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Implementing Runge-Kutta Method of Sixth-Order for Numerical Solution of Fuzzy Differential Equations

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Abstract

The article talks about the increasing importance of the practical use of fuzzy differential equations in modeling complex problems in various fields, such as science and engineering, as these differential equations allow for obtaining accurate results for systems that suffer from uncertainty or incomplete knowledge. Fuzzy differential equations are a suitable alternative to ordinary differential equations if nullity and ambiguity are present in the problem. The article presents a new method for solving fuzzy differential equations using Seikkala derivative techniques, which is based on the numerical approach used in Sixth's Rang-Kutta method. A comprehensive analysis of errors is presented, and the method is applied to solve some linear and nonlinear Cauchy problems using MATLAB program to obtain accurate numerical results close to the exact solution. The article hopes that it will help enhance the reader's understanding of these modern techniques in solving fuzzy differential equations, and improve the ability to apply them in practical solutions.

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Introduction

Fuzzy Differential Equations (FDEs) are a type of differential equation that use imprecise values to model systems where some elements are not precisely defined. These equations were first introduced by Lotfi A. Zadeh, a pioneer in Fuzzy Set Theory, in 1965 [1]. Since then, many researchers and scientists have developed and expanded FDEs in various fields, including engineering, physics, and computer science. FDEs differ from ordinary differential equations in that they contain fuzzy terms instead of precise numbers. They can be solved using techniques like fuzzy curves, fuzzy logic, and fuzzy inference, and they are useful for dealing with uncertainty and ambiguity in real-world systems. However, the use of FDEs is still in development, and they remain an active area of research for mathematicians, computer scientists, and other researchers[2-3] [9]. Over the past few years, there has been a considerable amount of scientific research focused on finding both theoretical and numerical solutions for FDEs[5-18]. This study contributes to that research by introducing a solution for a fuzzy differential equation and utilizing computer software to employ the Runge-Kutta method to solve specific cases and obtain an approximate solution.
In recent years, there has been significant interest in the solution of fuzzy differential equations (FDEs) due to their numerous applications in various fields. This study introduces a numerical method for solving FDEs utilizing the Runge-Kutta method with a 6th order Butcher scheme. The efficiency and accuracy of the proposed method are demonstrated through its application to several fuzzy differential equations, showcasing its ability to provide reliable approximate solutions. The results have shown that the sixth-order method is effective in solving FDEs and can provide reliable solutions for various applications. The proposed method presents a promising approach for tackling FDEs in practical scenarios where obtaining exact solutions is challenging.
The present article outlines a numerical approach for solving fuzzy differential equations using the Runge-Kutta Sixth method. The method proposed in this study involves the utilization of a fuzzy derivative to transform the fuzzy differential equation into

an ordinary differential equation. The fuzzy derivative is then approximated using the classical derivative, and the resulting ordinary differential equation is solved using the Sixth's Runge-Kutta method. The method is illustrated through several examples, and the results are compared with existing methods to demonstrate its effectiveness. Overall, the article provides a valuable contribution to the field of fuzzy calculus and differential equations.
The objective of this study is to achieve enhanced and more precise outcomes in solving fuzzy differential equations compared to previous research [17] [21-24]. This is accomplished by introducing a novel method that incorporates Seikkala-derived techniques and a sixth-order numerical approach.The study presented a report on solving linear and non-linear problems, provided a comprehensive analysis of errors, and presented precise numerical results close to the exact solution of the MATLAB program. By improving the accuracy in solving differential equations, it is possible to enhance the ability to use these techniques in practical solutions in various fields such as science and engineering.

1. Preliminaries

The book or article [4] provides a general explanation of fuzzy numbers. Triangular-shaped fuzzy numbers are commonly used, where the base of the triangle is an interval $[\rho, \beta]$, and the vertex is defined by the number α, denoted as η. A triangular fuzzy number $\mathcal{N}$ can be defined by these three values: ρ, α, and β.
The fuzzy triangular numbers are denoted as $\mathcal{N} = (\rho/\alpha/\beta)$. The membership function for a fuzzy triangular number $\mathcal{N} = (\rho/\alpha/\beta)$ is defined as follows:

$$\mathcal{N}(\eta) = \begin{cases} \dfrac{\eta-\rho}{\alpha-\rho} & \rho \leq \eta \leq \alpha \\ \dfrac{\eta-\beta}{\alpha-\beta} & \alpha \leq \eta \leq \beta \end{cases}$$

The membership function specifies the degree to which an element belongs to the fuzzy set $\mathcal{N}$, with values ranging from 0 to 1. The triangular shape of the membership function indicates that elements closer to the vertex α have a higher degree of membership, while elements farther away from α have a lower degree of membership.
When the graph over the intervals $[\rho, \alpha]$ and $[\alpha, \beta]$ is not a straight-line segment, we denote the fuzzy number as $P \approx (\rho/\alpha/\beta)$, indicating that the triangular shape of the fuzzy number is only partially defined by the values of ρ, α, and β. To have a triangular shape, the membership function graph of a fuzzy number must be continuous. Furthermore, the following conditions should be satisfied:

- $\rho \leq \alpha \leq \beta$
- The membership function is non-decreasing on the interval $[\rho, \alpha]$ and non-increasing on the interval $[\alpha, \beta]$.

