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Free Vibration of Thin Beams on Winkler Foundations Using Generalized Integral Transform Method

Charles C. Ike

Civil Engineering Dept., Enugu State University of Science and Technology, Agbani, Enugu State, Nigeria.
*Corresponding author Email: charles.ike@esut.edu.ng

HIGHLIGHTS

- Eigenfunctions of thin beam vibrations used as kernel functions
- Generalized integral transform method converts governing equations to integral and then algebraic equations
- Exact solutions obtained for an infinite spectrum of natural frequencies of beams on Winkler foundations
- Method provides accurate solutions compared to previous approaches

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ABSTRACT

The determination of free vibration frequencies of thin beams on Winkler foundations is important in their design to avoid resonant failures when the excitation frequency equals the least natural frequency. This article presents the determination of natural transverse vibration frequencies of Euler-Bernoulli beams on Winkler foundations using the Generalized Integral Transform Method (GITM). The problem is governed by a fourth-order partial differential equation (PDE) and boundary conditions dependent on the end restraints. The governing PDE is transformed into an algebraic eigenvalue problem for harmonic vibrations and harmonic response. Analytical solutions of the governing equations are difficult and complex. Hence, other solution methods are needed. The main merit of the GITM is the apriori selection of kernel functions as orthogonal functions of vibrating thin beams with equivalent boundary conditions. This prior selection of the kernel functions and their orthogonality properties simplify the resulting integral equation formulation to an algebraic problem in the GITM space. Eigenfunctions of freely vibrating thin beams with identical restraints are used in the GITM to construct the displacement function as infinite series in terms of unknown parameters. The solution of the algebraic equation yielded exact solutions to the candidate problem. Exact solutions for frequency parameters are obtained for the four cases of boundary conditions for the various values of foundation parameters considered. The effectiveness of the GITM in obtaining a simplification of the governing PDE and reducing the problem to an algebraic problem is illustrated.

1. Introduction

The natural vibration frequencies of Euler-Bernoulli (EB) beams resting on Winkler foundations is an important dynamic characteristic of vibrating thin beams necessary to design such structures to avoid resonant failures. Resonant failures occur in dynamic structures when the excitation frequencies due to applied loads from dynamic machines coincide with the system's natural frequencies. Various theories have been presented to describe beams under flexure, vibration, and buckling. They include the Euler-Bernoulli beam theory (EBBT) [1]. Timoshenko beam theory (TBT) [2]. Reddy beam theory, Levinson beam theory [3], Krisha Murty [4] beam theory, and shear deformation beam theories proposed by Ghugal [5], sayyad and Ghugal [6], other researchers. EBBT also called the classical thin beam theory, is constructed for beams with span to-depth ratio greater than 20, based on the hypothesis that plane cross-sections that are originally orthogonal to the middle surface of the beam before deformation would remain plane and orthogonal to the middle surface after deformation. The orthogonality hypothesis implies that transverse shear deformation is neglected in formulating the governing equation. Consequently, this limits the applicability to thin beams with insignificant transverse shear deformation effects.

TBT and other shear deformation beam theories consider transverse shear deformation in constructing the governing field equations by modifying the orthogonality hypothesis in the EBBT. This renders them suitable for moderately thick and thick beams with a span-to-depth ratio of less than 20. However, this paper considers thin beams and uses EBBT. Elastic foundation models have been developed using discrete parameter models and continuum parameter models. Winkler, Pasternak, Filonenko-

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Borodich, Vlasov, and Hetenyi proposed discrete parameter foundation models. The most common discrete parameter foundation model was proposed by Winkler as a bed of independent, linear elastic springs which cannot interact with adjoining springs and whose reaction is directly proportional to the deflection, w, of the beam. In the Winkler model, one parameter – the Winkler constant, k, is used to express the soil reaction p_s at a point as [1,2].

$$p_s = kw \tag{1}$$

The Winkler model is thus a one-parameter model, widely used due to its simplicity. The main issue of the Winkler model is its inability to consider shear interaction. This has prompted the development of other models to account for shear interaction.

Vlasov, Hetenyi, Pasternak, and Filonenko-Borodich's discrete parameter models consider shear interaction by introducing the springs' coupling. This results in the two-parameter representation of the soil reaction in terms of k_1, and k_2, where k_1 is a parameter analogous to the Winkler soil constant, and k_2 is the second foundation parameter representing the coupling interaction of the springs. The soil reaction p_s for two parameter discrete foundations becomes the more complex equation given by Mama et al., [7].

$$p_s = k_1 w - k_2 \frac{d^2 w}{dx^2} \tag{2}$$

This paper, however, considers EB beams resting on Winkler foundations.

