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New Concept of Finite Group(Z_{p^nq}) on the Sum Graph with Some Topological Indices

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Article information **Abstract**

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In this paper, we study the extended graph theory in the sum group via Z_{p^nq} which is Z_{p^nq}, by two distinct orders, the sum is greatest than the order of the group Z_{p^nq}, where p, q are prime numbers. We have some results that the group sum graph of Z_{p^nq} are connected, cyclic, etc., if they satisfy some properties of the graph theory and compute all the degrees of graphs. Furthermore, we shall calculate the famous topological indices via generalized it.

Keywords:
Group sum graph, Euler graph, Hamilton graph, Zagreb index, Forgotten index.

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1. INTRODUCTION

Assume that (G) is a graph, with $G(V(G),E(G))$ standing for G's collection of vertices and edges. Solely take into account straightforward finite graphs in this study. Deg(u) (or d(u)) of G is used to indicate the vertex u $V(G)$ degree. If an edge with the symbol xy connects two vertices x and y, those vertices are adjacent. The number of edges connected to an edge (i.e., the number of edges connected to both x and y except for the edge xy) is referred to the edge's degree. Refer to G as a graph having k vertices and l edges when we say that $G \in G_+(k,l)$.The notations and terminologies used in this article are not defined in [1]. The numbers that are connected to a graph structure are known as topological indices.

Numerous topological indices have been developed and examined throughout the years based on the degree, distance, and other graph properties. You can find some of them in [7, 11]. Zagreb indices, which have applications in many fields of study including economics, physics, and chemical research, might be regarded historically as the first degree-based topological indices. [22-25] Graph theory has been used in a number of research that relates to group or ring theory features of the Z modulo. For further information, check [3-11,20,21,26] and the references therein. The whole Zagreb indices of graphs were investigated in 2018 by Alwardi, et al. [2].

This paper introduces a new definition of the sum graph of a group Z_{p^nq} be a finite group of order p^nq, where p and q are prime numbers and $2 < p < q$ where the vertices of the graph represent the elements of Z_{p^nq} , such that there is an edge between the two vertices a and b if and only if $\mathcal{O}(a) + \mathcal{O}(b) > \mathcal{O}(Z_{p^nq})$, denoted by $G_+(Z_{p^nq})$.The topological indices have been well studied in the last years; some of them may be found in [12-16]. In 2013, the indices of Zagreb were re-defined by Ranjini et al. [3] as first, second, and third indices of Zagreb. Further, it can be considered as a particular case of the generalized inverse sum index $ISI_{(\gamma,\mu)}(G)$ of a graph G proposed by Buragohain et al. in [4].

In this paper, got some new properties of $G_+(Z_{p^nq})$ and computed topological indices such as (The Eccentric Connectivity Index, the First and second indices of Zagreb, the Sum-Connectivity index, the Randic index(or Connectivity index), and sum particular special cases indices of Z_{p^nq}.

1. Basic Concepts and TERMINOLOGY

The topological index required in the following considerations is our main concern.

The Eccentric Connectivity index $\mathfrak{J}^c$ is defined as
$\mathfrak{J}^c = \mathfrak{J}^c(G) = \sum_{i=1}^{k} e(u)d_i$, *where* $e(u) = \max\{d(u,v), v \in V(G)\}$. [14]

The Eccentric Connectivity index was generalized $\mathfrak{J}_\gamma^c$. Which is denoted by $\mathfrak{J}_\gamma^c$ and defined as
$\mathfrak{J}_\gamma^c(G) = \sum_{i=1}^{k} e(u)d_i^\gamma, \gamma \in \mathbb{R}$. [19]

The definition of the first generic Zagreb index, also known as the general Zeroth-order Randic index, which is
$Q_\gamma = Q_\gamma(G) = \sum_{i=1}^{k} d_i^\gamma = \sum_{i\sim j} \left(d_i^{\gamma-1} + d_j^{\gamma-1}\right)$,
Where k is the order of vertices and $\gamma \in \mathbb{R}$ [14].

The generalized Randic index (or Connectivity index) R_γ, which is defined as
$R_\gamma = R_\gamma(G) = \sum_{i\sim j} \left(d_i d_j\right)^\gamma$, and $\gamma \in \mathbb{R}$ [14].

The general Sum-Connectivity index H_μ as
$H_\mu = H_\mu(G) = \sum_{i\sim j}\left(d_i + d_j\right)^\mu$, and $\mu \in \mathbb{R}$ [15].

2. THE SUM GRAPHS OF GROUPS Z_{pq} , Z_{p^2q} , Z_{p^3q} AND Z_{p^nq}.

shall explore graph theory's understanding of the Z_n the group in the following section defines it in terms of order rules. Additionally, established the degree of Z_n at p^nq,where $p, and\ q$ are prime numbers raised to the power of positive integer number n.

