_n, 0) + (\alpha_n y_n x_n, 0) + (\alpha_n x_n y_n, 0) \\
&\quad + ({\alpha_n}^2 y_n, 0)
\end{aligned}
$$

The sequence $(\alpha_n y_n x_n)$ is convergent to zero as $n \to \infty$; there exist $k_0 \in \mathbb{N}$ such

$$|\alpha_n| < \frac{\epsilon}{k}, \qquad \text{for all } n > k_0$$

$$
\begin{aligned}
\|\alpha_n y_n x_n - 0\| &= \|\alpha_n y_n x_n\| \\
&\leq |\alpha_n| \|y_n\| \|x_n\|
\end{aligned}
$$

Since $\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} 1 - |\alpha_n| = 1$.

$$
\begin{aligned}
&< |\alpha_n| k \\
&< \frac{\epsilon}{k}\, k = \epsilon
\end{aligned}
$$

Make the sequence $(\alpha_n y_n x_n, 0)$ is convergent to zero. Similarity, we have the sequences $(\alpha_n x_n y_n, 0), ({\alpha_n}^2 y_n, 0)$ are convergent to zero,

$$\lim_{n\to\infty} (x_n, \alpha_n)(y_n, 0)(x_n, \alpha_n) = \lim_{n\to\infty} (x_n y_n x_n, 0)$$

The sequence $(x_n y_n x_n)$ does not converge to zero because $(z_n y_n z_n), n \in \mathbb{N}$, does not tend to zero.

Suppose the sequence (α_n) does not converge to zero. Since $\|x_n\| + |\alpha_n| = 1$, we get that $|\alpha_n| \leq 1$; let it converge to α. And let (x_n) convergent to $x \neq 0$. Now, the sequence $(z_n) = \frac{1}{1-|\alpha_n|}(x_n)$ is a sequence in A with $\|z_n\| = 1$. Since A is ultrasemiprime algebra by using Theorem 2.1, there is a sequence (y_n), $n \in \mathbb{N}$ in A which is bounded with $\|y_n\| < k$, a result of which the sequence $(z_n y_n z_n), n \in \mathbb{N}$ does not tend to zero and it converges to c.

There is a sequence. $(z_n y_n, 0)$ in $A \times \mathbb{F}$, $(y_n, 0)$ is bounded because

$\|(z_n y_n, 0)\| = \|z_n y_n\| = \|z_n\| \|y_n\| < k$. We get the following

$$
\begin{aligned}
&(x_n, \alpha_n)(z_n y_n z_n, 0)(x_n, \alpha_n) \\
&= (x_n z_n y_n z_n + 0x_n + \alpha_n z_n y_n z_n, 0)(x_n, \alpha_n) \\
&= (x_n z_n y_n z_n + \alpha_n z_n y_n z_n)x_n \\
&\qquad + \alpha_n(x_n z_n y_n z_n + \alpha_n z_n y_n z_n) + 0x_n, 0) \\
&= (x_n z_n y_n z_n x_n + \alpha_n z_n y_n z_n x_n + \alpha_n x_n z_n y_n z_n \\
&\qquad + {\alpha_n}^2 z_n y_n z_n, 0) \\
&= ((x_n z_n y_n z_n x_n, 0) + (\alpha_n z_n y_n z_n x_n, 0) \\
&\qquad + (\alpha_n x_n z_n y_n z_n, 0) + ({\alpha_n}^2 z_n y_n z_n, 0)
\end{aligned}
$$

The above sequence has parts. The first sequence part is the sequence $(x_n z_n y_n z_n x_n, 0)$ where we see the sequence $(x_n z_n y_n z_n x_n)$ is not tended, to zero because it is the multiplication of three sequences, so it is convergent to xcx. Similarly for $(\alpha_n z_n y_n z_n x_n, 0), (\alpha_n x_n z_n y_n z_n, 0), ({\alpha_n}^2 z_n y_n z_n, 0)$

We get that the sequence $(x_n, \alpha_n)(z_n y_n, 0)(x_n, \alpha_n)$ is does not converge to zero. Therefore, A is ultrasemiprime algebra.■

Conclusion

Ultrasemiprime algebra is not necessarily ultraprime algebra. When the ultrasemiprime algebra is prime, it will be ultraprime. The unitization of A is ultrasemiprime and it is also ultrasemiprime.

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التكافؤ بين الجبور شبه الأولية المميزة والجبور الأولية المميزة تحت شروط خاصة

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الملخص

أثبت ماثيو ان الجبر الأولي والذي يكون ايضاً جبراً من النمط C^* يكون جبرا اولياً مميزا. وقد بين برايسر ان الجبر من النمط C^* يكون جبرا شبه اوليا مميزا. في هذا البحث سيتم إعطاء شرط للجبور الشبه الأولية المميزة للحصول على الجبور الأولية المميزة. عرف ماثيو الجبر الاولي المميز عن طريق استخدام أربعة شروط متكافئة. حيث استخدم جبور القوى المميزة في تعريفه. بينما في الشروط المكافئة الأخرى استخدم مرة المتتابعات في البرهان ، ومرة أخرى استخدم مؤثر خطي خاص. في هذا البحث ، قمنا بإضافة شرط الجبر الاولي الى الجبر شبه الأولي المميز ليكون مكافئ اخر للجبور الأولية المميزة.

االكلمات المفتاحية. الجبر الاولي المميز ، الجبر الشبه الاولي المميز ، جبر القوى المميز