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Determining the Fundamental Conditions of the Soliton Solution for the New Nonlocal Discrete

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Abstract

The purpose of this research is to present a new model for the nonlocal reductions of the multi-component discrete Manakov system. In particular, the focusing solution is determined based on a special condition of the potential function. This study includes: solving the spectral problem and finding the eigenfunctions and the scattering data. The importance of our study lies in examining the conditions distinguishing the solution called a soliton. There are two cases of the potential functions: single and double excitations, if the Lax operator has no spectrum neither outside nor inside the unit circle then, there is no soliton solution, this happens with a single site case. On the other hand, the two-site case gives two soliton solutions. It is shown that the soliton is more likely to occur at the discrete eigenvalues outside or inside the unit circle, as the excitations are more than one. Each case introduced is supported by numerical simulations.

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Introduction

In applied mathematics and physics, wave dynamics is one of the most engaging and intriguing problems, where modern mathematical physics has been conducted to study solitary wave solutions and the related theory of integrable nonlinear evolution equations in one-dimensional physical systems [1]–[4], which began with the physical observation of the integrable Korteweg–de Vries (KdV) equation:

$$\mathcal{P}_t + \mathcal{P}_{xxx} - 6\mathcal{P}_x\mathcal{P}(x,t) = 0, \tag{1}$$

where, $\mathcal{P}(x,t)$ is the travelling wave solution. Also, the scalar nonlinear Schrödinger equation (NLS) is attracted researchers in recent years. Respectively, the local and nonlocal have the following form:

$$\mathrm{i}\,\mathcal{P}_t(x,t) = -\mathcal{P}_{xx}(x,t) - 2\sigma\,|\mathcal{P}|^2\mathcal{P}(x,t), \tag{2}$$

$$\mathrm{i}\,\mathcal{P}_t(x,t) = -\mathcal{P}_{xx}(x,t) - 2\sigma\,\mathcal{P}^*(-x,t)\,\mathcal{P}^2(x,t), \tag{3}$$

where, a complex value $\mathcal{P}(x,t)$ is a function with two real variables x and t, the symbol $|\ |$ called the norm of a function and in equation (2) represents $\mathcal{P}(x,t)\mathcal{P}^*(x,t) = |\mathcal{P}|^2(x,t)$. When $\sigma = +1$, equation (2 or 3) corresponds to the focusing case solution, while the defocusing case solution is when $\sigma = -1$, and $(*)$, represents the conjugate of a function. The nonlinear term in the NLS equation $\mathcal{N}(x,t) = \mathcal{P}(x,t)\mathcal{P}^*(-x,t)$, represents a self-induced potential that satisfies the PT-symmetry condition $\mathcal{N}(x,t) = \mathcal{N}^*(-x,t)$. Replacing $x \to -x$ and $t \to -t$, the complex conjugate on equation (3) remains invariant [5]. This

applies to the field of PT symmetry of quantum mechanics, PT-symmetric of optics, and research activities are currently taking place in these areas [6]–[8].
The discrete NLS (DNLS) equation has two types local and nonlocal are presented in books and papers in the following form [9]–[12]:

$$\mathrm{i}\,\frac{d}{d\tau}\mathcal{P}_n = -(\mathcal{P}_{n+1} - 2\mathcal{P}_n + \mathcal{P}_{n-1}) - \sigma|\mathcal{P}_n|^2(\mathcal{P}_{n+1} + \mathcal{P}_{n-1}), \tag{4}$$

$$\mathrm{i}\,\frac{d}{d\tau}\mathcal{P}_n = -(\mathcal{P}_{n+1} - 2\mathcal{P}_n + \mathcal{P}_{n-1}) - \sigma\mathcal{P}_n\mathcal{P}^*_{-n}(\mathcal{P}_{n+1} + \mathcal{P}_{n-1}), \tag{5}$$

where, $\mathcal{P}_n(\tau)$ is a complex function in equation (4-5). Here τ is the variable related to time and $n \in \mathbb{N}$ represents an infinite lattice. The nonlocal nonlinear term in equation (5) $\mathcal{P}_n\mathcal{P}^*_{-n}$ is replaced by the local nonlinear term $|\mathcal{P}_n|^2$ in equation (4) [5]. The DNLS equation is also in the group of PT-symmetric model [13], where the self-induced potential $\mathcal{N}_n = \mathcal{P}_n\mathcal{P}^*_{-n}$. According to classical optics, this condition guarantees that the equation is invariant under parity and time symmetry [14]. The DNLS equation is over a century old [15]–[20], but it is still actively studied by mathematicians and physicists, where it is a fundamental equation of quantum mechanics and is at the heart of several research areas, including nonlinear optics, quantum computing, and theoretical physics. It is a nonlinear, dispersive partial differential equation that describes the evolution of a wave function in time and space which means that small changes in its parameters can lead to changes in its solutions. A multi-component system (MCS) of partial differential equations (PDEs) that serves as a generalization of the scalar NLS equation is called the Manakov vector NLS system (MVNLS) [21]. This system is a member of a family of integrable systems. It has become a prominent model due to its wide range of applications in various fields of physics and mathematics, such as optical fiber propagation, plasma physics, and Bose-Einstein condensate [22], [23].
Solitary wave solution and also called soliton is a type of solution that can be obtained from an integrable system. Soliton is experimentally discovered in many fields for instance chemistry, biology, and physics phenomenon: plasma and nonlinear optics and many others. Bright and dark solitons are some types of solutions such as NLS, DNLS, and MVNLS. Numerical and analytical methods are used to find the soliton solution, like the variational iteration method and inverse scattering method [24] [25].

In this paper, we obtained another type of nonlocal MCS which is also a member of the family of discrete MVNLS (DMVNLS).
The paper is arranged as follows: Section 2 includes an exposition of the Lax representations, compatibility condition, the general class of the DMNKS type equations, and the nonlocal symmetry (involution) case. The direct scattering problem is discussed in section 3, which covers the Jost solutions and the scattering matrix. We conclude in Section 3 with two examples, corresponding to the nonlocal DMVNLS equation symmetrically and asymmetrically concerning the potential of barriers.