Definition 3.1 [19]: Let G be a finite cyclic group. The group sum graph, denoted by $G_+(V,E)$ of G_+ is a graph with $V(G_+) = \cup_{x\in G}\langle x\rangle$ and two distinct vertices x and y are adjacent in G_+, denoted by $\langle x\rangle \sim \langle y\rangle$ if and only if $\mathcal{O}(x) + \mathcal{O}(y) > \mathcal{O}(G)$, since $\mathcal{O}(G)$ is the order of the group G. (i.e.)
$V(G_+) = \bigcup_{x\in G}\langle x\rangle$,
$E(G_+) = \{xy \mid \langle x\rangle \sim \langle y\rangle\ if\ and\ only\ if\ \mathcal{O}(x) + \mathcal{O}(y) > \mathcal{O}(G)$, where $x, y \in G$ and $x \neq y\}$.

Remark 3.2 [19]:
If taking as (definition 3.1) $\mathcal{O}(x) + \mathcal{O}(y) \leq \mathcal{O}(G)$, where G is a finite group of order n. See that the graph is not connected because there exist at least one element a such that $\mathcal{O}(a) = \mathcal{O}(G)$. Therefore, $\mathcal{O}(a) + \mathcal{O}(a_i) > \mathcal{O}(G), \forall\ a_i \in G, 1 \leq i \leq n,$ (i.e.) a is isolated-vertex, hence G is not connected.

Lemma 3.3 [19]: Every finite cyclic group holds the group sum graph are connected and cyclic graphs.

Lemma 3.4 [19]: If Z_p Be a finite group of order a prime number p, then
$G_+\left(V(Z_p), E(Z_p)\right) \cong K_p$,
where K_p It is a complete graph of p vertices.

Theorem 3.5 [19]: If $G_+\left(V(Z_{p^n}), E(Z_{p^n})\right)$, then

$$\underset{u\in V(Z_{p^n})}{deg(u)} = \begin{cases} p^{n-1}(p-1) & \text{if } \mathcal{O}(u) \neq p^n \\ p^n - 1 & \text{if } \mathcal{O}(u) = p^n \end{cases},$$

where $p \geq 3$ is the prime number and $n \geq 2$ is a positive integer number.

Now, extended the sum group of Z_{p^nq} , as n=1 follows that:

Proposition 3.6.: If $G_+\left(V(Z_{pq}), E(Z_{pq})\right)$, then

$$deg(u) = \begin{cases} pq - (p+q) + 1 & if\ \mathcal{O}(u) \neq pq \\ pq - 1 & if\ \mathcal{O}(u) = pq \end{cases},$$

where $u \in V(Z_{pq})$ and $p, q \geq 3, p$ and q are distinct relative prime numbers.

Proof: Since $Z_{pq} = \{0, 1, 2, \dots, pq-1\}$, then Z_{pq} have four sets of distinct orders, which that

- $\mathcal{O}(0) = 1, \mathcal{O}(p) = \mathcal{O}(p, 2p, \dots, (q-1)p) = q,$
- $\mathcal{O}(q) = \mathcal{O}(q, 2q, \dots, (p-1)q) = p$
- $\mathcal{O}(a_1, a_2, \dots, a_\alpha) = pq,$

where $a_i \in Z_{pq}, 1 \leq i \leq \alpha, \alpha = pq - [1 + (q-1) + (p-1)] = pq - (p+q) + 1.$

Now, by (definition 3.1), we have

<u>**Case 1:**</u> $\mathcal{O}(0) + \mathcal{O}(p) = 1 + q < \mathcal{O}(Z_{pq}) = pq \Rightarrow 0p \notin E(Z_{pq})$,
$\mathcal{O}(0) + \mathcal{O}(q) = 1 + p < pq \Rightarrow 0q \notin E(Z_{pq})$,
$\mathcal{O}(0) + \mathcal{O}(a_i) = 1 + pq > pq \Rightarrow 0a_i \in E(Z_{pq}), \forall 1 \leq i \leq \alpha$.

<u>**Case 2:**</u> $\mathcal{O}(p) + \mathcal{O}(q) = q + p < pq \Rightarrow pq \notin E(Z_{pq})$,
$\mathcal{O}(p) + \mathcal{O}(a_i) = q + pq > pq \Rightarrow pa_i \in E(Z_{pq}),\ \forall 1 \leq i \leq \alpha$.

<u>**Case 3:**</u> $\mathcal{O}(q) + \mathcal{O}(a_i) = p + pq > pq \Rightarrow qa_i \in E(Z_{pq})$, $\forall 1 \leq i \leq \alpha$, which implies that, $deg(0) = deg(p) = deg(q) = pq - (p+q) - 1$,
While $deg(0) = deg(p) = deg(2p) = \dots = deg((q-1)p) = deg(q) = deg(2q) = \dots = deg((p-1)q)$.