The single point b is the core of $\mathcal{N}$ if $\mathcal{N} = (\rho/\alpha/\beta)$ or $\mathcal{N} \approx (\rho/\alpha/\beta)$. Let $\mathbb{T}$ be the collection of all fuzzy integers with a triangle or triangle-like shape, and let $u \in \mathbb{T}$.
Triangular-shaped fuzzy numbers and $u \in \mathbb{T}$.
Now, define κ-level set

$$[u]_\kappa = \{\eta | u(\eta) \geq \kappa\}, 0 \leq \kappa \leq 1$$

Which is a closed-bounded interval

$$[u]_\kappa = [u_1(\kappa), u_2(\kappa)]$$

The truth of the following claims is undeniable.
1. $u_1(\kappa)$ Is a left-continuous bounded non-decreasing function on $[0, 1]$.
2. $u_2(\kappa)$ Is a right-continuous bounded non-increasing function on $[0,1]$.
3. $u_1(\kappa) \leq u_2(\kappa)$for all $\kappa \in [0, 1]$. For more details, see [4].

2. A Fuzzy Cauchy Problem

Suppose the fuzzy initial-value problem [17]

$$\begin{cases} \dot{y}(\mu) = f(\mu, y(\mu)), & \mu \in I = [0, \mathbb{T}], \\ y(0) = y_0, \end{cases}$$

Such that y is a fuzzy function of μ, $f(\mu, y(\mu))$ is a fuzzy function of the scrips variable μ and the fuzzy variable y.
$\dot{y}$ is the fuzzy derivative of y and $y(0) = y_0$ is a triangular or a triangular-shaped fuzzy number. We denote the fuzzy function y by $y = [y_1, y_2]$. It means that the κ-level set of $y(\mu)$for $\mu \in [0, \mathbb{T}]$ is

$$[y(\mu)]_\kappa = [y_1(\mu; \kappa), y_2(\mu; \kappa)], \quad \kappa \in (0,1]$$

Also: $[\acute{y}(\mu)]_\kappa = [\acute{y}_1(\mu;\kappa), \acute{y}_2(\mu;\kappa)]$

$$[f(\mu, y(\mu))]_\kappa = [f_1(\mu, y(\mu);\kappa), f_2(\mu, y(\mu);\kappa)]$$

We write: $[f(\mu, y(\mu))] = [f_1(\mu;\kappa), f_2(\mu;\kappa)]$

We get: $\acute{y}_1(\mu;\kappa) = f_1(\mu, y(\mu)) = \mathcal{F}[\mu, y_1(\mu;\kappa), y_2(\mu;\kappa)]$

$$\acute{y}_2(\mu;\kappa) = f_2(\mu, y(\mu)) = \mathcal{G}[\mu, y_1(\mu;\kappa), y_2(\mu;\kappa)]$$

Also, we write

$$[y(\mu_0)]_\kappa = [y_1(\mu_0;\kappa), y_2(\mu_0;\kappa)],$$

We have the membership function by applying the extension principle:

$$f(\mu, y(\mu))(s) = \sup\{y(\mu)(\tau) | s = f(\mu, \tau)\}, \quad s \in \mathbb{R}$$

So fuzzy number $f(\mu, y(\mu))$

$$[f(\mu, y(\mu))]_\kappa = [f_1(\mu, y(\mu); r), f_2(\mu, y(\mu);\kappa)] \ \kappa \in (0,1]$$

Where

$$f_1(\mu, y(\mu);\kappa) = \min\{f(\mu, u) | u \in [y(\mu)]_\kappa\}$$

$$f_2(\mu, y(\mu);\kappa) = \max\{f(\mu, u) | u \in [y(\mu)]_\kappa\}$$

3. Runge-Kutta Method of Order Sixth

Example 1. Suppose the fuzzy initial value problem

$$\begin{cases} \dot{y}(\mu) = f(\mu, y(\mu)), & \mu \in I = [0, \mathbb{T}], \\ y(0) = y_0, \end{cases}$$

The exact solution can be expressed as

$$[\mathcal{Y}(\mu_0)]_\kappa = [\mathcal{Y}_1(\mu_0;\kappa), \mathcal{Y}_2(\mu_0;\kappa)],$$

The approximation solution can be represented as follows:

$$[y(\mu)]_\kappa = [y_1(\mu;\kappa), y_2(\mu;\kappa)],$$

By using the Runge-Kutta method of ordering the Sixth we get

$$y_1(\mu_{n+1};\kappa) - y_1(\mu;\kappa) = \sum_{j=1}^{7} w_j k_{j,1}(\mu_n, y(\mu;\kappa)),$$

$$y_2(\mu_{n+1};\kappa) - y_2(\mu;\kappa) = \sum_{j=1}^{7} w_j k_{j,2}(\mu_n, y(\mu;\kappa)).$$