1. Preliminaries

a. Lax representation

It is possible to represent the DMVNLS as a compatibility condition for two linear operators known as the Lax pair L_n and M_n, where the first operator L_n, which is also called the spectral problem:

$$\Psi_{n+1} = L_n\Psi_n, \quad L_n = (Z + Q_n), \tag{6}$$

here,

$$Z = \left(\begin{array}{c|cc} z & 0 & 0 \\ \hline 0 & z^{-1} & 0 \\ 0 & 0 & z^{-1} \end{array}\right), \qquad Q_n = \left(\begin{array}{c|cc} 0 & \mathcal{P}_{1,n} & \mathcal{P}_{2,n} \\ \hline \mathcal{R}_{1,n} & 0 & 0 \\ \mathcal{R}_{2,n} & 0 & 0 \end{array}\right), \tag{7}$$

where $\mathcal{P}_{i,n}$ and $\mathcal{R}_{i,n}$, $i = 1,2$ are complex value functions. The second operator is M_n, so the time evolution is the second operator of $\Psi_{n,t}$, where, Ψ_n is eigenfunctions of M_n and L_n:

$$M_n = \left(\frac{z - z^{-1}}{2}\right)^2 \mathcal{D} + \frac{\mathcal{D}}{2}(Z^{-1}Q_n - ZQ_{n-1}) - \frac{1}{2}\mathcal{D}Q_nQ_{n-1}, \tag{8}$$

where,

$$\mathcal{D}=\left(\begin{array}{c|cc}-1 & 0 & 0\\ \hline 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right), \tag{9}$$

then, the compatibility condition between the two operators L_n and M_n is:

$$M_{n+1}L_n = L_{n,t} + L_n\, M_n \tag{10}$$

2.2 The eigenvalue problem

The eigenfunctions are defined by the following boundary conditions, when $-\infty < n < \infty$, the functions $\mathcal{P}_{i,n}$ and $\mathcal{R}_{i,n}$, tends to zero, as a result, equation (6) satisfies:

$$\Psi_n(z) \sim \left(\begin{array}{c|cc} z^n & 0 & 0\\ \hline 0 & z^{-n} & 0\\ 0 & 0 & z^{-n}\end{array}\right), \quad \text{as} \quad n\to+\infty \tag{11}$$

$$\Phi_n(z) \sim \left(\begin{array}{c|cc} z^n & 0 & 0\\ \hline 0 & z^{-n} & 0\\ 0 & 0 & z^{-n}\end{array}\right). \quad \text{as} \quad n\to-\infty \tag{12}$$

The pairs $\Phi_n(z) = (\phi_n^+, \boldsymbol{\phi}_n^-)$ and $\Psi_n(z) = (\psi_n^-, \boldsymbol{\psi}_n^+)$ have linear combinations,

$$\Phi_n(z) = \Psi_n(z)T(z),\ \ T(z) = \begin{pmatrix} a^+(z) & -\boldsymbol{b}^-(\boldsymbol{z})\\ \boldsymbol{b}^+(\boldsymbol{z}) & \boldsymbol{a}^-(\boldsymbol{z})\end{pmatrix} \quad \text{when} \quad |z|=1 \tag{13}$$

the coefficients of these linear combinations depend on z, with the relation holding on $|z| = 1$, the coefficients $a^+(z), 1\times 1, \boldsymbol{a}^-(\boldsymbol{z}), 2\times 2,\ \boldsymbol{b}^+(\boldsymbol{z}), 2\times 1$ and $\boldsymbol{b}^-(\boldsymbol{z}), 1\times 2$ matrices. They called the scattering dates. The first component of equation (13) is

$$(\Phi_1)_n = a^+(z)(\Psi_1)_n + \boldsymbol{b}^+(\boldsymbol{z})(\boldsymbol{\Psi_2})_{\boldsymbol{n}}, \tag{14}$$

here $(\Phi_1)_n$ is the first component of the Φ_n matrix. Similarly, we can define $(\Psi_1)_n$ and $(\boldsymbol{\Psi_2})_{\boldsymbol{n}} = 2\times 2$ matrix. As we mentioned before, $\Phi_n \sim Z^n\ as\ n$ approches $-\infty$, then $(\Phi_1)_n \sim z^n(1,0,0)^T$, where T denotes the matrix transpose. For decay, $(\Phi_1)_n$ approaches 0 when n approaches $-\infty$, we require $|z|>1$. Also, we can write equation (13) when n approches $+\infty$.

$$\Psi_n = ((\Psi_1)_n, (\boldsymbol{\Psi_2})_{\boldsymbol{n}}) \sim \begin{pmatrix} z^n & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{z^{-n}}\end{pmatrix}, \tag{15}$$

where, $\boldsymbol{z^{-n}} = 2\times 2$ matrix. Hence, if $|z|>1$ then $(\Psi_1)_n$ approaches $+\infty$ and $(\boldsymbol{\Psi_2})_{\boldsymbol{n}}$ approaches 0 ,as $n\to+\infty$. Now, from equation (13), this leads to $(\Phi_1)_n$ approaches $+\infty$ as n approaches $+\infty$, with one condition, when $a^+(z) = 0$.

Now, for $N \in \mathbb{N}$, let $z_1, z_2, \dots, z_N$ be solutions of the following equation

$$a^+(z_k) = 0, \quad k = 1,2,\dots,N \tag{16}$$

such that $|z_k| > 1$. Then, $(\Phi_1)_n(z_k) = \boldsymbol{b}^+(\boldsymbol{z_k})(\boldsymbol{\Psi_2})_{\boldsymbol{n}}(\boldsymbol{z_k}), k = 1,2,\dots,N$. From this, it follows that $(\Phi_1)_n(z_k)$ decays $((\Phi_1)_n(z_k)$ approches $0\ as\ n$ approches $\pm\infty)$. Then, we can approve that $(\Phi_1)_n(z_k)$ is an eigenfunction with the corresponding eigenvalue z_k. Same discussion for $\boldsymbol{a}^-(\boldsymbol{z})$, but in this case, we can calculate it by taking the det $(\boldsymbol{a}^-(\boldsymbol{z}))$ because it is a matrix of 2×2.