<u>**Case 4:**</u> $\mathcal{O}(a_i) + \mathcal{O}(a_j) = pq + pq > pq,\ \forall\ 1 \leq i, j \leq \alpha, i \neq j$, where $a_i \in Z_{pq}, 1 \leq i \leq \alpha$, then $a_ia_j \in E(Z_{pq}), \forall i, j, i \neq j$, which implies that $deg(a_i) = pq - 1, \forall 1 \leq i \leq \alpha$. (Since the graph is a simple graph)

$$\underset{u\in V(Z_{pq})}{deg(u)} = \begin{cases} pq - (p+q) + 1 & \text{if } \mathcal{O}(u) \neq pq \\ pq - 1 & if\ \mathcal{O}(u) = pq \end{cases}$$

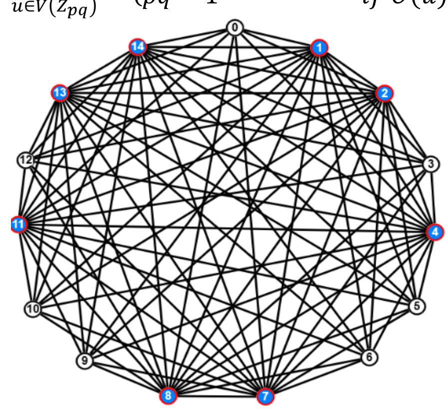

Figure 3.1: Sum Graph (Z_{15}).

Example 3.7: $Z_{15} = \{0, \dots, 14\}$
$\mathcal{O}(0) = 1, \mathcal{O}(p = 3) = \mathcal{O}(3,6,9,12) = 5, \mathcal{O}(q = 5) = \mathcal{O}(5,10) = 3$

$\mathcal{O}(1,2,4,7,8,11,13,14) = 15$

$$\underset{u\in V(Z_{15})}{deg(u)} = \begin{cases} 8 & \text{if } \mathcal{O}(u) \neq 15 \\ 14 & \text{if } \mathcal{O}(u) = 15 \end{cases}$$

As figures (3.1) & (3.2)

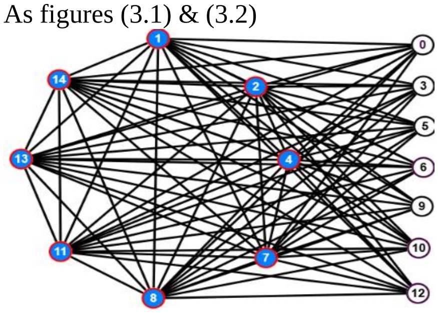

Figure 3.2.: Sum Graph (Z_{15}) =(K_8+S).

Now, notice the degree difference when $n \geq 2$ follows as.

Proposition 3.8.: If $G_+\left(V(Z_{p^2q}), E(Z_{p^2q})\right)$, then

$$\underset{u\in V\left(Z_{p^2q}\right)}{deg(u)} = \begin{cases} p(pq-(p+q)+1) & \text{if } \mathcal{O}(u) \neq p^2q \\ p^2q-1 & \text{if } \mathcal{O}(u) = p^2q \end{cases},$$

Where $2 < p < q$ are distinct relative prime numbers.

Proof: Since $Z_{p^2q} = \{0,1,2,\dots,p^2q-1\}$, then Z_{p^2q} Have six-sets distinct orders that

$\mathcal{O}(0) = 1, \mathcal{O}(p) = \mathcal{O}(p, 2p, \dots, \alpha_1 p) = pq$, $\mathcal{O}(p^2) = \mathcal{O}(p^2, 2p^2, \dots \alpha_2 p^2) = q$,

$\mathcal{O}(pq) = \mathcal{O}(pq, 2pq, \dots, \alpha_3 pq) = p, \mathcal{O}(q) = o(q, 2q, \dots, \alpha_4 q) = p^2$ and $\mathcal{O}(a_1, a_2, \dots, a_\alpha) = p^2q$, where $a_i \in Z_{p^2q}, \mathcal{O}(a_i) = p^2q, \forall 1 \leq i \leq \alpha$. Thus,

$\alpha_1 = (pq-(p+q)+1), \alpha_2 = (q-1), \alpha_3 = (p-1)$, $\alpha_4 = p(p-1)$ and $\alpha = p^2q - \left[1 + \sum_{i=1}^{4} \alpha_i\right] = p(pq-(p+q)+1)$.

For the cases. Notice $\mathcal{O}(0), \mathcal{O}(p), \mathcal{O}(p^2), \mathcal{O}(pq)$ and $\mathcal{O}(q)$ which for any two distinict orders of them. It's addictive a least than the order of the group Z_{p^2q}.

This means, $0(p), 0(p^2), 0(pq), 0(q), p(p^2), p(pq), p(q), p^2(pq), (p^2)q$ and $(pq)q \notin E(Z_{p^2q})$. But all this satisfying with $\mathcal{O}(a_i), \forall 1 \leq i \leq \alpha$. (i.e.) $0a_i, pa_i, (p^2)a_i, (pq)a_i$ and $q(a_i) \in E(Z_{p^2q}), \forall 1 \leq i \leq \alpha$, which implies that,

$deg(0) = deg(p) = deg(p^2) = deg(pq) = deg(q) = p(pq-(p+q)+1))$,

so,$deg(p) = deg(2p) = \cdots = deg(\alpha_1 p) = deg(p^2) = deg(2p^2) = \cdots = deg(\alpha_2 p^2) = deg(pq) = deg(2pq) = \cdots = deg(\alpha_3 pq) = deg(q) = deg(2q) = \cdots = deg(\alpha_4 q)$.

