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On a Partial Integro-Differential Equation on Time Scales

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Abstract

This paper aims to study the derivation of estimation of some integral inequality in three variables over time scales. The result of this estimation is used as a tool to investigate some properties for solving partial integro-differential equations with initial-boundary conditions on time scales, such as estimating the difference between two approximate solutions and the closeness between the solutions. This difference has many applications to in various scientific fields in some branches of mathematics, physics, economics, electricity, and biology. In this paper we study the problem from two points review, firstly we deal with estimating the difference between two ε-approximate solutions for the given certain nonlinear integro-differential equation with initial-boundary conditions, through which a suitable estimation for the approximate solutions can be obtained, and secondly, we provide the conditions for the closeness of solution of the problem under the study.

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1. Introduction

In 1988, Stefan Hilger has been started the time scales calculus. He fills the gap between continuous and discrete analysis and extends both theories. [1], [2], [3]. This field was interesting to many authors for investigating the solving of integral and integro-differential equations on time scales. [4], [5], [6]. In this article, we develop the findings which were obtained from [7], [8], [9]. Assuming that $\mathbb{T}_1$, $\mathbb{T}_2$ and $\mathbb{T}_3$ are arbitrary time scales such that $\mathrm{A} = \mathbb{T}_1 \times \mathbb{T}_2 \times \mathbb{T}_3$. The partial delta derivative of $\mathrm{D}(ɱ, ᶖ, ɽ)$ for $ɱ$, $ᶖ$, $ɽ$, and $ɱ, ᶖ, ɽ$ respectively are denoted by $\mathrm{D}^{\Delta_1}(ɱ, ᶖ, ɽ)$, $\mathrm{D}^{\Delta_2}(ɱ, ᶖ, ɽ)$, $\mathrm{D}^{\Delta_3}(ɱ, ᶖ, ɽ)$ and $\mathrm{D}^{\Delta_3\Delta_2\Delta_1}(ɱ, ᶖ, ɽ)$, for $(ɱ, ᶖ, ɽ) \in \mathrm{A}$. The set of right-dense continuous functions is denoted by C_{rd}. In this paper, we investigate the non-linear partial integro-differential equation on time scales of the form

$$\mathrm{D}^{\Delta_3\Delta_2\Delta_1}(ɱ, ᶖ, ɽ) = F(ɱ, ᶖ, ɽ, \mathrm{D}(ɱ, ᶖ, ɽ), \mathrm{D}^{\Delta_3\Delta_2\Delta_1}(ɱ, ᶖ, ɽ), (H\mathrm{D})(ɱ, ᶖ, ɽ), (G\mathrm{D})(ɱ, ᶖ, ɽ)), \quad (1)$$

with the given initial-boundary conditions

$$\mathrm{D}(ɱ, ᶖ, z_0) = \rho(ɱ, ᶖ),\ \mathrm{D}(ɱ, y_0, ɽ) = \delta(ɱ, ɽ),\ \mathrm{D}(x_0, ᶖ, ɽ) = \alpha(ᶖ, ɽ),\ \mathrm{D}(ɱ, y_0, z_0) = \beta(ɱ),$$
$$\mathrm{D}(x_0, ᶖ, z_0) = \sigma(ᶖ),\ \mathrm{D}(x_0, y_0, ɽ) = \xi(ɽ),\ \mathrm{D}(x_0, y_0, z_0) = 0 \quad (2)$$

where

$$(H\mathrm{D})(ɱ, ᶖ, ɽ) = \int_{x_0}^{ɱ}\int_{y_0}^{ᶖ}\int_{z_0}^{ɽ} h\left(ɱ, ᶖ, ɽ, \zeta, \lambda, \tau, \mathrm{D}(\zeta, \lambda, \tau), \mathrm{D}^{\Delta_3\Delta_2\Delta_1}(\zeta, \lambda, \tau)\right)\Delta\tau\Delta\lambda\Delta\zeta, \quad (3)$$

$$(G\mathrm{D})(ɱ, ᶖ, ɽ) = \int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} g\left(ɱ, ᶖ, ɽ, \zeta, \lambda, \tau, \mathrm{D}(\zeta, \lambda, \tau), \mathrm{D}^{\Delta_3\Delta_2\Delta_1}(\zeta, \lambda, \tau)\right)\Delta\tau\Delta\lambda\Delta\zeta, \quad (4)$$