Euler-Bernoulli beam theory accurately describes thin beams where transverse shear deformation effects are disregarded. Timoshenko beam theories and other shear deformable beam theories developed variously by Reddy, Levinson, and other scholars properly model moderately thick and thick beams where transverse shear deformation effects are accounted for. Elastic foundation models for the effect of the elastic foundation include models proposed by Winkler, Vlasov, Pasternak, and Hetenyi, simplified elastic continuum models, and elastic continuum models.

This article studies the vibration of the EB beam resting on the Winkler foundation using the Generalised Integral Transform Method (GITM). The governing equation for the problem is a partial differential equation (PDE) of the fourth order. Generally, the problem has been solved in the literature using analytical and numerical methods for solving boundary value problems (BVPs).

Adair et al. [8], have used the Variational Iteration Method to solve the free transverse vibration problems of Timoshenko beams resting on Pasternak foundations. Balkaya et al. [9], and Agboola and Gbadeyan [10], have used the Differential Transform Method (DTM) to study the vibrations of elastic beams resting on an elastic foundation. The DTM was found as an amenable transformation tool based on Taylor series expansions, resulting in acceptable accuracy.

Kacar et al. [11], have studied the natural transverse vibration problems of thin beams resting on variable one-parameter foundations using the Differential Transform Method (DTM). Their formulation assumed constant linear and parabolic variations along the major axis of the Winkler foundation parameter. They considered both ends simply supported (SS) or clamped (CC) and found solutions for the cantilever beam on an elastic foundation.

Boudaa et al. [12], have used the Spectral Element Method to study the eigenfrequency problems of beams on elastic foundations. Khnaijar and Benamar [13], have studied the discrete physical model for nonlinear foundation beam problems for various elastic foundation models. Yayli et al. [14], have presented analytical solutions to the vibration problem of a beam on an elastic foundation with elastically restrained ends. Al-Azzawi and Daud [15], have studied the natural transverse vibrations of non-prismatic beams resting on nonhomogeneous elastic one-parameter foundations.

Motaghian et al. [16], have presented a new solution based on the Fourier Series theory for solving natural, free transverse flexural vibration problems of non-uniform beams on variable elastic foundations. Soltani and Asgarian [17], have presented a new hybrid method for natural transverse vibration analysis of functionally graded Euler-Bernoulli beams with non-prismatic cross-sections resting on a two-parameter elastic foundation. Coskun [18], presented a study on the response of a finite beam on a tensionless two-parameter (Pasternak) foundation subjected to a sinusoidal loading.

Chen [19], investigated the vibration of the prismatic beam on an elastic foundation using the Differential Quadrature Element Method (DQEM) and obtained accurate solutions for the eigenfrequencies for various boundary conditions. Mutman and Coskun [20] have solved the natural vibration problems of non-uniform thin beams on elastic foundations using Homotopy Perturbation Method (HPM) and obtained satisfactory results for the natural frequencies of transverse vibrations for different boundary conditions considered.

Franciosi and Masi [21], have studied the natural transverse vibrations of beams resting on two-parameter elastic foundations. Rahbar-Ranji and Shahbaztabar [22], have also presented studies of free vibration problems of beams resting on Pasternak foundations using Legendre polynomials interpolating functions and the Rayleigh-Ritz method. Zhou [23], presented a general solution to the transverse free vibrations of beams resting on variable Winkler elastic foundations.

Ike [24], used the Fourier sine transform method to solve the natural vibration problem of thin beams on a one-parameter foundation and obtained closed-form solutions for Dirichlet boundary conditions. Ike [25], used the Sumudu transform method to solve the free transverse vibration problems for thin beams and obtained closed-form solutions for the eigenfrequencies for various boundary conditions.

The literature review reveals that GITM has not been used to study the free vibration problems of EB beams resting on Winkler foundations. Thus, This paper aims to use the GITM to determine the free vibration frequencies of thin beams on

Winkler foundations for various boundary conditions. The research is motivated by the successful applications of the GITM by Ike et al. [26], and Ike [27], to the bending and buckling analysis of thin plates and the flexural analysis of clamped rectangular thin plates, respectively.