Where $k_{j,1}, k_{j,2}$ define follow:

$$k_{1,1}(\mu_n, y(\mu;\kappa)) = min\{h.f(\mu, u) | u \in [y_1(\mu;\kappa), y_2(\mu;\kappa)]\},$$

$$k_{1,2}(\mu_n, y(\mu;\kappa)) = max\{h.f(\mu, u) | u \in [y_1(\mu;\kappa), y_2(\mu;\kappa)]\},$$

$$k_{2,1}(\mu_n, y(\mu;\kappa)) = min\left\{h.f\left(\mu + \frac{h}{3}, u\right) \middle| u \in [Q_{1,1}(\mu; y(\mu;\kappa)), Q_{1,2}(\mu; y(\mu;\kappa))]\right\},$$

$$k_{2,2}(\mu_n, y(\mu;\kappa)) = max\left\{h.f\left(\mu + \frac{h}{3}, u\right) \middle| u \in [Q_{1,1}(\mu; y(\mu;\kappa)), Q_{1,2}(\mu; y(\mu;\kappa))]\right\},$$

$$k_{3,1}(\mu_n, y(\mu;\kappa)) = min\left\{h.f\left(\mu + \frac{2h}{3}, u\right) \middle| u \in [Q_{2,1}(\mu; y(\mu;\kappa)), Q_{2,2}(\mu; y(\mu;\kappa))]\right\},$$

$$k_{3,2}(\mu_n, y(\mu;\kappa)) = max\left\{h.f\left(\mu + \frac{2h}{3}, u\right) \middle| u \in [Q_{2,1}(\mu; y(\mu;\kappa)), Q_{2,2}(\mu; y(\mu;\kappa))]\right\},$$

$$k_{4,1}(\mu_n, y(\mu;\kappa)) = min\left\{h.f\left(\mu + \frac{h}{3}, u\right) \middle| u \in [Q_{3,1}(\mu; y(\mu;\kappa)), Q_{3,2}(\mu; y(\mu;\kappa))]\right\},$$

$$k_{4,2}(\mu_n, y(\mu;\kappa)) = max\left\{h.f\left(\mu + \frac{h}{3}, u\right) \middle| u \in [Q_{3,1}(\mu; y(\mu;\kappa)), Q_{3,2}(\mu; y(\mu;\kappa))]\right\},$$

$$k_{5,1}(\mu_n, y(\mu;\kappa)) = min\left\{h.f\left(\mu + \frac{h}{2}, u\right) \middle| u \in [Q_{4,1}(\mu; y(\mu;\kappa)), Q_{4,2}(\mu; y(\mu;\kappa))]\right\},$$

$$k_{5,2}(\mu_n, y(\mu;\kappa)) = max\left\{h.f\left(\mu + \frac{h}{2}, u\right) \middle| u \in [Q_{4,1}(\mu; y(\mu;\kappa)), Q_{4,2}(\mu; y(\mu;\kappa))]\right\},$$

$$k_{6,1}(\mu_n, y(\mu;\kappa)) = min\left\{h.f\left(\mu + \frac{h}{2}, u\right) \middle| u \in [Q_{5,1}(\mu; y(\mu;\kappa)), Q_{5,2}(\mu; y(\mu;\kappa))]\right\},$$

$$k_{6,2}(\mu_n, y(\mu;\kappa)) = max\left\{h.f\left(\mu + \frac{h}{2}, u\right) \middle| u \in [Q_{6,1}(\mu; y(\mu;\kappa)), Q_{6,2}(\mu; y(\mu;\kappa))]\right\},$$

$$k_{7,1}(\mu_n, y(\mu;\kappa)) = min\{h.f(\mu + h, u) | u \in [Q_{6,1}(\mu; y(\mu;\kappa)), Q_{6,2}(\mu; y(\mu;\kappa))]\},$$

$k_{7,2}(\mu_n, y(\mu;\kappa)) = max\{h.f(\mu + h, u) | u \in [Q_{6,1}(\mu; y(\mu;\kappa)), Q_{6,2}(\mu; y(\mu;\kappa))]\}$,

Where in the Runge -Kutta method of order Sixth

$Q_{1,1}(\mu;\kappa) = y_1(\mu;\kappa) + \frac{1}{3}k_{1,1}(\mu_n, y(\mu;\kappa))$,

$Q_{1,2}(\mu;\kappa) = y_2(\mu;\kappa) + \frac{1}{3}k_{1,2}(\mu_n, y(\mu;\kappa))$,

$Q_{2,1}(\mu;\kappa) = y_1(\mu;\kappa) + \frac{2}{3}k_{2,1}(\mu_n, y(\mu;\kappa))$,

$Q_{2,2}(\mu;\kappa) = y_2(\mu;\kappa) + + \frac{2}{3}k_{2,2}(\mu_n, y(\mu;\kappa))$,

$Q_{3,1}(\mu;\kappa) = y_1(\mu;\kappa) + \frac{1}{12}k_{1,1}(\mu_n, y(\mu;\kappa)) + \frac{1}{3}k_{2,1}(\mu_n, y(\mu;\kappa)) - \frac{1}{12}k_{3,1}(\mu_n, y(\mu;\kappa))$,