$F \in C_{rd}(\mathrm{A} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $h, g \in C_{rd}(\mathrm{A} \times \mathrm{A} \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $\rho, \delta, \alpha \in C_{rd}(\mathbb{T} \times \mathbb{T}, \mathbb{R}^n)$, $\beta, \sigma, \xi \in (\mathbb{T}, \mathbb{R}^n)$ and

$$(H0)(ɱ, ᶖ, ɽ) = \int_{x_0}^{ɱ}\int_{y_0}^{ᶖ}\int_{z_0}^{ɽ} h(ɱ, ᶖ, ɽ, \zeta, \lambda, \tau, 0,0)\Delta\tau\Delta\lambda\Delta\zeta, \quad (5)$$

$$(G0)(ɱ, ᶖ, ɽ) = \int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} g(ɱ, ᶖ, ɽ, \zeta, \lambda, \tau, 0,0)\Delta\tau\Delta\lambda\Delta\zeta. \quad (6)$$

The approximate solution of problems (1) and (2) for F, h, and g are given functions and D is the unknown function, $\mathbb{R}^n$ is n-dimensional real Euclidean space with norm $|\cdot|$.

2. Preliminaries

The time scale is defined to be an arbitrary closed subset of real numbers, and it is denoted by $\mathbb{T}$. The forward jump operator $\sigma: \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := inf\ \{\ell \in \mathrm{T}: \ell > t\} \text{ for } t \in \mathbb{T},$$

the backward jump operator $\rho: \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\ \{\ell \in \mathrm{T}: \ell < t\} \text{ for } t \in \mathbb{T},$$

and the empty set is denoted by $\emptyset$ such that

$$inf\emptyset = sup\mathbb{T} \text{ and } sup\emptyset = inf\mathbb{T}.$$

We defined a function η as $\eta: \mathbb{T} \to [0, \infty)$, which is called the graininess function

$$\eta(t) = \psi(t) - t, \quad \text{for } t \in \mathbb{T}.$$

A function $F: \mathbb{T} \to \mathbb{R}$ is said to be an antiderivative of $S: \mathbb{T} \to \mathbb{R}$ provided then $F^{\Delta}(t) = S(t)$ for $t \in \mathbb{T}$, we determine the (Hilger) delta integral in this instance by

$$\int_a^c S(t)\Delta t = F(c) - F(a), \quad \text{for } a, c \in \mathbb{T}. \text{ [10], [11], [12]}$$

Definition 2.1: A point t is said to be left-scattered if $\rho(t) < t$, left-denes if $t > inf\mathbb{T}$ and $\rho(t) = t$, right-scattered if $\sigma(t) > t$, right-dense if $t < sup\ \mathbb{T}$ and $\sigma(t) = t$, we defined $\mathbb{T}^k = \mathbb{T} - M$ if $\mathbb{T}$ has a Left-scattered maximum M, otherwise $\mathbb{T}^k = \mathbb{T}$. If a function $F: \mathbb{T} \to \mathbb{R}$ is continuous at every right-dense point and has a finite left-sided limit at every left-dense point, it is said to be right-dense continuous (rd-continuous). The letter C_{rd} stands for the set of all rd-continuous functions $\mathrm{C}_{rd}(\mathbb{T})$. [11], [13].

Definition 2.2: Assume that the function $F: \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^k$, the delta derivative $F^{\Delta}(t)$ to be the number (provide if exist) with the condition that for any $\varepsilon > 0$, there is a neighborhood U of t i.e., $\mathrm{U} = (t - \bar{\delta}, t + \bar{\delta}) \cap \mathbb{T}$ for some $\bar{\delta} > 0$, s.t

$$\left|F(\sigma(t)) - F(\ell) - F^{\Delta}(t)\right|[\sigma(t) - \ell] \leq \varepsilon|\sigma(t) - \ell|, \quad \text{for } \ell \in \mathrm{U}.$$

So, we say $F^{\Delta}(t)$ is the delta derivative of F at t.