Furthermore, we see that $\mathcal{O}(a_i) + \mathcal{O}(a_j) = p^2q + p^2q \geq p^2q$. Therefore, $a_i a_j \in E(Z_{p^2q})$, $\forall 1 \leq i, j \leq \alpha, i \neq j$, which implies that $deg(a_i) = p^2q - 1, \forall 1 \leq i \leq \alpha$ (since the graph is a simple graph). We have the result.

Example3.9.: $Z_{3^2\cdot 5} = \{0,1,\dots,44\}$, we get

$\mathcal{O}(0) = 1$, $\mathcal{O}(p) = \mathcal{O}(3,6,12,21,24,33,39,42) = 15$,

$\mathcal{O}(p^2) = \mathcal{O}(9,18,27,36)$=5,

$\mathcal{O}(pq) = o(15,30) = 3$, $\mathcal{O}(q) = \mathcal{O}(5,10,20,25,35,40) = 9$ and

$\mathcal{O}(1,2,4,7,8,11,13,14,16,17,19,22,23,26,28,29,31,32,34,37,38,41,43,44) = 45$.

As Figure (3.3).

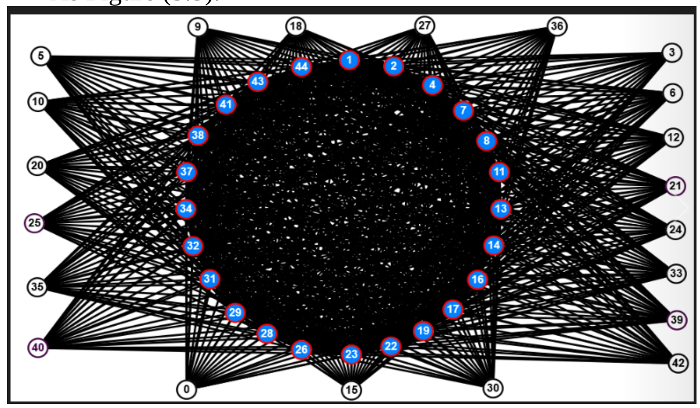

Figure 3.3: Sum Graph (Z_{45}).

Remark 3.10.: In the previous example, notice that this graph expresses this formal, $K_{24} + S$, where any $v \in K_{24}$ have $\mathcal{O}(v) = 45$, $u \in S$ have $\mathcal{O}(u) \neq 45$.

In general, we can express the graph $G_+\left(V(Z_{p^nq}), E(Z_{p^nq})\right)$ by the formal $K_{p^{n-1}(pq-(p+q)+1)} + S$, where any $v \in K_{p^{n-1}(pq-(p+q)+1)}$ have $\mathcal{O}(v) = p^nq$ and $u \in S$ have $\mathcal{O}(u) \neq p^nq, 3 \leq p < q$ are relative prime numbers and $n \geq 1$ is positive integer number. as figure (3.3).

Proposition 3.11: If $G_+\left(V(Z_{p^3q}), E(Z_{p^3q})\right)$, then

$$\underset{u\in V\left(Z_{p^3q}\right)}{deg(u)} = \begin{cases} p^2(pq-(p+q)+1) & \text{if } \mathcal{O}(u) \neq p^3q \\ p^3q-1 & \text{if } \mathcal{O}(u) = p^3q \end{cases},$$

Where $2 < p < q$ are distinct relative primes numbers.

Proof: Since $Z_{p^3q} = \{0,1,\dots,p^3q-1\}$. Then Z_{p^3q} have eight sets of distinct orders which that,

$\mathcal{O}(0) = 1, \mathcal{O}(p, 2p, \dots, \alpha_1 p) = p^2q$,

$\mathcal{O}(p^2, 2p^2, \dots, \alpha_2 p^2) = pq, \mathcal{O}(p^3, 2p^3, \dots, \alpha_3 p^3) = q$,

$\mathcal{O}(q, 2q, \dots, \alpha_4 q) = p^3, \mathcal{O}(pq, 2pq, \dots, \alpha_5 pq)p^2, \mathcal{O}(p^2q, 2p^2q, \dots \alpha_6 p^2q) = p$ and $\mathcal{O}(a_1, a_2, \dots, a_\alpha) = p^3q$, where $a_i \in Z_{p^3q}, \mathcal{O}(a_i) = p^3q, \forall 1 \leq i \leq \alpha$. So that

$\alpha_1 = p(pq-(p+q)+1), \alpha_2 = (p-1)(q-1), \alpha_3 = (q-1), \alpha_4 = p^2(p-1), \alpha_5 = p(p-1), \alpha_6 = p-1$ and

$$\alpha = p^3q - \left[1 + \sum_{i=1}^{6} \alpha_i\right] = p^2(pq-(p+q)+1).$$

The remaining may be proven in a similar manner for (Proposition 3.8); we have the result.