GITM is adopted in this research because it does not require pre-selection of the basis functions but uses the eigenfunctions of vibrating thin beams with equivalent boundaries as the kernel functions, thus rendering the problems as integral equations, which further simplify to algebraic equations in the transformed space. The orthogonality properties of the kernel functions are also useful properties that aid in further simplification of the resulting calculus issues involved in the GITM.

The innovative aspect of the paper is the first principle, a systematic presentation of the GITM for solving free vibration problems of EB beams resting on Winkler foundations for various boundary conditions.

2. Theory

The governing equation of the vibrating Euler-Bernoulli beam on the Winkler foundation is the inhomogeneous partial differential equation (PDE) given as Equation 1 by [10].

$$EI\frac{\partial^4 w}{\partial x^4}(x,t)+kw(x,t)+\rho A\frac{\partial^2 w(x,t)}{\partial t^2}=q(x,t);\qquad 0\le x\le l \tag{3}$$

In Equation (3) $w(x,t)$ is the transverse deflection, x is the longitudinal axial coordinate of the beam, t is time, l is the length of the beam, A is the cross-sectional area of the beam, ρ is the mass density of the beam, k is the Winkler modulus or Winkler constant, E is Young's modulus of elasticity, I is the moment of inertia, $q(x,t)$ is the applied dynamic load.

For free harmonic vibrations, $q(x,t)=0$ and the governing equation simplifies to the homogeneous PDE expressed in Equation 4.

$$EI\frac{\partial^4 w}{\partial x^4}+kw+\rho A\frac{\partial^2 w}{\partial t^2}=0 \tag{4}$$

For harmonic vibrations, the dynamic displacement $w(x,t)$ is expected to be harmonic. Hence, let

$$w(x,t)=\bar{W}(x)\exp(i\omega_n t) \tag{5}$$

where i is the imaginary number,

$$i=\sqrt{-1} \tag{6}$$

ω_n is the natural frequency

$\bar{W}(x)$ is the displacement shape function.

Using Equation (5), Equation (4) becomes Equation (7)

$$\left(EI\frac{d^4\bar{W}}{dx^4}+(k-\rho A\omega_n^2)\bar{W}(x)\right)\exp i\omega_n t=0 \tag{7}$$

Hence,

$$EI\frac{d^4\bar{W}(x)}{dx^4}+(k-\rho A\omega_n^2)\bar{W}(x)=0 \tag{8}$$

Dividing Equation (8) by *EI* gives Equation (9)

$$\frac{d^4\bar{W}(x)}{dx^4}+\left(\frac{k-\rho A\omega_n^2}{EI}\right)\bar{W}(x)=0 \tag{9}$$

Alternatively, Equation (9) can be expressed as Equation (10).

$$\frac{d^4\bar{W}(x)}{dx^4}+\left(4\beta^4-\frac{\rho A\omega_n^2}{EI}\right)\bar{W}(x)=0 \tag{10}$$

where
$$4\beta^4 = \frac{k}{EI} \tag{11}$$

Let
$$\frac{\rho A\omega_n^2}{EI} = \frac{\bar{m}\omega_n^2}{EI} = \Omega^4 \tag{12}$$

where $4\beta^4$ is the beam on the Winkler foundation parameter, Ω^4 is the frequency parameter; $\bar{m}$ is the mass per unit length of the beam. Then,

$$\frac{d^4\bar{W}(x)}{dx^4} + (4\beta^4 - \Omega^4)\bar{W}(x) = 0 \tag{13}$$

3. Formulation With GITM

In the Generalized Integral Transform Method (GITM) $\bar{W}(x)$ is expressed using linear combinations of the eigenfunctions $f_n(x)$ of a vibrating EB beam on the Winkler foundation with identical end restraints. This is usually constructed as infinite series of the general form given by Equation (14).

$$\bar{W}(x) = \sum_{n=1}^{\infty} c_n f_n(x) \tag{14}$$

where c_n are the unknown parameters of the function $\bar{W}(x)$. The generalized integral transform of the governing equation becomes:

$$\int_0^l \left\{ \frac{d^4}{dx^4}\sum_{n=1}^{\infty} c_n f_n(x) + (4\beta^4 - \Omega^4)\sum_{n=1}^{\infty} c_n f_n(x) \right\} f_m(x)dx = 0 \tag{15}$$

Simplifying, Equation (15) yields Equation (16).