If $\mathbb{T} = \mathbb{R}$ then the usual derivative $F^{\Delta}(t) = F'(t)$ whereas If $\mathbb{T} = \mathbb{Z}$ then the forward difference operator

$$F^{\Delta}(t) = \Delta F(t) = F(t+1) - F(t). \text{ [14]}$$

3. Main Results

Let $\mathrm{D}(ɱ, ᶖ, ɽ) \in \mathrm{C}_{rd}(\mathrm{A}, \mathbb{R}^n)$, $\mathrm{D}^{\Delta_3\Delta_2\Delta_1}(ɱ, ᶖ, ɽ)$ exist and the following inequality is satisfied:

$$|\mathrm{D}^{\Delta_3\Delta_2\Delta_1}(ɱ, ᶖ, ɽ) - F(ɱ, ᶖ, ɽ, \mathrm{D}(ɱ, ᶖ, ɽ), \mathrm{D}^{\Delta_3\Delta_2\Delta_1}(ɱ, ᶖ, ɽ), (H\mathrm{D})(ɱ, ᶖ, ɽ), (G\mathrm{D})(ɱ, ᶖ, ɽ))| \leq \varepsilon, \tag{7}$$

a given constant $\varepsilon \geq 0$, then the function $\mathrm{D}(ɱ, ᶖ, ɽ)$ that achieves relation (7) is called the ε-approximate solution of the initial-boundary values problem (1), (2).

We present our main results in the following Lemma.

Lemma 3.1: Let $\mathrm{D}(ɱ, ᶖ, ɽ), Q(ɱ, ᶖ, ɽ) \in \mathrm{C}_{rd}(\mathrm{A}, \mathbb{R}_+)$ and $p_1, p_2 \in \mathrm{C}_{rd}(\mathrm{A} \times \mathrm{A}, \mathbb{R}_+)$, non-decreasing functions concerning $ɱ, ᶖ, ɽ$ where $x_0, \ell_1, y_0, \ell_2, z_0, \ell_3$, and $\mathrm{C} \geq 0$ are constants such that $x_0 \leq \ell_1$, $y_0 \leq \ell_2$, $z_0 \leq \ell_3$, if

$$\begin{aligned}\mathrm{D}(ɱ, ᶖ, ɽ) \leq \mathrm{C} + \int_{x_0}^{ɱ}\int_{y_0}^{ᶖ}\int_{z_0}^{ɽ}\{\mathrm{Q}(v, p, e)\ \mathrm{D}(v, p, e) + \\ \int_{x_0}^{v}\int_{y_0}^{p}\int_{z_0}^{e} p_1(v, p, e, \zeta, \lambda, \tau)\ \mathrm{D}(\zeta, \lambda, \tau)\Delta\tau\Delta\lambda\Delta\zeta + p_1(ɱ, ᶖ, ɽ, v, p, e)\mathrm{D}(v, p, e) + \\ \int_{x_0}^{\ell}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} p_2(v, p, e, \zeta, \lambda, \tau)\ \mathrm{D}(\zeta, \lambda, \tau)\Delta\tau\Delta\lambda\Delta\zeta\}\Delta e\Delta p\Delta v + \int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} p_2(ɱ, ᶖ, ɽ, \zeta, \lambda, \tau)\ \mathrm{D}(\zeta, \lambda, \tau)\Delta\tau\Delta\lambda\Delta\zeta,\end{aligned} \tag{8}$$

$(ɱ, ᶖ, ɽ) \in \mathrm{A}$, then

$$\mathrm{D}(ɱ, ᶖ, ɽ) \leq \frac{\mathrm{C}}{1 - G_1(ɱ, ᶖ, ɽ)} + \frac{\mathrm{C}\ G_2(ɱ, ᶖ, ɽ, \ell_1, \ell_2, \ell_3)}{(1 - G_1(ɱ, ᶖ, ɽ))(1 - G_1(\ell_1, \ell_2, \ell_3) - G_2(\ell_1, \ell_2, \ell_3, \ell_1, \ell_2, \ell_3))}. \tag{9}$$