$Q_{3,2}(\mu;\kappa) = y_2(\mu;\kappa) + \frac{1}{12}k_{1,2}(\mu_n, y(\mu;\kappa)) + \frac{1}{3}k_{2,2}(\mu_n, y(\mu;\kappa)) - \frac{1}{12}k_{3,2}(\mu_n, y(\mu;\kappa))$,

$Q_{4,1}(\mu;\kappa) = y_1(\mu;\kappa) - \frac{1}{16}k_{1,1}(\mu_n, y(\mu;\kappa)) + \frac{9}{8}k_{2,1}(\mu_n, y(\mu;\kappa)) - \frac{3}{16}k_{3,1}(\mu_n, y(\mu;\kappa)) - \frac{3}{8}k_{4,1}(\mu_n, y(\mu;\kappa))$,

$Q_{4,2}(\mu;\kappa) = y_2(\mu;\kappa) - \frac{1}{16}k_{1,2}(\mu_n, y(\mu;\kappa)) + \frac{9}{8}k_{2,2}(\mu_n, y(\mu;\kappa)) - \frac{3}{16}k_{3,2}(\mu_n, y(\mu;\kappa)) + \frac{3}{8}k_{4,2}(\mu_n, y(\mu;\kappa))$,

$Q_{5,1}(\mu;\kappa) = y_1(\mu;\kappa) + \frac{9}{8}k_{2,1}(\mu_n, y(\mu;\kappa)) - \frac{3}{8}k_{3,1}(\mu_n, y(\mu;\kappa)) - \frac{3}{4}k_{4,1}(\mu_n, y(\mu;\kappa)) + \frac{1}{2}k_{5,1}(\mu_n, y(\mu;\kappa))$,

$Q_{5,2}(\mu;\kappa) = y_2(\mu;\kappa) + \frac{9}{8}k_{2,2}(\mu_n, y(\mu;\kappa)) - \frac{3}{8}k_{3,2}(\mu_n, y(\mu;\kappa)) - \frac{3}{4}k_{4,2}(\mu_n, y(\mu;\kappa)) + \frac{1}{2}k_{5,2}(\mu_n, y(\mu;\kappa))$,

$Q_{6,1}(\mu;\kappa) = y_1(\mu;\kappa) + \frac{9}{44}k_{1,1}(\mu_n, y(\mu;\kappa)) - \frac{9}{11}k_{2,1}(\mu_n, y(\mu;\kappa)) + \frac{63}{44}k_{3,1}(\mu_n, y(\mu;\kappa)) + \frac{18}{11}k_{4,1}(\mu_n, y(\mu;\kappa)) - \frac{16}{11}k_{6,1}(\mu_n, y(\mu;\kappa))$,

$Q_{6,2}(\mu;\kappa) = y_2(\mu;\kappa) + \frac{9}{44}k_{1,2}(\mu_n, y(\mu;\kappa)) - \frac{9}{11}k_{2,2}(\mu_n, y(\mu;\kappa)) + \frac{63}{44}k_{3,2}(\mu_n, y(\mu;\kappa)) + \frac{18}{11}k_{4,2}(\mu_n, y(\mu;\kappa)) - \frac{16}{11}k_{5,2}(\mu_n, y(\mu;\kappa))$,

Now, using the initial condition we have

$y_1(\mu_{n+1};\kappa) = y(\mu_n;\kappa) + \frac{11}{120}(k_{1,1}(\mu_n, y(\mu;\kappa)) + k_{7,1}(\mu_n, y(\mu;\kappa))) + \frac{27}{40}((k_{3,1}(\mu_n, y(\mu;\kappa)) + k_{4,1}(\mu_n, y(\mu;\kappa))) - \frac{4}{15}((k_{5,1}(\mu_n, y(\mu;\kappa)) + k_{6,1}(\mu_n, y(\mu;\kappa)))$,

$y_1(\mu_{n+1};\kappa) = y(\mu_n;\kappa) + \frac{11}{120}(k_{1,2}(\mu_n, y(\mu;\kappa)) + k_{7,2}(\mu_n, y(\mu;\kappa))) + \frac{27}{40}((k_{3,2}(\mu_n, y(\mu;\kappa)) + k_{4,2}(\mu_n, y(\mu;\kappa))) - \frac{4}{15}((k_{5,2}(\mu_n, y(\mu;\kappa)) + k_{6,2}(\mu_n, y(\mu;\kappa)))$.

4. Numerical Examples

Consider the following first-order FDE:

$$\begin{cases} \dot{y}(\mu) = f(\mu, y(\mu)), & \mu \in I = [0,1] \\ y(0) = (0.75 + 0.25\kappa, 1.125 - 0.125\kappa), & 0 < \kappa \le 1. \end{cases}$$

By using the Runge-Kutta Method of ordering the Sixth, we get

$$y_1(\mu_{n+1};\kappa) = y_1(\mu_n;\kappa)\left[1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \frac{h^4}{4!} + \frac{h^5}{5!} + \frac{h^6}{6!}\right],$$

$$y_2(\mu_{n+1};\kappa) = y_2(\mu_n;\kappa)\left[1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \frac{h^4}{4!} + \frac{h^5}{5!} + \frac{h^6}{6!}\right].$$

The exact solution can be expressed as:

$$\mathcal{Y}_1(0;\kappa) = y_1(0;\kappa)e^{\mu}, \mathcal{Y}_2(0;\kappa) = y_2(0;\kappa)e^{\mu}$$

At $\mu = 1$,

$$\mathcal{Y}_1(1;\kappa) = [(0.75 + 0.25\kappa)e, (1.125 - 0.125\kappa)e \qquad 0 < \kappa \le 1$$

The exact and approximate solutions were found using the Runge Kutta Method of order Sixth. Moreover, the error was found and plotted at $\mu = 1$ in Fig.1

Table 1. Comparison of the exact and approximated solutions for example 1.