2. The multi-component discrete system

In this section, the MC discrete system was presented first for the continuous case (scalar) (17-18). Generalization of equations (2-3) are the two-component vector NLS equation of the local and nonlocal scalar MVNLS systems respectively [21], [26]–[28]

$$\mathrm{i}\,\mathcal{P}_t(x,t) = -\mathcal{P}_{xx}(x,t) - 2\sigma\,(|\mathcal{P}|^2 + |\mathcal{R}|^2)\mathcal{P}(x,t), \tag{17}$$

$$\mathrm{i}\,\mathcal{R}_t(x,t) = -\mathcal{R}_{xx}(x,t) - 2\sigma\,(|\mathcal{P}|^2 + |\mathcal{R}|^2)\mathcal{R}(x,t), \tag{18}$$

$$\mathrm{i}\,\mathcal{P}_t(x,t) = -\mathcal{P}_{xx}(x,t) - 2\sigma\left(\mathcal{P}(x,t)\mathcal{P}^*(-x,t) + \mathcal{R}(x,t)\mathcal{R}^*(-x,t)\right)\mathcal{P}(x,t), \tag{19}$$

$$\mathrm{i}\,\mathcal{R}_t(x,t) = -\mathcal{R}_{xx}(x,t) - 2\sigma\left(\mathcal{P}(x,t)\mathcal{P}^*(-x,t) + \mathcal{R}(x,t)\mathcal{R}^*(-x,t)\right)\mathcal{R}(x,t), \tag{20}$$

Following the generalization of the NLS equations (15-18), the generalization of the local DMVNLS system is represented in [29] which has this formula

$$\mathrm{i}\,\frac{d}{d\tau}\mathcal{P}_n = -\frac{1}{2}(\mathcal{P}_{n+1} - 2\mathcal{P}_n + \mathcal{P}_{n-1}) - \sigma\,(|\mathcal{P}_n|^2 + |\mathcal{R}_n|^2)\left(\frac{\mathcal{P}_{n+1} + \mathcal{P}_{n-1}}{2}\right), \tag{21}$$

$$\mathrm{i}\,\frac{d}{d\tau}\mathcal{R}_n = -\frac{1}{2}(\mathcal{R}_{n+1} - 2\mathcal{R}_n + \mathcal{R}_{n-1}) - \sigma\,(|\mathcal{P}_n|^2 + |\mathcal{R}_n|^2)\left(\frac{\mathcal{R}_{n+1} + \mathcal{R}_{n-1}}{2}\right), \tag{22}$$

while the form of the nonlocal generalization of the DMVNLS system (DMVNLS) as a coupled two-components equation is [25]:

$$\mathrm{i}\,\frac{d}{d\tau}\mathcal{P}_n = -(\mathcal{P}_{n+1} - 2\mathcal{P}_n + \mathcal{P}_{n-1}) - \sigma\left(\mathcal{P}_n(\tau)\mathcal{P}^*_{-n}(\tau) + \mathcal{R}_n(\tau)\mathcal{R}^*_{-n}(\tau)\right)(\mathcal{P}_{n+1} + \mathcal{P}_{n-1}), \tag{23}$$

$$\mathrm{i}\,\frac{d}{d\tau}\mathcal{R}_n = -(\mathcal{R}_{n+1} - 2\mathcal{R}_n + \mathcal{R}_{n-1}) - \sigma\left(\mathcal{P}_n(\tau)\mathcal{P}^*_{-n}(\tau) + \mathcal{R}_n(\tau)\mathcal{R}^*_{-n}(\tau)\right)(\mathcal{R}_{n+1} + \mathcal{R}_{n-1}). \tag{24}$$

The solution of the scalar NLS and DMVNLS is studied in [26] and [25] numerically and analytically.

3. The new reduction nonlocal discrete Manakov system (NE-NDMS)

We obtained another type of nonlocal two-component equation which is also a member of the family of DMVNLS, and we estimate the type of solution based on the location of the discrete eigenvalues. The goal of this work is to apply nonlocal reduction and to calculate the necessary condition for the creation of soliton solutions. This requires the evaluation of the discrete eigenvalues which correspond to the integrable DMVNLS system. We have introduced different cases of the potential functions of barriers concerning nonlocal DMVNLS.

The NE-NDMS comes from setting the relation as $\mathcal{R}_{2,n} = -\rho\mathcal{P}^*_{1,-n}$ and $\mathcal{R}_{1,n} = -\rho\mathcal{P}^*_{2,-n}$ on the (1-4), so the matrix Q_n in equation (7) becomes

$$M_n = \left(\begin{array}{c|cc} 0 & \mathcal{P}_{1,n} & \mathcal{P}_{2,n} \\ \hline -\rho\mathcal{P}^*_{2,-n} & 0 & 0 \\ -\rho\mathcal{P}^*_{1,-n} & 0 & 0 \end{array}\right)$$

Therefore, the NE-NDMS has the form

$$\mathrm{i}\frac{d\mathcal{P}_{1,n}}{d\tau} = -\frac{1}{2}\left(\mathcal{P}_{1,n+1} - 2\mathcal{P}_{1,n} + \mathcal{P}_{1,n-1}\right) - \rho\left(\mathcal{P}_{1,n}\mathcal{P}^*_{2,-n} + \mathcal{P}_{2,n}\mathcal{P}^*_{1,-n}\right)\left(\frac{\mathcal{P}_{1,n+1} + \mathcal{P}_{1,n-1}}{2}\right), \tag{25}$$

$$\mathrm{i}\frac{d\mathcal{P}_{2,n}}{d\tau} = -\frac{1}{2}\left(\mathcal{P}_{2,n+1} - 2\mathcal{P}_{2,n} + \mathcal{P}_{2,n-1}\right) - \rho\left(\mathcal{P}_{1,n}\mathcal{P}^*_{2,-n} + \mathcal{P}_{2,n}\mathcal{P}^*_{1,-n}\right)\left(\frac{\mathcal{P}_{2,n+1} + \mathcal{P}_{2,n-1}}{2}\right), \tag{26}$$