$$\sum_{n=1}^{\infty} c_n \int_0^l (f_n^{iv}(x) f_m(x) + (4\beta^4 - \Omega^4) f_n(x) f_m(x))dx = 0 \tag{16}$$

Further simplification yields Equation (17):

$$\sum_{n=1}^{\infty} c_n \int_0^l (\alpha_n^4 f_n(x) f_m(x) + (4\beta^4 - \Omega^4) f_n(x) f_m(x))dx = 0 \tag{17}$$

The orthogonality considerations are given by Equation (18):

$$\int_0^1 f_n(x) f_m(x)\, dx = 0 \text{ if } n \neq m$$
$$\int_0^{\infty} f_n(x) f_m(x)\, dx \neq 0 \text{ if } n = m \tag{18}$$

From orthogonality considerations, Equation (17) simplifies further to Equation (19):

$$\sum_{n=1}^{\infty} c_n \left(\alpha_n^4 \int_o^l f_n^2(x)dx + (4\beta^4 - \Omega^4)\int_0^l f_n^2(x)dx \right) = 0 \tag{19}$$

Let
$$\int_0^l f_n^2(x)dx = I_n \tag{20}$$

$$\sum_{n=1}^{\infty} c_n \left(\alpha_n^4 I_n + (4\beta^4 - \Omega^4)\right) I_n = 0 \quad (21)$$

For nontrivial solutions,

$$\alpha_n^4 + (4\beta^4 - \Omega^4) = 0 \quad (22)$$

$$\Omega_n^4 = \alpha_n^4 + 4\beta^4 = \frac{m\omega_n^2}{EI} \quad (23)$$

$$\omega_n = \sqrt{\frac{EI}{m}}\sqrt{(\alpha_n^4 + 4\beta^4)} = \lambda_n^2\sqrt{\frac{EI}{\rho A}} \quad (24)$$

$$\text{where } \lambda_n^2 = \sqrt{(\alpha_n^4 + 4\beta^4)} \quad (25)$$

4. Results

4.1 Case 1

Euler-Bernoulli (EB) beams with both ends $x = 0,\ x = l$ clamped (CC), as shown in Figure 1 is considered. The boundary conditions are: $w(x = 0, t) = w(x = l, t) = 0$

$$\frac{\partial w}{\partial x}(x = 0, t) = \frac{\partial w}{\partial x}(x = l, t) = 0 \quad (26)$$

$f_n(x)$, which satisfies the boundary conditions, is the eigenfunction for EB on the Winkler foundation with clamped edges at x=0, and x=l, f_n of x is given by [23]:

$$f_n = \cos\alpha_n x - \cosh\alpha_n x - \beta_n(\sin\alpha_n x - \sinh\alpha_n x) \quad (27)$$

where

$$\beta_n = \frac{\cos\alpha_n l - \cosh\alpha_n l}{\sin\alpha_n l - \sinh\alpha_n l} \quad (28)$$

α_n is the nth root of the transcendental Equation:

$$\cos\alpha l \cosh\alpha l = 1 \quad (29)$$

Solving using Mathematica software, the roots of Equation (29) are given by Equation (30):

$$\begin{aligned} \alpha_1 l &= 4.73004 \\ \alpha_2 l &= 7.85321 \\ \alpha_3 l &= 10.9956 \\ \alpha_4 l &= 14.13717 \\ \alpha_5 l &= 17.27876 \\ \alpha_n l &= \pi\left(n + \frac{1}{2}\right) \end{aligned} \quad (30)$$

Using Equation (25), the natural frequency parameters of clamped-clamped EB beams resting on the Winkler foundation are presented in Table 1 for various values of K0 ranging from $K_0 = 1,\ K_0 = 10,\ K_0 = 100,\ K_0 = 1000$ and $K_0 = 10{,}000$.

The natural fequnecy parameters for the first four modes are shown in Table 2 alongside previous solutions obtained using HPM, DTM, ADM, and DQEM.