κ	Runge Kutta of order sixth Butcher		Exact Solution		Error	
	$y_1(\mu_i;\kappa)$	$y_2(\mu_i;\kappa)$	$\mathcal{Y}_1(\mu_i;\kappa)$	$\mathcal{Y}_2(\mu_i;\kappa)$	$E_1(\mu_i;\kappa)$	$E_2(\mu_i;\kappa)$
0.1	2.1073614	3.0250833	2.1066684	3.0240885	6.9296722e-04	9.9474326e-04
0.2	2.2433202	2.9910936	2.1746255	2.9901100	7.1532091e-04	9.8356637e-04
0.3	2.2473644	2.9571039	2.2425825	2.9561315	7.3767478e-04	9.7238948e-04
0.4	2.3112996	2.9231142	2.3105395	2.9221521	7.6002856e-04	9.6121259e-04
0.5	2.3792789	2.8891245	2.3784966	2.8881744	7.8238234e-04	9.5003570e-04
0.6	2.4472585	2.8551348	2.4464536	2.8541951	8.0473612e-04	9.3885880e-04
0.7	2.5152378	2.8211451	2.5144107	2.8202174	8.2708990e-04	9.2768192e-04
0.8	2.5152378	2.8211451	2.5823677	2.7862389	8.2708990e-04	9.2768192e-04
0.9	2.5832173	2.7871554	2.6503248	2.7522603	8.4944368e-04	9.1650503e-04
1	2.7191751	2.7191751	2.0387114	3.0580670	8.9415124e-04	8.9415125e-04

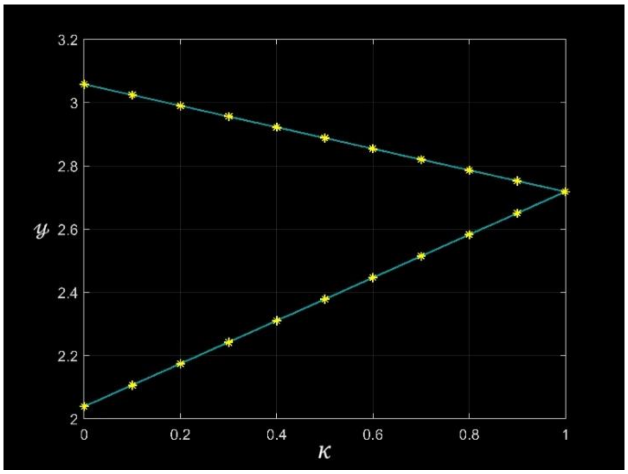

Figure 2. Graphical representation of the solution of the equation presented in Example.1 when $\hbar = 0.01$.

Example 2. Consider the Fuzzy initial value problem

$$\dot{y}(\mu) = \vartheta_1 y(\mu)^2 + \vartheta_2 \qquad y(0) = 0,$$

where $\vartheta_i > 0$, for $\mathrm{i} = 1, 2$ are triangular fuzzy numbers, [21].
The exact given by

$$\mathcal{Y}_1(\mu;\kappa) = \xi_1(\kappa)\tan(\psi_1(\kappa)),$$
$$\mathcal{Y}_2(\mu;\kappa) = \xi_2(\kappa)\tan(\psi_2(\kappa)).$$

$$\xi_1(\kappa) = \sqrt{\vartheta_{2,1}(\kappa)/\vartheta_{1,1}(\kappa)}, \qquad \xi_2(\kappa) = \sqrt{\vartheta_{2,2}(\kappa)/\vartheta_{1,2}(\kappa)},$$

$$\psi_1(\kappa) = \sqrt{\vartheta_{1,1}(\kappa)\vartheta_{2,1}(\kappa)}, \qquad \psi_2(\kappa) = \sqrt{\vartheta_{1,2}(\kappa)\vartheta_{2,2}(\kappa)},$$

Where

$$[\vartheta_1]_\kappa = [\vartheta_{1,1}(\kappa), \vartheta_{1,2}(\kappa)] \quad \text{and} \quad [\vartheta_2]_\kappa = [\vartheta_{2,1}(\kappa), \vartheta_{2,2}(\kappa)].$$

$$\vartheta_{1,1}(\kappa) = 0.5 + 0.5\kappa, \quad \vartheta_{2,1}(\kappa) = 1.5 + 0.5\kappa,$$
$$\vartheta_{2,1}(\kappa) = 0.75 + 0.25\kappa, \quad \vartheta_{2,2}(\kappa) = 1.25 - 0.25\kappa.$$