Remark 3.12:

$$\underset{u\in V(Z_{pq})}{\#(Z_{pq})(u)} = \begin{cases} 1+\alpha_1+\alpha_2 = 1+(p-1)+(q-1) = p+q-1 & \text{if } \mathcal{O}(u) \neq pq \\ pq-[1+\alpha_1+\alpha_2] = (pq-(p+q)+1) & \text{if } \mathcal{O}(u) = pq \end{cases}$$

$$\underset{u\in V\left(Z_{p^2q}\right)}{\#(Z_{p^2q})(u)} = \begin{cases} 1+\sum_{i=1}^{4}\alpha_i = p(p+q1) & \text{if } \mathcal{O}(u) \neq p^2q \\ p^2q - \left[1+\sum_{i=1}^{4}\alpha_i\right] = p(pq-(p+q)+1) & \text{if } \mathcal{O}(u) = p^2q \end{cases}$$

$$\underset{u\in V(Z_{p^3q})}{\#(Z_{p^3q})(u)} = \begin{cases} 1+\sum_{i=1}^{6}\alpha_i = p^2(p+q-1) & \text{if } \mathcal{O}(u) \neq p^3q \\ p^3q - \left[1+\sum_{i=1}^{6}\alpha_i\right] = p^2(pq-(p+q)+1) & \text{if } \mathcal{O}(u) = p^3q \end{cases}$$

So, in general, we have that

$$\underset{u\in V(Z_{p^nq})}{\#(Z_{p^nq})(u)} = \begin{cases} 1+\sum_{i=1}^{2n}\alpha_i = p^{n-1}(p+q-1) & \text{if } \mathcal{O}(u) \neq p^nq \\ p^nq - \left[1+\sum_{i=1}^{2n}\alpha_i\right] = p^{n-1}(pq-(p+q)+1) & \text{if } \mathcal{O}(u) = p^nq \end{cases}$$

where $2 < p < q$ are distinct relative prime numbers, $n \geq 1, n \in \mathbb{Z}^+$.

Theorem 3.13: If $G_+\left(V(Z_{p^nq}), E(Z_{p^nq})\right)$, then

$$\underset{u\in V(Z_{p^nq})}{deg(u)} = \begin{cases} p^{n-1}(pq-(p+q)+1) & \text{if } \mathcal{O}(u) \neq p^nq \\ p^nq-1 & \text{if } \mathcal{O}(u) = p^nq \end{cases},$$

where $2 < P < q$ are distinct relative prime numbers, $n \geq 1, n \in \mathbb{Z}^+$.

Proof: The proof follows immediately from (Proposition 3.8,3.11) and (Remark 3.12).

Remark 3.14:

$\sum_{u\in V(Z_{p^nq})} deg(u) =$
$p^{n-1}(pq-(p+q)+1)(p^{n-1}(pq+p+q-1)-1)$,
$2 < p < q, n \in \mathbb{Z}^+, n \geq 1$.

$X = \frac{p^{n-1}(pq-(p+q)+1)(p^{n-1}(pq+p+q-1)-1)}{2}$, where X is the size of the graph

Remarks 3.15

1-The sum graph $G_+(Z_{p^nq})$ is Hamilton's graph because the degree of any vertex is greater or equal to $p^nq/2$.

2- The sum graph $G_+(Z_{p^nq})$ is an Euler's graph because all vertices have an even degree.

3. Topological index of G+(Z_{p^nq})

Notice in this section, we will compute the famous generalized topological index with special cases.

Remark 4.1: The first general Zagreb index (or general Zeroth-order Randic index) Q_γ is defined as

$$Q_\gamma = Q_\gamma(G) = \sum_{i=1}^{o} d_i^\gamma = \sum_{i\sim j} \left(d_i^{\gamma-1} + d_j^{\gamma-1}\right),$$

Where o is an order of vertices.

Theorem 4.2: If $G_+\left(V(Z_{p^nq}), E(Z_{p^nq})\right)$, then the first general Zegrab index is $Q_\gamma = p^{n-1}(pq-(p+q)+1)\left[p^{n-1}(p+q-1)\left(p^{n-1}(pq-(p+q)+1)\right)^{\gamma-1} + (p^nq-1)^\gamma\right]$

Proof: The graph of $G_+\left(V(Z_{p^nq}), E(Z_{p^nq})\right)$ have degree by (theorem 3.13), where $2 < p < q$ and n is a positive integer number, $n \geq 1$, we get

$$Q_\gamma = Q_\gamma(Z_{p^nq}) = \sum_{i=1}^{p^nq} d_i^\gamma$$

$$= \underbrace{\left(p^{n-1}(pq-(p+q)+1)\right)^\gamma + \left(p^{n-1}(pq-(p+q)+1)\right)^\gamma + \dots + \left(p^{n-1}(pq-(p+q)+1)\right)^\gamma}_{p^{n-1}(p+q-1)-times}$$
$$+ \underbrace{(p^nq-1)^\gamma + (p^nq-1)^\gamma + \cdots + (p^nq-1)^\gamma}_{p^{n-1}(pq-(p+q)+1)-times}$$
$$= p^{n-1}(p+q-1)(p^{n-1}(pq-(p+q)+1))^\gamma + p^{n-1}(pq-(p+q)+1)(p^nq-1)^\gamma$$
$$= p^{n-1}(pq-(p+q)+1)\left[p^{n-1}(p+q-1)\left(p^{n-1}(pq-(p+q)+1)\right)^{\gamma-1} + \left(p^nq-1\right)^\gamma\right]$$