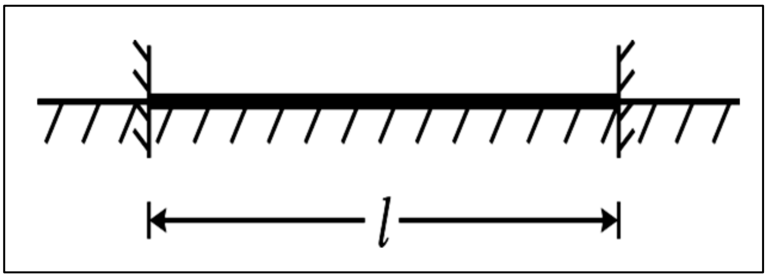

Figure 1: Euler-Bernoulli beam on Winkler foundation with clamped ends

Table 1: Natural frequency parameters λ_n of clamped-clamped EB beam on Winkler foundation (first four modes of vibration)

$K_0 = 4\beta^4 l^4$	λ_1 **Present study GITM**	λ_1 **(Balkaya et al. [9])**	λ_2 **Present study GITM**	λ_2 **(Balkaya et al. [9])**
1	4.7324	4.7324	7.8537261	7.85372
Exact (1)	4.730042		7.853203	
10	4.7534886	4.75349	7.858366685	7.85836
100	4.9503938	4.95039	7.9043264	7.90432
1000	6.223914255	6.22391	8.325120385	8.32512
10,000	10.12285818		10.8392112	
$K_0 = 4\beta^4 l^4$	λ_3 **Present study GITM**	λ_3 **(Balkaya et al. [9])**	λ_4 **Present study GITM**	λ_4 **(Balkaya et al. [9])**
1	10.995788	10.9958	14.13725848	14.1373
Exact (1)	10.99559			
10	10.99748	10.9975	14.13805473	14.13805
100	11.01435738	10.0144	14.14600986	14.14601
1000	11.17901383	11.179	14.22483276	14.2248
10,000	12.52597028		14.94928785	14.9493

Table 2: ω_n for clamped-clamped EB beam resting on Winkler foundation for $A = I = E = l = 1$, $K_0 = 4\beta^4 l^4 = 1$, $\omega_n = \lambda_n^2$

Method	ω_1	ω_2	ω_3	ω_4
Present method	22.395615	61.681014	120.907355	199.86208
HPM (Mutman and Coskun [13])	22.3956	61 .6809	120.908	199.862
DTM (Balkaya et al. [9])	22.3733	61.6728	120.903	199.859
ADM (Mutman and Coskun [13])	22.3956	61.6809	120.908	199.862
DQEM (Chen [12])	22.3956	61.6811	120.910	199.885

4.2 Case 2

Euler-Bernoulli beam on Winkler foundation simply supported at $x = 0$, clamped at $x = l$. The case of simply supported-clamped EB beam on Winkler foundation shown in Figure 2 is considered.

The boundary conditions are:

$$\begin{aligned} w(x = 0, t) &= 0 \\ w(x = l, t) &= 0 \\ w''(x = 0, t) &= 0 \\ w'(x = l, t) &= 0 \end{aligned} \quad (31)$$

where the prime over $w(x,t)$ denotes the partial derivative with respect to x coordinate.

$f_n(x)$ which satisfies the boundary conditions, is the eigenfunction of SC thin beam on Winkler foundation at the nth eigenvalue:

$$f_n(x) = \sin \alpha_n x - \beta_n \sinh \alpha_n x \quad (32)$$

where

$$\beta_n = \frac{\sin \alpha_n l}{\sinh \alpha_n l} \quad (33)$$

α_n are the roots of the transcendental Equation

$$\tanh \alpha_n l = \tan \alpha_n l \quad (34)$$

The exact solutions for $\alpha_n l$ are:

$$\alpha_1 l = 3.92660246314$$
$$\alpha_2 l = 7.068582745629 \quad (35)$$
$$\alpha_3 l = 10.21076122813$$

The solutions for the natural frequency parameters of SC thin beams are presented in Table 3 for the first four modes and various values of $4\beta^4 l^4$.