Example 1

The initial condition for the system (6) corresponds to the two-components $\mathcal{P}_{1,n}$ and $\mathcal{P}_{2,n}$ of the NE-NDMS equations (25-26) (single site) is

$$\mathcal{P}_{1,n} = \begin{cases} U_0, & n = -1 \\ 0, & otherwise \end{cases}, \qquad \mathcal{P}_{2,n} = \begin{cases} V_0, & n = 1 \\ 0, & otherwise \end{cases},$$

Solution: recall the spectral problem equation (6),

$$\Psi_{n+1} = (Z + M_n)\Psi_n,$$

where,

$$M_n = \begin{pmatrix} 0 & \mathcal{P}_{1,n} & \mathcal{P}_{2,n} \\ -\mathcal{P}^*_{2,-n} & 0 & 0 \\ -\mathcal{P}^*_{1,-n} & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} z & 0 & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & z^{-1} \end{pmatrix},$$

$$M_{-1} = \begin{pmatrix} 0 & U_0 & 0 \\ -V_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 0 & V_0 \\ 0 & 0 & 0 \\ -U_0 & 0 & 0 \end{pmatrix},$$

$$\Psi_{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} z & 0 & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & z^{-1} \end{pmatrix},$$

when $n = -1$, $\Psi_0 = (Z + M_{-1})\Psi_{-1}$

$$\Psi_0 = \begin{pmatrix} z & U_0 & 0 \\ -V_0 & z^{-1} & 0 \\ 0 & 0 & z^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} z & U_0 & 0 \\ -V_0 & z^{-1} & 0 \\ 0 & 0 & z^{-1} \end{pmatrix},$$

when $n = 0$, $\Psi_1 = (Z + M_0)\Psi_0$

$$\Psi_1 = \begin{pmatrix} z & 0 & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & z^{-1} \end{pmatrix} \begin{pmatrix} z & U_0 & 0 \\ -V_0 & z^{-1} & 0 \\ 0 & 0 & z^{-1} \end{pmatrix} = \begin{pmatrix} z^2 & zU_0 & 0 \\ -z^{-1}V_0 & z^{-2} & 0 \\ 0 & 0 & z^{-2} \end{pmatrix},$$

when $n = 1$, $\Psi_2 = (Z + M_1)\Psi_1$

$$\Psi_2 = \begin{pmatrix} z & 0 & V_0 \\ 0 & z^{-1} & 0 \\ -U_0 & 0 & z^{-1} \end{pmatrix} \begin{pmatrix} z^2 & zU_0 & 0 \\ -z^{-1}V_0 & z^{-2} & 0 \\ 0 & 0 & z^{-2} \end{pmatrix} = \begin{pmatrix} z^3 & z^2U_0 & z^{-2}V_0 \\ -z^{-2}V_0 & z^{-3} & 0 \\ -z^2U_0 & -zU_0^2 & z^{-3} \end{pmatrix}.$$

To find the $a^+(z)$, we need to compare it with equation (14), then

$$a^+(z) = 1 \neq 0.$$

Similarly, when we need to calculate the $\boldsymbol{a}^-(\boldsymbol{z}) = 1 \neq 0$. For both functions, there are no eigenvalues which mean there is no soliton solution. Figure (1) shows that there are no discrete eigenvalues outside nor inside the unit circle $|z| = 1$, while Figure (2) shows that there is no solitary wave solution (soliton solution).