In particular:

If $\gamma = 1 \Longrightarrow Q_1 = \sum_{i=1}^{p^nq} d_i = 2X$ (X is the size of the graph)

$$\Longrightarrow X = \frac{1}{2}Q_1 = \frac{1}{2}\left[p^{n-1}(pq-(p+q)+1)\left[p^{n-1}(p+q-1) + (p^nq-1)\right]\right].$$

If $\gamma = 2 \Rightarrow Q_2 = \sum_{i=1}^{p^nq} d_i^2 = \sum_{i\sim j} d_i + d_j = M_1$

$$= p^{n-1}(pq-(p+q)+1)\left[p^{n-1}(p+q-1)\left(p^{n-1}(pq-(p+q)+1)\right) + \left(p^nq-1\right)^2\right]$$

If $\gamma = 3 \Rightarrow Q_3 = \sum_{i=1}^{p^nq} d_i^3 = \sum_{i\sim j} d_i^2 + d_j^2 = F_1$

$$= p^{n-1}\left(pq-(p+q)+1\right)\left[p^{n-1}(p+q-1)\left(p^{n-1}(pq-(p+q)+1)\right)^2 + \left(p^nq-1\right)^3\right]$$

Remarks 4.3:

(1) Let $u \in V(G)$, where G_+ is a finite simple graph of order ($n \neq p$), where p is a prime number.

$$e(u) = \begin{cases} 1 & \text{if } \mathcal{O}(u) = |Z_n| \\ 2 & \text{if } \mathcal{O}(u) \neq |Z_n| \end{cases}.$$

(2) The general the Eccentric Connectivity index $\mathfrak{J}_\gamma^c$ Is define as

$$\mathfrak{J}_\gamma^c = \mathfrak{J}_\gamma^c(G) = \sum_{i=1}^{k} e(u)d_i^\gamma, \gamma \in \mathbb{R}.$$

Theorem 4.4: The general Eccentric connectivity index of the graph $G_+\left(V(Z_{p^nq}), E(Z_{p^nq})\right)$ is $\mathfrak{J}_\gamma^c(Z_{p^nq}) = \sum_{i=1}^{k} e(u)d_i^\gamma =$

$$p^{n-1}(pq-(p+q)+1)\left[2p^{n-1}(p+q-1) + (p^nq-1)\right], \gamma \in \mathbb{R}$$

Proof: From (Remark 4.3) and by (theorem 3.11), we have $\mathfrak{I}_\gamma^c(Z_{p^nq})$

$$= 2\left[\underbrace{[p^{n-1}(pq-(p+q)+1)]^\gamma + \cdots + [p^{n-1}(pq-(p+q)+1)]^\gamma}_{p^{n-1}(p+q-1)-times}\right]$$
$$+ 1\left[\underbrace{[p^nq-1]^\gamma + [p^nq-1]^\gamma + \cdots + [p^nq-1]^\gamma}_{p^{n-1}(pq-(p+q)-1)-times}\right]$$
$$= 2p^{n-1}(p+q-1)[p^{n-1}(pq-(p+q)+1)]^\gamma + p^{n-1}(pq-(p+q)+1)(p^nq-1)^\gamma$$
$$= p^{n-1}(pq-(p+q)+1)\left[2p^{n-1}(p+q-1)\left(p^{n-1}(pq-(p+q)+1)\right)^{\gamma-1} + \left(p^nq-1\right)^\gamma\right]$$

Remark 4.5: The generalized Randic index (or Connectivity index) R_γ, is defined as

$$R_\gamma = R_\gamma(G) = \sum_{i\sim j} \left(d_i d_j\right)^\gamma.$$

Theorem 4.6: If $G_+\left(V(Z_{p^nq}), E(Z_{p^nq})\right)$, then

$$M_2(Z_{p^nq}) = R_1 = \sum_{i\sim j} d_i d_j = p^{n-1}(pq-(p+q)-1)(p^nq-1)\left[\left(p^{n-1}(pq-(p+q)+1)\right)\left(p^{n-1}(pq+p+q-1)-1\right) - (p^nq-1)\frac{p^{n-1}(pq-(p+q)-1)+1}{2}\right]$$

Proof: Suppose that $S = \{a_1, a_2, \cdots, a_\alpha\}, \alpha = p^{n-1}(pq-(p+q)+1), S \subseteq Z_{p^nq}$, where $a_i \in Z_{p^nq}$, $O(a_i) = p^nq, 1 \le i \le \alpha$.