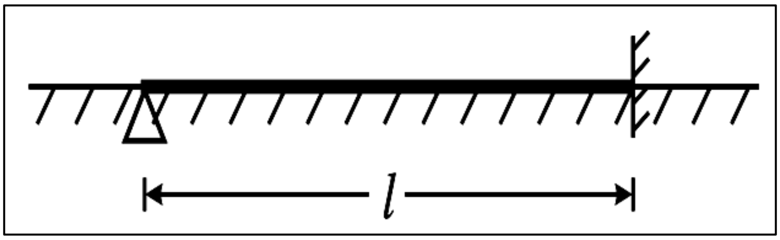

Figure 2: Simply supported-clamped (SC) EB resting on Winkler foundation

Table 3: Natural frequency parameters of SC thin beams on Winkler foundation

$4\beta^4 l^4$	λ_1 GITM Present study	HPM (Coskun [11]; Mutman and Coskun, [13])	λ_2 GITM Present study	HPM (Coskun [11]; Mutman and Coskun, [13])
1	3.930723	3.930776	7.0693077	7.0693
10	3.967258		7.0756679	
100	4.2868606		7.1383457	
1000	5.9313786		7.689689	
$4\beta^4 l^4$	λ_3 GITM Present study	HPM (Coskun [11]; Mutman and Coskun, [13])	λ_4 GITM Present study	HPM (Coskun [11]; Mutman and Coskun, [13])
1	10.210435	10.210436	13.351905	13.35189
10	10.212548		13.35285	
100	10.233607		13.36229	
1000	10.43738		13.455615	

4.3 Case 3

Cantilever beam on Winkler foundation

The natural vibration characteristics of a cantilever beam on the Winkler foundation shown in Figure 3 are considered. The boundary conditions are:

At the clamped end, $x = 0$,

$$w(x=0,t)=0 \quad (36a)$$

$$\theta(x=0,t)=w'(x=0,t)=0 \quad (36b)$$

At the free end, $x = l$

$$M(x=l,t) = -EI\frac{d^2w}{dx^2}(x=l,t)=0 \quad (36c)$$

Hence, $w''(x=l,t)=0$ (36d)

$$Q(x=l,t) = -EI\frac{d^3w}{dx^3}(x=l,t)=0 \quad (36e)$$

Hence, $w'''(x=l,t)=0$ (36f)

$f_n(x)$ given by Equation (37) satisfies the boundary conditions of the clamped-free beam, since

$$f_n(x=0) = f_n'(x=0) = f_n''(x=l) = f_n'''(x=l) = 0 \quad (36g)$$

Where Equation (36g) follows from Equations (36a), (36b), (36d), and (36f)

The eigenfunction for the cantilever beam on Winkler foundation is:

$$f_n(x) = (\cosh \alpha_n x - \cos \alpha_n x) - \beta_n(\sinh \alpha_n x - \sin \alpha_n x) \quad (37)$$

$$\beta_n = \left(\frac{\cosh \alpha_n l + \cos \alpha_n l}{\sinh \alpha_n l + \sin \alpha_n l}\right) \quad (38)$$

where α_n are the roots of

$$\cosh \alpha_n l \cos \alpha_n l = -1 \quad (39)$$

The eigenvalues found using computer software tools Mathcad, Wolfram Mathematica, and iterative methods for solving transcendental Equations are:

$$\alpha_1 l = 1.87510$$

$$\alpha_2 l = 4.69409$$

$$\alpha_3 l = 7.85476$$

$$\alpha_4 l = 10.99554 \quad (40)$$

$$\alpha_n l = \frac{(2n-1)\pi}{2} \text{ for } n \geq 5$$

The frequency parameter for the first four vibration modes and various values of non-dimensional foundation parameters are presented in Table 4.

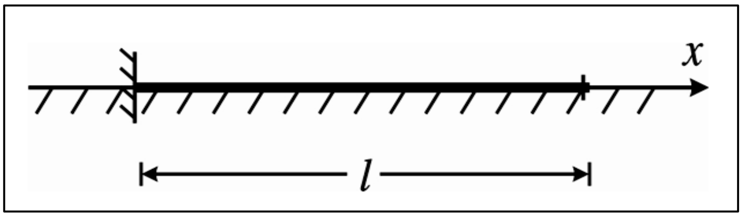

Figure 3: Cantilever beam on Winkler foundation

Table 4: Frequency parameters for the natural vibrations of cantilever beams resting on Winkler foundation

$4\beta^4 l^4$	λ_1 **Present study**	λ_1 **(Kacar et al. [4])**	λ_2 **Present study**	λ_2 **(Kacar et al. [4])**	λ_3 **Present study**	λ_3 **(Kacar et al. [4])**	λ_4 **Present study**	λ_4 **(Kacar et al. [4])**
1	1.91192	1.91192	4.696505	4.69651	7.855276	7.85527	10.995728	10.995728
10	2.174598	2.1746	4.718076	4.71808	7.859914	7.85991	10.99742	10.99742
100	3.25578	3.25578	4.919094	4.91910	7.9058466	7.90584	11.0142977	11.0142977
1000	5.64071	5.64071	6.208254	6.20825	8.3264215	8.32642	11.178957	11.178957