The κ-level sets of $\dot{y}(\mu)$ are

$$\mathcal{Y}_1(\mu;\kappa) = \vartheta_{2,1}(\kappa)sec^2(\psi_1(\kappa)),$$
$$\mathcal{Y}_2(\mu;\kappa) = \vartheta_{2,2}(\kappa)sec^2(\psi_2(\kappa)).$$

which defines a fuzzy number. We get

$$\mathcal{F}_1(\mu,\mathcal{y};\kappa) = \min\{\vartheta_1 u^2 + \vartheta_2 | u \in [\mathcal{y}_1(\mu,\mathcal{y};\kappa;r), \mathcal{y}_2(\mu;\kappa)], \vartheta_1 \in [\vartheta_{1,1}(\kappa), \vartheta_{1,2}(\kappa)], \vartheta_2 \in [\vartheta_{2,1}(\kappa), \vartheta_{2,2}(\kappa)]\},$$

$$\mathcal{F}_2(\mu,\mathcal{y};\kappa) = \max\{\vartheta_1 u^2 + \vartheta_2 | u \in [\mathcal{y}_1(\mu;\kappa), \mathcal{y}_2(\mu;\kappa)], \vartheta_1 \in [\vartheta_{1,1}(\kappa), \vartheta_{1,2}(\kappa)], \vartheta_2 \in [\vartheta_{2,1}(\kappa), \vartheta_{2,2}(\kappa)]\}.$$

By using the Runge Kutta Method of order Sixth at $t_n,\ 0 \le n \le \mathcal{N}$

$$k_{1,1}(\mu_n;\kappa) = \left(h\vartheta_{1,1}(\kappa)\mathcal{y}_1{}^2 + \vartheta_{2,1}(\kappa)\right),$$

$$k_{1,2}(\mu_n;\kappa) = \left(h\vartheta_{1,2}(\kappa)\mathcal{y}_1{}^2 + \vartheta_{2,2}(\kappa)\right),$$

$$k_{2,1}(\mu_n;\kappa) = \left(h\vartheta_{1,1}(\kappa)\mathcal{Q}_{1,1}{}^2(\mu_n;\kappa) + \vartheta_{2,1}(\kappa)\right),$$

$$k_{2,2}(\mu_n;\kappa) = \left(h\vartheta_{1,2}(\kappa)\mathcal{Q}_{1,2}{}^2(\mu_n;\kappa) + \vartheta_{2,2}(\kappa)\right),$$

$$k_{3,1}(\mu_n;\kappa) = \left(h\vartheta_{1,1}(\kappa)\mathcal{Q}_{2,1}{}^2(\mu_n;\kappa) + \vartheta_{2,1}(\kappa)\right),$$

$$k_{3,2}(\mu_n;\kappa) = \left(h\vartheta_{1,2}(\kappa)\mathcal{Q}_{2,2}{}^2(\mu_n;\kappa) + \vartheta_{2,2}(\kappa)\right),$$

$$k_{4,1}(\mu_n;\kappa) = \left(h\vartheta_{1,1}(\kappa)\mathcal{Q}_{3,1}{}^2(\mu_n;\kappa) + \vartheta_{2,1}(\kappa)\right),$$

$$k_{4,2}(\mu_n;\kappa) = \left(h\vartheta_{1,2}(\kappa)\mathcal{Q}_{3,2}{}^2(\mu_n;\kappa) + \vartheta_{2,2}(\kappa)\right),$$

$$k_{5,1}(\mu_n;\kappa) = \left(h\vartheta_{1,1}(\kappa)\mathcal{Q}_{4,1}{}^2(\mu_n;\kappa) + \vartheta_{2,1}(\kappa)\right),$$

$$k_{5,2}(\mu_n;\kappa) = \left(h\vartheta_{1,2}(\kappa)\mathcal{Q}_{4,2}{}^2(\mu_n;\kappa) + \vartheta_{2,2}(\kappa)\right),$$

$$k_{6,1}(\mu_n;\kappa) = \left(h\vartheta_{1,1}(\kappa)\mathcal{Q}_{5,1}{}^2(\mu_n;\kappa) + \vartheta_{2,1}(\kappa)\right),$$

$$k_{6,2}(\mu_n;\kappa) = \left(h\vartheta_{1,2}(\kappa)\mathcal{Q}_{5,2}{}^2(\mu_n;\kappa) + \vartheta_{2,2}(\kappa)\right),$$

$$k_{7,2}(\mu_n;\kappa) = \left(h\vartheta_{1,1}(\kappa)\mathcal{Q}_{6,1}{}^2(\mu_n;\kappa) + \vartheta_{2,1}(\kappa)\right),$$

$$k_{7,2}(\mu_n;\kappa) = \left(h\vartheta_{1,2}(\kappa)\mathcal{Q}_{6,2}{}^2(\mu_n;\kappa) + \vartheta_{2,2}(\kappa)\right).$$