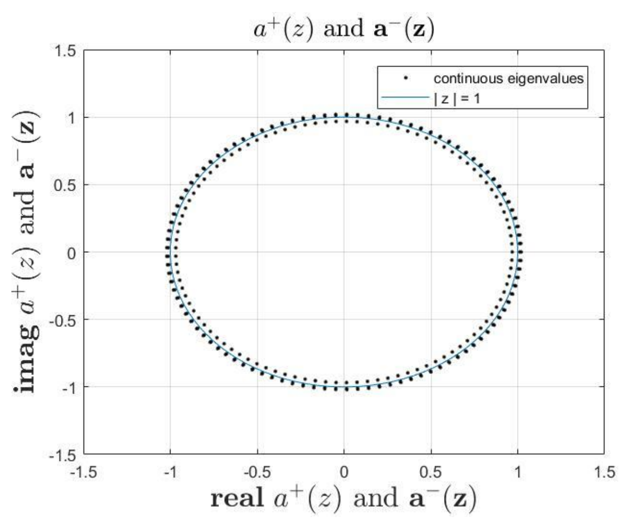
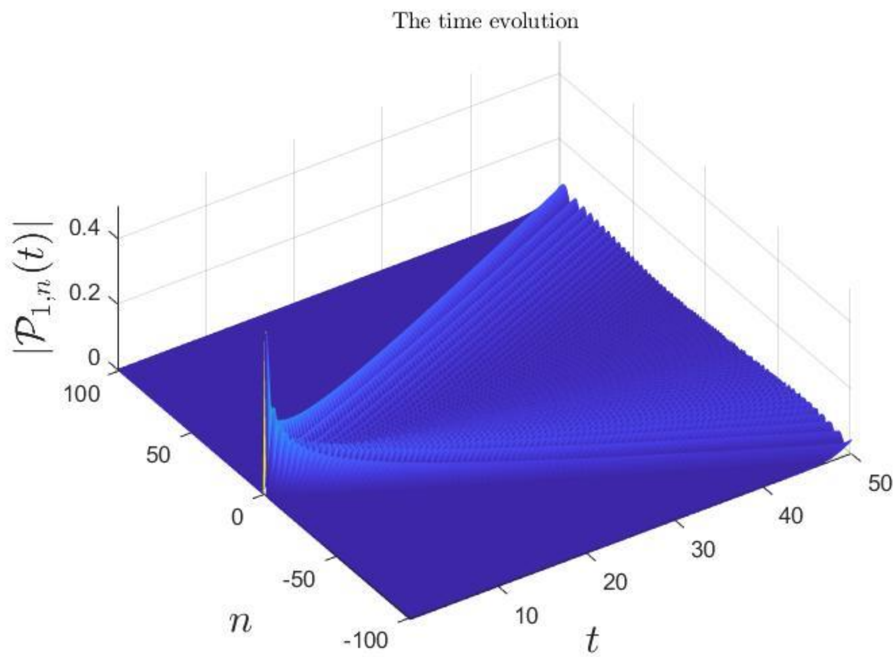
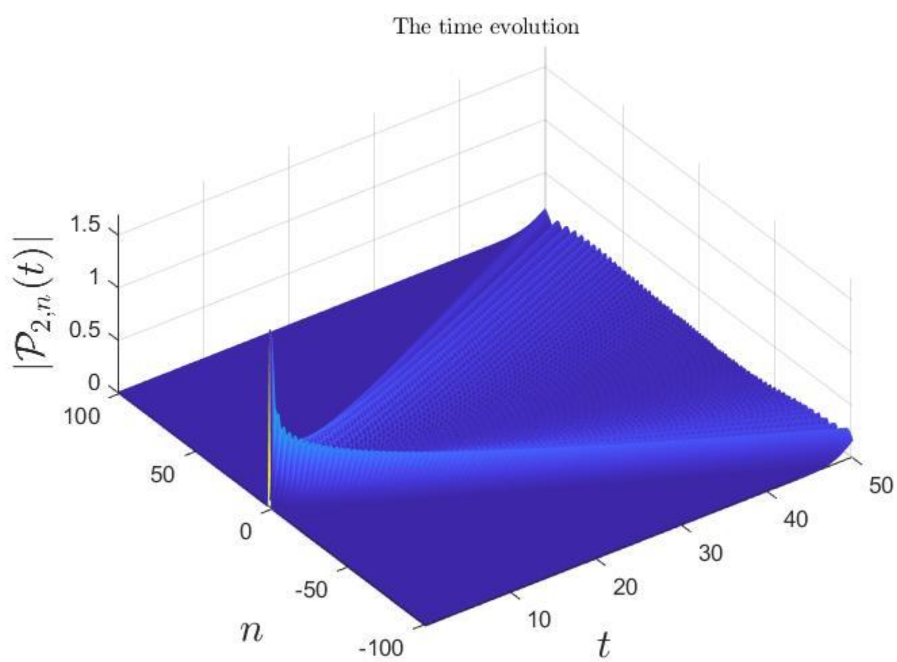


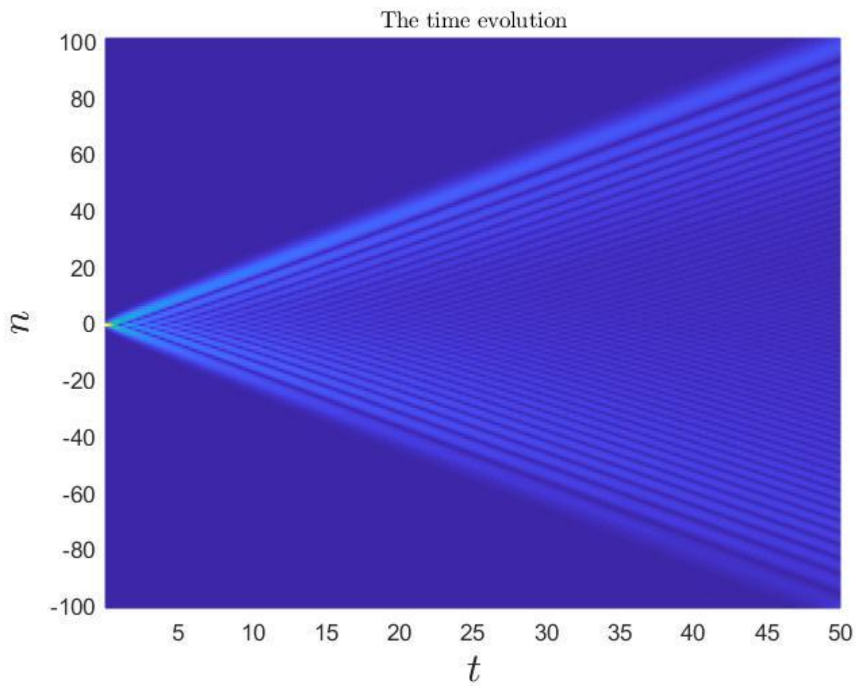
Figure 1. The plot of the scattering data $a^+(z)$ and $\boldsymbol{a}^-(\boldsymbol{z})$. This figure shows, the dots around the unit circle $|z| = 1$, which are continuous eigenvalues. As a result, we cannot predict a soliton solution.


The time evolution
$|\mathcal{P}_{1,n}(t)|$
0.4
0.2
0
100
50
0
-50
-100
n
10
20
30
40
50
t

(a)


The time evolution
$|\mathcal{P}_{2,n}(t)|$
1.5
1
0.5
0
100
50
0
-50
-100
n
10
20
30
40
50
t

(b)



(c)

Figure 2. Figure **(a)** and Figure **(b)** represent the plot of the $|\mathcal{P}_{1,n}(t)|$ and $|\mathcal{P}_{2,n}(t)|$, respectively. The vision of the figures **(a)** and **(b)**, from the front. While the vision in Figure **(c)** is from the top of Figure **(a)**.