4.4 Case 4

Simply supported (at $x = 0$, and $x = l$) EB beam on Winkler foundation.The simply supported thin beam on the Winkler foundation shown in Figure 4 is considered. The simply supported boundary conditions are :

$$w(x=0,t) = w(x=l,t) = 0 \quad (41a)$$

$$w''(x=0,t) = w''(x=l,t) = 0 \tag{41b}$$

Hence

$$f_n(x=0) = f_n''(x=0) = 0 \tag{41c}$$

$$f_n(x=l) = f_n''(x=l) = 0 \tag{41d}$$

The eigenfunctions for the simply supported thin beam on the Winkler foundation problem is

$$f_n(x) = \sin \alpha_n x \tag{42}$$

$$\text{where } \alpha_n = \frac{n\pi}{l}, \quad n = 1, 2, 3, 4 \ldots \tag{42a}$$

The frequency parameters for the simply supported thin beam on the Winkler foundation are presented in Table 5.

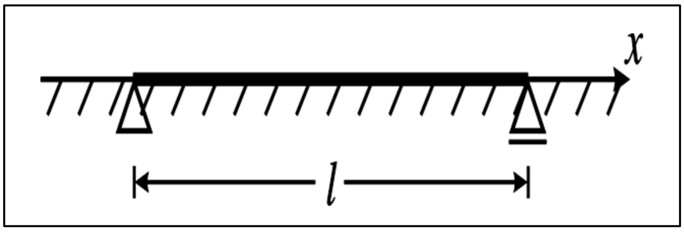

Figure 4: Simply supported thin beam on Winkler foundation

Table 5: Frequency parameters λ_n for the free vibration of simply supported Euler-Bernoulli beam on Winkler foundation

Foundation parameter $4\beta^4 l^4$	Present study λ_1	(Rahbar-Ranji and Shahbaztabar [15]) λ_1	(Zhou [16]) λ_1	Present study λ_2	(Rahbar-Ranji and Shahbaztabar [15]) λ_2	(Zhou [16]) λ_2
1	3.149624682	3.1496		6.284192925	6.28426	
10	3.219291184	3.2193	3.220	6.293239752	6.2932	6.293
100	3.74836425	3.7484	3.748	6.381633293	6.3816	6.382
1000	5.755620336	5.7556	5.755	7.11210704	7.1121	7.112
10,000	10.02426382	10.0243		10.36873551	10.3687	
Foundation parameter $4\beta^4 l^4$	**The present study (GITM) λ_3**	**(Rahbar-Ranji and Shahbaztabar [15]) λ_3**	**(Zhou, [16]) λ_3**	**The present study(GITM) λ_4**	**(Rahbar-Ranji and Shahbaztabar [15]) λ_4**	**(Zhou, [16]) λ_4**
1	9.425076572	9.4251		12.5664966	12.5665	
10	9.427762796	9.4277	9.427	12.56763024	12.5676	12.568
100	9.454499103	9.4545	9.454	12.57894997	12.5790	12.579
1000	9.710176091	9.7102	9.710	12.69050177	12.6905	12.690
10,000	11.56520706	11.5652		13.67163814	13.6716	

Foundation parameter $4\beta^4 l^4$	The present study (GITM) λ_5	(Rahbar-Ranji and Shahbaztabar [15]) λ_5	(Zhou [16]) λ_5
1	15.70802772	15.7080	
10	15.70860826	15.7086	15.709
100	15.71440961	15.7144	15.715
1000	15.77207279	15.7721	
10,000	16.31668659	16.3167	

5. Discussion

The fourth-order linear PDE governing the free transverse harmonic vibrations of EB beams resting on Winkler foundations has been solved in this work using the GITM. The assumption of harmonic response simplified the governing PDE to a fourth-order Ordinary Differential Equation (ODE) shown in Equation (13) in terms of the displacement shape function $\bar{W}(x)$. Using the GITM and linear combinations of eigenfunctions of EB beams with identical boundary conditions, $\bar{W}(x)$ is constructed as the infinite series given in Equation (14). The GITM formulation of the problem results in the integral equation presented in Equation (15). Simplifying the problem using the orthogonality of the eigenfunctions used in formulating gives the algebraic eigenvalue problem in Equation (19). Solving the eigenvalue problem gives the general solution for the natural frequency parameter λ_n for any n mode of vibration as Equation (25). Eigenfunctions of EB beams with clamped ends are employed in the general solution to obtain the natural frequency parameters λ_n presented in Table 1 for the first four modes of vibration and for various values of the foundation parameter $4\beta^4 l^4$ ranging from $4\beta^4 l^4 = 1$ to $4\beta^4 l^4 = 10,000$. Table 1 shows that the vibration frequencies increase as the foundation parameter $4\beta l^4$ increases.