We see that, every $a_i \in S, 1 \le i \le \alpha$ is adjacent to all vertices belonging to Z_{p^nq} except itself. (Since the graph is a simple graph).

$$R_1 = d(a_1)\left[\sum_{i=1}^{p^nq} d_i - d(a_1)\right] + d(a_2)\left[\sum_{i=1}^{p^nq} d_i - \left(d(a_1)+d(a_2)\right)\right] + \overset{\alpha-times}{\cdots} + d(a_\alpha)\left[\sum_{i=1}^{p^nq} d_i - \sum_{i=1}^{\alpha} d(a_i)\right]$$

Now, since $d(a_1) = d(a_2) = \cdots = d(a_\alpha) = d(a) = p^nq-1$ and

$\sum_{i=1}^{p^nq} d_i = 2X$ (X is the size of the graph).

$$\Rightarrow R_1 = d(a)\left[\underbrace{(2X-d(a)) + (2X-2d(a)) + \cdots + (2X-\alpha d(a)}_{\alpha-times}\right] \ldots (1)$$
$$= d(a)\left[\alpha\cdot 2X - \sum_{i=1}^{\alpha} id(a)\right]$$
$$= d(a)\left[\alpha\cdot 2X - d(a)\frac{\alpha(\alpha+1)}{2}\right], \quad \left[\sum_{i=1}^{\alpha} i = \frac{\alpha(\alpha+1)}{2}\right]$$
$$R_1 = \alpha d(a)\left[2X - d(a)\frac{(\alpha+1)}{2}\right]$$
$$= -(pq-(p+q)-1)(p^nq-1)\left[\left(p^{n-1}(pq-(p+q)+1)\right)\left(p^{n-1}(pq+p+q-1)-1\right) - (p^nq-1)\frac{p^{n-1}(pq-(p+q)-1)+1}{2}\right]$$

So, in general, we get by Eg(1).

$$R_\gamma = (d(a))^\gamma\left[\underbrace{(2X-d(a))^\gamma + (2X-2d(a))^\gamma + \cdots + (2X-\alpha d(a))^\gamma}_{\alpha\text{-times}}\right]$$
$$= (d(a))^\gamma \sum_{i=1}^{\alpha} (2X - id(a))^\gamma$$
$$R_\gamma = (p^nq-1)^\gamma \sum_{i=1}^{p^{n-1}(pq-(p+q)+1)} \left[\left(p^{n-1}(pq-(p+q)+1)\right)\left(p^{n-1}(pq+p+q-1)-1\right) - i\left(p^nq-1\right)\right]^\gamma$$

In particular:

(1) $\gamma = -1 \to R_{-1} = \sum_{i\sim j} \frac{1}{d_i d_j}$

$$= \frac{1}{(p^nq-1)} \sum_{i=1}^{p^{n-1}(pq-(p+q)+1)} \left[\left(p^{n-1}(pq-(p+q)+1)\right)\left(p^{n-1}(pq+p+q-1)-1\right) - i(p^nq-1)\right]^{-1}$$

(2) $\gamma = -1/2 \to R_{-1/2} = X(G) = \sum_{i\sim j} \frac{1}{\sqrt{d_i d_j}}$

$$= \frac{1}{\sqrt{(p^nq-1)}} \sum_{i=1}^{p^{n-1}(pq-(p+q)+1)} \left[\left(p^{n-1}(pq-(p+q)+1)\right)\left(p^{n-1}(pq+p+q-1)-1\right) - i(p^nq-1)\right]^{-1/2}$$

(3) $\gamma = 1 \to R_1 = M_2 = \sum_{i\sim j} d_i d_j =$

$$p^{n-1}(pq-(p+q)-1)(p^nq-1)\left[\left(p^{n-1}(pq-(p+q)+1)\right)\left(p^{n-1}(pq+p+q-1)-1\right) - (p^nq-1)\frac{p^{n-1}(pq-(p+q)-1)+1}{2}\right]$$

Remark 4.7: Zhou Trinajstic defined the general Sum-Connectivity index k_μ as

$$k_\mu = k_\mu(G) = \sum_{i\sim j} (d_i + d_j)^\mu.$$

Theorem 4.8: If $G_+\left(V(Z_{p^nq}), E(Z_{p^nq})\right)$, then

$$k_\mu = \sum_{i\sim j} (d_i + d_j)^\mu = p^{n-1}(pq-(p+q)+1)\left[2^{\mu-1}(p^nq-1)^\mu(p^{n-1}(pq-(p+q)+1)-1) + p^{n-1}(p+q-1)\left(p^nq-1+p^{n-1}(pq-(p+q)+1)\right)^\mu\right]$$