Example 2

The initial condition for the system (6) corresponds to the two-components $\mathcal{P}_{1,n}$ and $\mathcal{P}_{2,n}$ of the NE-NDMS equations (25-26) (two site excitations) is

$$\mathcal{P}_{1,n} = \begin{cases} k, & n = -1 \\ l, & n = 1 \end{cases}, \qquad \mathcal{P}_{2,n} = \begin{cases} l, & n = -1 \\ k, & n = 1 \end{cases}$$

Solution: Recall the scattering problem equation (6)

$$\Psi_{n+1} = (Z + M_n)\Psi_n,$$

where,

$$M_n = \begin{pmatrix} 0 & \mathcal{P}_{1,n} & \mathcal{P}_{2,n} \\ -\mathcal{P}^*_{2,-n} & 0 & 0 \\ -\mathcal{P}^*_{1,-n} & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} z & 0 & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & z^{-1} \end{pmatrix},$$

$$M_{-1} = \begin{pmatrix} 0 & k & l \\ -k & 0 & 0 \\ -l & 0 & 0 \end{pmatrix}, \quad M_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & l & k \\ -l & 0 & 0 \\ -k & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Psi_{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

when $n = -1$, $\Psi_0 = (Z + M_{-1})\Psi_{-1}$,

$$\Psi_0 = \begin{pmatrix} z & k & l \\ -k & z^{-1} & 0 \\ -l & 0 & z^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} z & k & l \\ -k & z^{-1} & 0 \\ -l & 0 & z^{-1} \end{pmatrix},$$

when, $n = 0$, $\Psi_1 = (Z + M_0)\Psi_0$,

$$\Psi_1 = \begin{pmatrix} z & 0 & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & z^{-1} \end{pmatrix}\begin{pmatrix} z & k & l \\ -k & z^{-1} & 0 \\ -l & 0 & z^{-1} \end{pmatrix} = \begin{pmatrix} z^2 & zk & zl \\ -z^{-1}k & z^{-2} & 0 \\ -z^{-1}l & 0 & z^{-2} \end{pmatrix},$$

when, $n = 1$, $\Psi_2 = (Z + M_1)\Psi_1$,

$$\Psi_2 = \begin{pmatrix} z & l & k \\ -l & z^{-1} & 0 \\ -k & 0 & z^{-1} \end{pmatrix}\begin{pmatrix} z^2 & zk & zl \\ -z^{-1}k & z^{-2} & 0 \\ -z^{-1}l & 0 & z^{-2} \end{pmatrix},$$

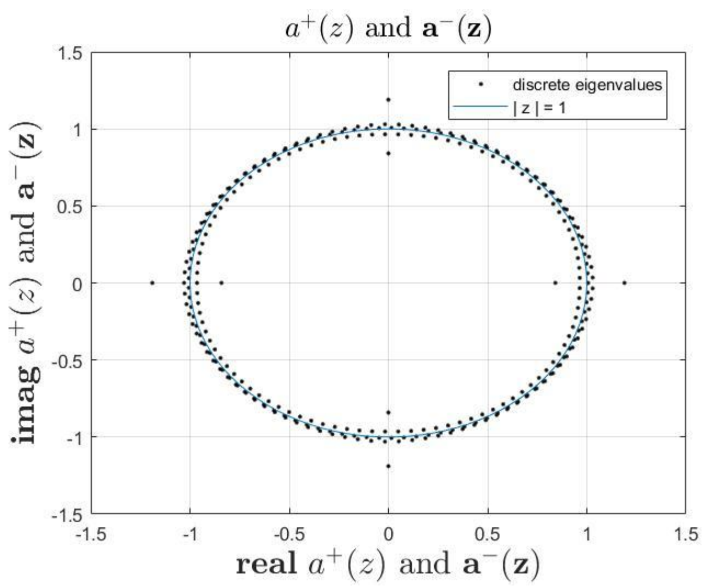
$$= \begin{pmatrix} z^3 - 2lkz^{-1} & z^2k + lz^{-2} & z^2l + kz^{-2} \\ -lz^2 - kz^{-2} & -lkz + z^{-3} & -l^2z \\ -kz^2 - lz^{-2} & -k^2z & -klz + z^{-3} \end{pmatrix},$$

when $n = 2$, $\Psi_3 = (Z + M_2)\Psi_2$,

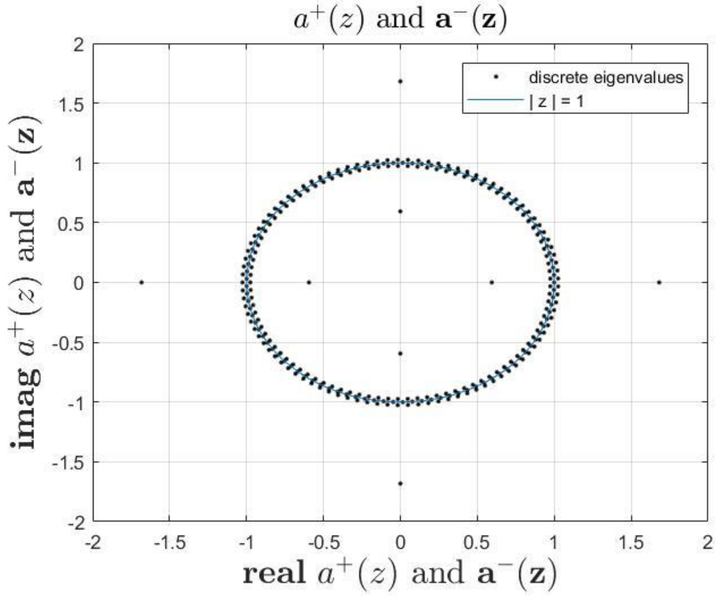
$$\Psi_3 = \begin{pmatrix} z & 0 & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & z^{-1} \end{pmatrix}\begin{pmatrix} z^3 - 2lkz^{-1} & z^2k + lz^{-2} & z^2l + kz^{-2} \\ -lz^{-2} - kz^{-2} & -lkz + z^{-3} & -l^2z \\ -kz^{-2} - lz^{-2} & -k^2z & -klz + z^{-3} \end{pmatrix},$$

$$\Psi_3 = \begin{pmatrix} z^4 - 2lk & z^3k + lz^{-1} & z^3l + kz^{-1} \\ -lz - kz^{-3} & -lk + z^{-4} & -l^2 \\ -kz - lz^{-3} & -k^2 & -kl + z^{-4} \end{pmatrix}.$$