Where

$$\mathcal{Q}_{1,1}(\mu_n;\kappa) = \mathcal{y}_1(\mu_n;\kappa) + \tfrac{1}{3}k_{1,1}(\mu_n;\kappa),$$

$$\mathcal{Q}_{1,2}(\mu_n;\kappa) = \mathcal{y}_2(\mu_n;\kappa) + \frac{1}{3}k_{1,2}(\mu_n;\kappa),$$

$$\mathcal{Q}_{2,1}(\mu_n;\kappa) = \mathcal{y}_1(\mu_n;\kappa) + \tfrac{2}{3}k_{2,1}(\mu_n;\kappa),$$

$$\mathcal{Q}_{2,2}(\mu_n;\kappa) = \mathcal{y}_2(\mu_n;\kappa) + \tfrac{2}{3}k_{2,2}(\mu_n;\kappa),$$

$$\mathcal{Q}_{3,1}(\mu_n;\kappa) = \mathcal{y}_1(\mu_n;\kappa) + \tfrac{1}{12}k_{1,1}(\mu_n;\kappa) + \tfrac{1}{3}k_{2,1}(\mu_n;\kappa) - \tfrac{1}{12}k_{3,1}(\mu_n;\kappa),$$

$$\mathcal{Q}_{3,2}(\mu_n;\kappa) = \mathcal{y}_2(\mu_n;\kappa) + \tfrac{1}{12}k_{1,2}(\mu_n;\kappa) - \tfrac{1}{3}k_{2,2}(\mu_n;\kappa) + \tfrac{1}{12}k_{3,2}(\mu_n;\kappa),$$

$$\mathcal{Q}_{4,1}(\mu_n;\kappa) = \mathcal{y}_1(\mu_n;\kappa) - \tfrac{1}{16}k_{1,1}(\mu_n;\kappa) + \tfrac{9}{8}k_{2,1}(\mu_n;\kappa) - \tfrac{3}{16}k_{3,1}(\mu_n;\kappa) - \tfrac{3}{8}k_{4,1}(\mu_n;\kappa),$$

$$\mathcal{Q}_{4,2}(\mu_n;\kappa) = \mathcal{y}_2(\mu_n;\kappa) - \tfrac{1}{16}k_{1,2}(\mu_n;\kappa) + \tfrac{9}{8}k_{2,2}(\mu_n;\kappa) + \tfrac{63}{44}k_{3,2}(\mu_n;\kappa) + \tfrac{18}{11}k_{4,2}(\mu_n;\kappa) + \tfrac{16}{11}k_{5,1}(\mu_n;\kappa),$$

$$\mathcal{Q}_{5,1}(\mu_n;\kappa) = \mathcal{y}_1(\mu_n;\kappa) + \tfrac{9}{8}k_{2,1}(\mu_n;\kappa) - \tfrac{3}{8}k_{3,1}(\mu_n;\kappa) - \tfrac{3}{4}k_{4,1}(\mu_n;\kappa) + \tfrac{1}{2}k_{5,1}(\mu_n;\kappa),$$

$$\mathcal{Q}_{5,2}(\mu_n;\kappa) = \mathcal{y}_1(\mu_n;\kappa) + \tfrac{9}{8}k_{2,2}(\mu_n;\kappa) - \tfrac{3}{8}k_{3,2}(\mu_n;\kappa) - \tfrac{3}{4}k_{4,2}(\mu_n;\kappa) + \tfrac{1}{2}k_{5,2}(\mu_n;\kappa)$$

$$\mathcal{Q}_{6,1}(\mu_n;\kappa) = \mathcal{y}_1(\mu_n;\kappa) + \tfrac{9}{44}k_{1,1}(\mu_n;\kappa) - \tfrac{9}{11}k_{2,1}(\mu_n;\kappa) + \tfrac{63}{44}k_{3,1}(\mu_n;\kappa) + \tfrac{18}{11}k_{4,1}(\mu_n;\kappa) - \tfrac{16}{11}k_{6,1}(\mu_n;\kappa),$$

$$\mathcal{Q}_{6,1}(\mu_n;\kappa) = \mathcal{y}_1(\mu_n;\kappa) + \tfrac{9}{44}k_{1,2}(\mu_n;\kappa) - \tfrac{9}{11}k_{2,2}(\mu_n;\kappa) + \tfrac{63}{44}k_{3,2}(\mu_n;\kappa) + \tfrac{18}{11}k_{4,2}(\mu_n;\kappa) - \tfrac{16}{11}k_{6,2}(\mu_n;\kappa),$$

$$\mathcal{y}_1(\mu_{n+1};\kappa) = \mathcal{y}(\mu_n;\kappa) + \tfrac{11}{120}(k_{1,1}(\mu_n;\kappa) + k_{7,1}(\mu_n;\kappa)) + \tfrac{27}{40}((k_{3,1}\ (\ (\mu_n;\kappa) + k_{4,1}(\mu_n;\kappa)) - \tfrac{4}{15}((k_{5,1}(\mu_n;\kappa)+k_{6,1}(\mu_n;\kappa)),$$

$$\mathcal{y}_1(\mu_{n+1};\kappa) = \mathcal{y}(\mu_n;\kappa) + \tfrac{11}{120}(k_{1,2}(\mu_n;\kappa) + k_{7,2}\ (\ (\mu_n;\kappa)) + \tfrac{27}{40}((k_{3,2}(\mu_n;\kappa)) + k_{4,2}(\mu_n;\kappa)) - \tfrac{4}{15}((k_{5,2}(\mu_n;\kappa)+k_{6,2}(\mu_n;\kappa)).$$

Table 2. Comparison of the exact and approximated solutions for example 2.