Table 1 shows that values obtained in the present study for λ_n for $n = 1, 2, 3, 4$ are identical with results obtained by Balkaya et al., [9]. Table 2 further confirms the agreement of the GITM with solutions using other methods.

A similar procedure was used to obtain λ_n for $n = 1, 2, 3, 4$ for thin beams with SC boundaries, and the results shown in Table 3 agree with results obtained by Coskun [18] and Mutman and Coskun [20] for $4\beta^4 l^4 = 1$.

The eigenfunction for the cantilever beam on the Winkler foundation was used to obtain λ_n for various values of the foundation parameter and for the first four vibration modes, as given in Table 4. Table 4 also shows identical results from Kacar et al., [11].

The GITM solution for simply supported thin beam on the Winkler foundation is presented in Table 5 for various values of the foundation parameter $4\beta^4 l^4$ and for the first five modes. The results agree with the solutions presented by Rahbar-Ranji and Shahbaztabar [22], and Zhou [23]. Tables 1 – 5 show that as the foundation parameter increases, the natural frequencies of vibration increase.

6. Conclusion

In conclusion, this paper has studied the free vibration frequency analysis of Euler-Bernoulli beams resting on Winkler foundations using the Generalized Integral Transform Method (GITM). Unlike other methods of solving BVPs, GITM does not require a pre-selection of the basis functions as the eigenfunctions of vibrating thin beams with equivalent boundary conditions are selected as the kernel functions, thus converting the BVPs to integral equations and ultimately to algebraic equations.

1) GITM transforms the problem of free transverse vibrations of the EB beam on the Winkler foundation to an algebraic eigenvalue problem.
2) The method uses the eigenfunctions of vibrating EB beams on the Winkler foundation with identical boundary conditions. Thus the boundary conditions are aproiri satisfied by the resulting algebraic problem.
3) The solution can give all the vibrating frequencies for all the possible vibration modes and hence gives closed-form solutions for λ_n for $n = 1, 2, 3, 4\ldots$
4) The present solutions for the natural frequency parameters λ_n are exact and agree with previously obtained solutions that used the Homotopy Perturbation Method, (HPM), Adomian Decomposition Method (ADM), Differential Transform Method (DTM) and Differential Quadrature Element Method (QQEM).
5) The natural frequencies increase as the foundation parameter $4\beta l^4$ increases for all the cases of boundary conditions studied.

Notations

E	Young's modulus of elasticity
I	moment of inertia
x	longitudinal coordinate variable
t	time
A	cross-sectional area of the beam
$w(x, t)$	transverse deflection
k	Winkler modulus or Winkler constant
$q(x, t)$	applied dynamic load
i	imaginary number
$\bar{W}(x)$	displacement shape function
ω_n	natural frequency
$4\beta^4$	beam on Winkler foundation parameter
Ω^4	frequency parameter
$\bar{m}$	mass per unit length of the beam
c_n	unknown parameters of the function $\bar{W}(x)$
f_n	eigenfunction of a vibrating Euler-Bernoulli beam resting on the Winkler foundation for the nth mode.
n	vibration mode
α_n	nth root of transcendental characteristic vibration equation
β_n	parameter defined in terms of α_n
K_0	dimensionless foundation beam parameter defined as $4\beta^4 l^4$
λ_n	Natural frequency parameter for the nth vibration mode
$\sum$	summation
$\leq$	less than or equal to

$\geq$ greater than or equal to

$\frac{d^n}{dx^n}$ nth ordinary derivative with respect to x

$\int$ integral
GITM Generalized Integral Transform Method
PDE(s) Partial Differential Equation(s)
ODE(s) Ordinary Differential Equation(s)
DQEM Differential Quadrature Element Method
ADM Adomian Decomposition Method
HPM Homotopy Perturbation Method
BVP(s) Boundary Value Problem(s)
EB Euler-Bernoulli

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Data availability statement

The data that support the findings of this study are available on request from the corresponding author.

Conflicts of interest

The authors declare that there is no conflict of interest.

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