The transmission coefficient is $a^+(z) = z^{-4}(z^4 - 2lk)$, for $z^{-1} \neq 0$, such that $a^+(z) = 0$ if $z^4 - 2lk = 0$ by solving $a^+(z)$ for z, we obtain four roots, $z^+_{1,2} = \pm\sqrt{+\sqrt{2lk}}$, $z^+_{3,4} = \pm\sqrt{-\sqrt{2lk}}$, under the condition $|z| > 1$, verify $2lk > 0$, from which follows that $z^+_{1,2,3,4}$ are discrete eigenvalues, and the transmission coefficient is $\boldsymbol{a}^-(\boldsymbol{z}) = z^{-8}(1 - 2lkz^4)$, such that $\boldsymbol{a}^-(z) = 0$ if $1 - 2lkz^4 = 0$, then solving for z, we get four roots, $z^-_{1,2} = \pm\sqrt{+\sqrt{\frac{1}{2lk}}}$, $z^-_{3,4} = \pm\sqrt{-\sqrt{\frac{1}{2lk}}}$, from which follows that $z^+_{1,2,3,4}$ are discrete eigenvalues. In this example, when both values of l and k are < 1, then we do not have discrete eigenvalues as in example 1. So, we do not have soliton solution. However, we have two soliton solutions when either the values of l and $k > 1$ or at least one of them. The following figures display all these cases.

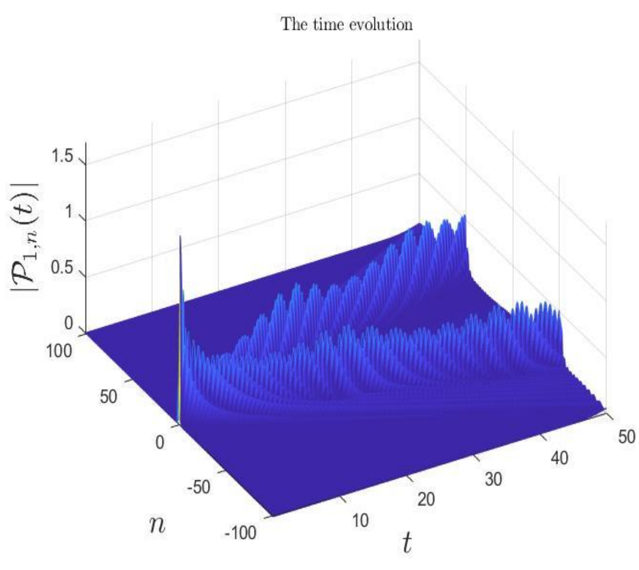


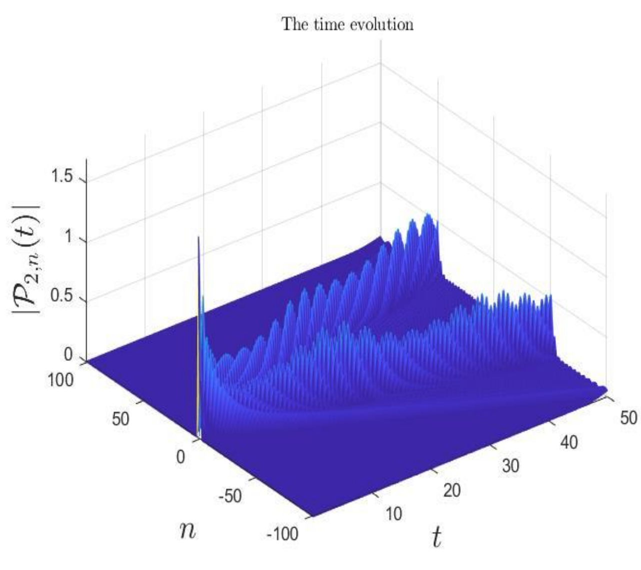
(a)



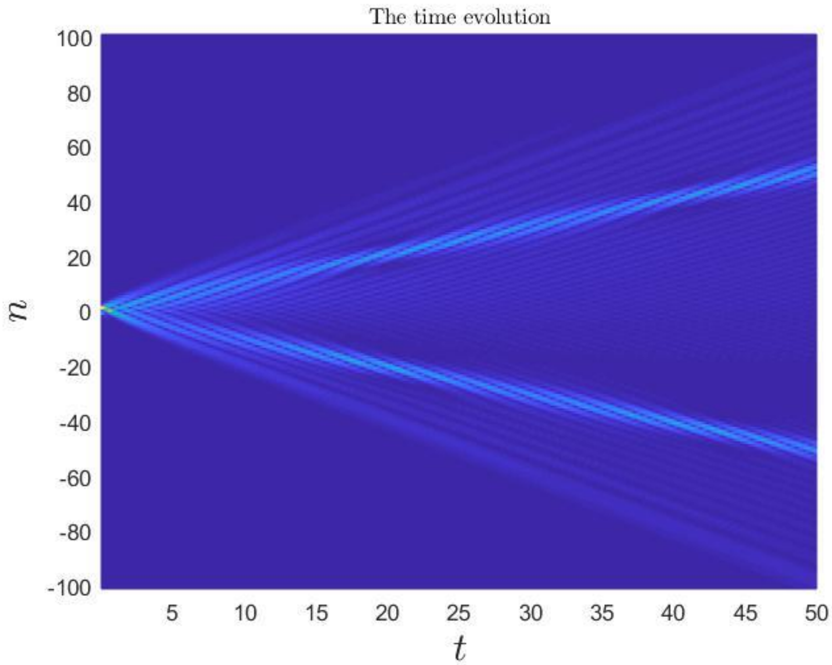
(b)

Figure 3. Figure **(a)** and figure **(b)** shows the discrete eigenvalues outside and inside the unit circle.