κ	Runge Kutta of order sixth Butcher		Exact Solution		Error	
	$y_1(\mu_i;\kappa)$	$y_2(\mu_i;\kappa)$	$Y_1(\mu_i;\kappa)$	$Y_2(\mu_i;\kappa)$	$E_1(\mu_i;\kappa)$	$E_2(\mu_i;\kappa)$
0.1	0. 9704689	3. 7880958	0.9078046	3.7881624	3.5662200e-4	6.6576100e-04
0.2	0.9586021	3.2858412	0.9585038	3.2857434	9.8256700e-4	9.7781100e-04
0.3	1.0129561	2.8913855	1.0128729	2.8912856	8.4085800e-4	9.9837500e-04
0.4	1.0715058	2.5913453	1.0714393	2.5919439	6.6581800e-4	5.986600e-04
0.5	1.1348995	2.3416595	1.1348316	2.3415333	6.7935300e-4	1.2628500e-04
0.6	1.2039571	2.1332124	1.2038061	2.1331433	1.5104400e-4	6.9088100e-04
0.7	1.2793052	1.9568565	1.2792807	1.9567137	2.252900e-04	1.4286800e-04
0.8	1.3624589	1.8051566	1.3623814	1.8051545	7.7536800e-4	2.0164000e-05
0.9	1.4544626	1.6783412	1.4545053	1.6733254	4.2745200e-4	8.6955000e-04
1	1.5574458	1.5574458	1.5574077	1.5574077	3.3809800e-5	3.3809000e-05

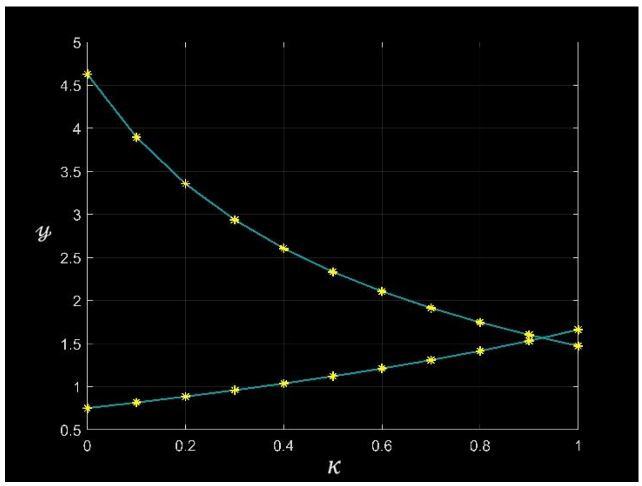

Figure 2. Comparison of the exact solution and approximate solution for example.2 when $\hbar = 0.02$.

5. Conclusion

The analysis of errors in the proposed method highlights its effectiveness, with convergence improving as the step size decreases. The article provides a valuable contribution to the field of fuzzy calculus and differential equations, emphasizing the importance of these modern techniques in modeling and solving complex problems with uncertainty or incomplete knowledge. The method's simplicity and accuracy make it a useful tool for researchers and practitioners in various fields.

Overall, the article presents a clear and comprehensive explanation of the proposed method and its potential practical applications. The method's effectiveness and accuracy make it a valuable addition to the existing approaches for solving fuzzy differential equations. The article provides a solid foundation for future research in the field, and the proposed method can be further improved and optimized for specific applications.

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7. References

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تطبيق طريقة رانج ـ كوتا من الرتبة السادسة لحل للمعادلات التفاضلية المضببة عدديا

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الخلاصة:

يتحدث المقال عن الأهمية المتزايدة للاستخدام العملي للمعادلات التفاضلية الضبابية في نمذجة المشكلات المعقدة في مختلف المجالات، مثل العلوم والهندسة، حيث تتيح هذه المعادلات التفاضلية الحصول على نتائج دقيقة للأنظمة التي تعاني من عدم اليقين أو المعرفة غير المكتملة. وتعد المعادلات التفاضلية الغامضة بديلاً مناسبًا للمعادلات التفاضلية العادية في حالة تواجد العدم والغموض في المشكلة. ويقدم المقال طريقة جديدة لحل المعادلات التفاضلية الضبابية باستخدام تقنيات مشتقة Seikkala، والتي تستند إلى النهج العددي المستخدم في طريقة رانج-كوتا من الرتبة السادسة . كما يتم تقديم تحليل شامل للأخطاء، وتطبيق الطريقة على حل بعض المشاكل Cauchy الخطية وغير الخطية باستخدام برنامج MATLAB للحصول على نتائج عددية دقيقة قريبة من الحل الدقيق. ويأمل المقال بأن يساعد في تعزيز فهم القارئ لهذه التقنيات الحديثة في حل المعادلات التفاضلية الغامضة، وتحسين القدرة على تطبيقها في الحلول العملية.