Proof: Suppose that $S = \{a_1, a_2, \dots, a_\alpha\}, \alpha = p^{n-1}(pq-(p+q)+1,\ S \subseteq Z_{p^nq}$, $a_i \in Z_{p^nq}$, where $O(a_i) = p^nq, \forall\, 1 \le i \le \alpha$ since every a_i adjacent to all vertices to Z_{p^nq}, $\forall\, 1 \le i \le \alpha$ except itself, then

$$k_\mu = \underbrace{(d(a_1)+d(a_1))^\mu + (d(a_1)+d(a_2))^\mu + \cdots + (d(a_1)+d(a_\alpha))^\mu}_{\alpha-times} - (d(a_1)+d(a_1))^\mu + \underbrace{(d(a_1)+d(0))^\mu + \cdots + (d(a_1)+d(a_{2n}p^{n-1}q))^\mu}_{p^{n-1}((p+q)-1)-times} + \underbrace{(d(a_\alpha)+d(a_1))^\mu + \cdots + (d(a_\alpha)+d(a_\alpha))^\mu}_{\alpha-times} - \left[\underbrace{(d(a_\alpha)+d(a_1))^\mu + \cdots + (d(a_\alpha)+d(a_\alpha))^\mu}_{\alpha-times}\right] + \underbrace{(d(a_\alpha)+d(0))^\mu + \cdots + (d(a_\alpha)+d(a_{2n}p^{n-1}q))^\mu}_{p^{n-1}((p+q)-1)-times}$$

Now, since $d(a_1) = d(a_2) = \cdots = d(a_\alpha) = d(a) = p^nq - 1$

$$d(0) = d(p) = \cdots = d(a_{2n}p^{n-1}q) = p^{n-1}(pq-(p+q)+1)$$

$$k_\mu = [(\alpha-1)(2d(a))^\mu + p^{n-1}(p+q-1)(d(a)+\alpha)^\mu + [(\alpha-2)(2d(a))^\mu + p^{n-1}(p+q-1)(d(a)+\alpha)^\mu] + \cdots + [(\alpha-\alpha)(2d(a))^\mu + p^{n-1}(p+q-1)(d(a)+\alpha)^\mu]$$

$$k_\mu = (2d(a))^\mu \sum_{i=1}^{\alpha} (\alpha - i) + \alpha p^{n-1}(p+q-1)(d(a)+\alpha)^\mu$$

Since $\sum_{i=1}^{\alpha} (\alpha - i) = \frac{\alpha(\alpha-1)}{2}$

$$k_\mu = \alpha\left[\frac{(\alpha-1)}{2}(2d(a))^\mu + p^{n-1}(p+q-1)(d(a)+\alpha)^\mu\right] = p^{n-1}(pq-(p+q)+1)\left[\frac{p^{n-1}(pq-(p+q)+1)-1}{2}(2p^nq-1)^\mu + p^{n-1}(p+q-1)\left(p^nq-1+p^{n-1}(pq-(p+q)+1)\right)^\mu\right]$$

$$k_\mu = p^{n-1}(pq-(p+q)+1)\left[2^{\mu-1}(p^nq-1)^\mu(p^{n-1}(pq-(p+q)+1)-1) + p^{n-1}(p+q-1)\left(p^nq-1+p^{n-1}(pq-(p+q)+1)\right)^\mu\right]$$

In particular

(1) If $\mu = 1 \Rightarrow k_1 = M_1 = Q_2 = p^{n-1}(pq-(p+q)+1)[(p^nq-1)(p^{n-1}(pq-(p+q)+1)-1) + p^{n-1}(p+q-1)(p^nq-1+p^{n-1}(pq-(p+q)-1)]$

(2) If $\mu = -1 \Rightarrow 2k_{-1} = H = \sum_{i\sim j} \frac{2}{d_i+d_j} =$

$$= 2p^{n-1}(pq-(p+q)+1)\left[\frac{1}{4(p^nq-1)}p^{n-1}(pq-(p+q)+1)-1 + \frac{p^{n-1}(p+q-1)}{(p^nq-1+p^{n-1}(pq-(p+q)+1)}\right]$$

(3) If $\mu = -\frac{1}{2} \Rightarrow \chi(Z_{p^nq}) = k_{-1/2} = \sum_{i\sim j} \frac{1}{\sqrt{d_i+d_j}} =$

$$= p^{n-1}(pq-(p+q)+1)\left[\frac{p^{n-1}(pq-(p+q)+1)-1}{(\sqrt{2})^3\cdot\sqrt{p^nq-1}} + \frac{p^{n-1}(p+q-1)}{\sqrt{(p^nq-1+p^{n-1}(pq-(p+q)+1)}}\right]$$

Conclusion

The study uncovers some explicit properties of graphs of a group Z_{p^nq} such as the Eccentric Connectivity index, the first and second Zagreb indices, the Sum-Connectivity index, Randic' index (or Connectivity index), and sum particular special cases indices of a group Z_{p^nq}.

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مفهوم جديد للزمرة المنتهية ($Z_{p^n q}$) على بيان الجمع مع بعض المؤشرات الطوبولوجية

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الملخص

في هذا البحث قدمنا مفهوم جديد لبيان الجمع للزمرة ($Z_{p^n q}$) حيث حاصل جمع رتبة اي راسين متجاورين اكبر من رتبة الزمرة نفسها حيث ان q و p اعداد اولية. وتوصلنا الى بعض النتائج التي تشير ان بيان المجموعة للزمرة يكون متصل ودوري وما الى ذلك من الخصائص مع حساب بعض الخصائص الطوبولوجيا الشهيرة وتعميمها.

الكلمات المفتاحية : بيان جمع الزمرة، بيان اويلر ،بيان هاملتون ،مؤشر زغرب،مؤشر فوركتن .