(a) **(b)**



(c)

Figure 4. Figures **(a)** and **(b)** represent the plot of the $|\mathcal{P}_{1,n}(t)|$ and $|\mathcal{P}_{2,n}(t)|$. These figures show equations (25-26) have breather soliton solutions. Figures **(a)** and **(b)**, are the vision from the front but Figure **(c)** is from the top.

4. Discussion

The discrete eigenvalues of equation (4) are $\{\pm z, \pm \frac{1}{z^*}\}$, this was set in Ablowitz and Musslimani's book [20]. The local MS equation (21-22) in [29], shows also some symmetry, but this depends on the number of site excitation in each component. In this paper, we present the new nonlocal MS equation (25-26), we obtained $4\mathcal{N}$ eigenvalues: $\{z, -z, z^*, -z^*\}$.
In example 1, the excitation of a single site in each component is given as an initial condition. After the calculation, the scattering data $a^+(z) = 1$ and $\boldsymbol{a}^-(\boldsymbol{z}) = 1$, in this case, we cannot find a z which makes the $a^+(z)$ and $\boldsymbol{a}^-(\boldsymbol{z})$, equals zero. On figure 1 the plot of the scattering data $a^+(z)$ and $\boldsymbol{a}^-(\boldsymbol{z})$. This figure shows, for both cases, when U_0 and $V_0 > 1$ or < 1, the $a^+(z)$ and $\boldsymbol{a}^-(\boldsymbol{z})$ have no zeroes outside or inside the unit circle. The dots around the unit circle $|z| = 1$ are continuous eigenvalues. As a result, we cannot predict a soliton solution. Figures (2) **(a)** and **(b)** represent the plot of the $|\mathcal{P}_{1,n}(t)|$ and $|\mathcal{P}_{2,n}(t)|$, respectively. These figures show that the integrable discrete lattice has no solitary wave when the values for both $U_0 = 1.5$ and $V_0 = 2$ are > 1 or $U_0 = 0.5$ and $V_0 = 0.2$ are < 1 and even when one of the U_0, V_0 is > 1.
In example 2, The discrete eigenvalues are clear in Figure **(a)**. When one of the values of $k = 2 > 1$ and $l = 0.5 < 1$, we have 4 discrete eigenvalues for $a^+(z)$: 1.189207 , -1.189207 , $1.189207\mathrm{I}, -1.189207\mathrm{I}$ and 4 discrete eigenvalues for $\boldsymbol{a}^-(\boldsymbol{z})$: $0.84089, -0.84089, 0.84089\mathrm{I}, -0.84089\mathrm{I}$. Therefore, two soliton solutions have been found. Figure **(b)** shows, when both of the values of $k = 2$ and $l = 2 > 1$, we have also 4 discrete eigenvalues but different than in Figure **(a)** where, the discrete eigenvalues for $a^+(z)$ are: $1.681793, -1.681793$, $1.681793\mathrm{I}, -1.681793\mathrm{I}$ and 4 discrete eigenvalues for $\boldsymbol{a}^-(\boldsymbol{z})$: $0.594603, -0.594603, 0.594603\mathrm{I}, -0.594603\mathrm{I}$. Figures (4) **(a)** and **(b)** represent the plot of the $|\mathcal{P}_{1,n}(t)|$ and $|\mathcal{P}_{2,n}(t)|$. These figures show that the integrable discrete lattices (25-26) have two solitary waves, that is mean two soliton solutions (breath soliton).

5. Conclusion

In this paper, it is worth saying that the new nonlocal MS which corresponds to the eigenvalue problem (6), show a symmetry case in the potential functions. In the examples, the results indicate that if z is the eigenvalue of (6), then $-z$ is also the

eigenvalue of (6). In addition, z^* is also eigenvalue because of $a^{\pm^*}(z^*) = a^{\pm}(z) = 0$. Taking into account that the symmetry case allows the system to have -z^* as an eigenvalue too. So, we have $4N$ eigenvalues for both outside and inside the unit circle: $\{z, -z, z^*, -z^*\}$, which gives us N soliton-type solutions. It is shown that the type of symmetry in the examples allows us to study the roots of the equations (25-26).
The conditions on a special type of initial condition represented in the form of a square-barrier to obtain the soliton type of the DMVNLS have been studied. In other words, the scattering matrix calculated the roots for each of $a^+(z)$ and $\boldsymbol{a}^-(\boldsymbol{z})$ when $|z| > 1$ and $|z| < 1$, respectively, to find the condition which generates the soliton-type solution. This work used the iteration method on the equation $\Psi_{n+1} = (Z + M_n)\Psi_n$, to calculate the zeros for the scattering data $a^+(z)$ and $\boldsymbol{a}^-(\boldsymbol{z})$ with $|z| \neq 1$.

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تحديد الشروط الأساسية للحل من نوع سوليتون للنظام الجديد من النوع المتقطع غير المحلي

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المستخلص:

الغرض من هذا البحث هو تقديم نموذج جديد للتخفيضات غير المحلية لنظام منكوف المنفصل متعدد المكونات .على وجه الخصوص ، يتم تحديد حل التركيز بناءً على حالة خاصة للدالة المحتملة .تتضمن هذه الدراسة: حل المسألة الطيفية وإيجاد الدوال الذاتية وبيانات التشتت .تكمن أهمية دراستنا في فحص الشروط التي تميز الحل المسمى سوليتون. هناك حالتان من دالتي القوى: الإثارة الفردية والمزدوجة ، إذا لم يكن لدى لمؤثر لاكس طيف خارج دائرة الوحدة ولا داخلها ، فلا يوجد حل من نوع سوليتون ، وهذا يحدث مع حالة الحاجز الواحد .من ناحية أخرى ، يعطي الحاجز المربع ذو الموقعين حلين منفصلين .يتضح أن السوليتون يحدث على الأرجح عند القيم الذاتية المتقطعة خارج دائرة الوحدة أو داخلها ، حيث تكون الحواجز أكثر من واحدة .كل حالة يتم تقديمها مدعومة بمحاكاة عددية.