Where

$$G_1(ɱ, ᶖ, ɽ) = \int_{x_0}^{ɱ}\int_{y_0}^{ᶖ}\int_{z_0}^{ɽ}\{\mathrm{Q}(v, p, e) + \int_{x_0}^{v}\int_{y_0}^{p}\int_{z_0}^{e} p_1(v, p, e, \zeta, \lambda, \tau)\ \Delta\tau\Delta\lambda\Delta\zeta + p_1(ɱ, ᶖ, ɽ, v, p, e,)\}\Delta e\Delta p\Delta v, \tag{10}$$

$$G_2(ɱ, ᶖ, ɽ, \ell_1, \ell_2, \ell_3) = \int_{x_0}^{ɱ}\int_{y_0}^{ᶖ}\int_{z_0}^{ɽ}\int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} p_2(v, p, e, \zeta, \lambda, \tau)\ \Delta\tau\Delta\lambda\Delta\zeta\Delta e\Delta p\Delta v + \int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} p_2\ (ɱ, ᶖ, ɽ, \zeta, \lambda, \tau)\Delta\tau\Delta\lambda\Delta\zeta, \tag{11}$$

Assume that

$1 - G_1(ɱ, ᶖ, ɽ) > 0$, $x_0 \leq ɱ \leq \ell_1$, $y_0 \leq ᶖ \leq \ell_2$, $z_0 \leq ɽ \leq \ell_3$,

$1 - G_1(\ell_1, \ell_2, \ell_3) - G_2(\ell_1, \ell_2, \ell_3, \ell_1, \ell_2, \ell_3) > 0$,

Proof: Since the function D is a non-decreasing function concerning the first, second, and third variables, let $x_0 \leq ɱ \leq \ell_1, y_0 \leq ᶖ \leq \ell_2, z_0 \leq ɽ \leq \ell_3$, and from equation (8) we get

$$\mathrm{D}(ɱ,ị,ɽ) \le \mathrm{C} + \int_{x_0}^{ɱ}\int_{y_0}^{ị}\int_{z_0}^{ɽ}[\mathrm{Q}(v,p,e)\,\mathrm{D}(ɱ,ị,ɽ) + \mathrm{D}(ɱ,ị,ɽ)\int_{x_0}^{v}\int_{y_0}^{p}\int_{z_0}^{e} p_1(v,p,e,\zeta,\lambda,\tau)\,\Delta\tau\Delta\lambda\Delta\zeta +$$
$$p_1(ɱ,ị,ɽ,v,p,e)\mathrm{D}(ɱ,ị,ɽ) + \mathrm{D}(\ell_1,\ell_2,\ell_3)\int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} p_2(v,p,e,\zeta,\lambda,\tau)\,\Delta\tau\Delta\lambda\Delta\zeta]\Delta e\Delta p\Delta v +$$
$$\mathrm{D}(\ell_1,\ell_2,\ell_3)\int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} p_2(ɱ,ị,ɽ,\zeta,\lambda,\tau)\,\Delta\tau\Delta\lambda\Delta\zeta,$$
$$= \mathrm{C} + \mathrm{D}(ɱ,ị,ɽ)\{\int_{x_0}^{ɱ}\int_{y_0}^{ị}\int_{z_0}^{ɽ}[\mathrm{Q}(v,p,e) + \int_{x_0}^{v}\int_{y_0}^{p}\int_{z_0}^{e} p_1(v,p,e,\zeta,\lambda,\tau)\Delta\tau\Delta\lambda\Delta\zeta + p_1(ɱ,ị,ɽ,v,p,e)]\Delta e\Delta p\Delta v\} +$$
$$\mathrm{D}(\ell_1,\ell_2,\ell_3)\{\int_{x_0}^{ɱ}\int_{y_0}^{ị}\int_{z_0}^{ɽ}\int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} p_2(v,p,e,\zeta,\lambda,\tau)\Delta\tau\Delta\lambda\Delta\zeta\Delta e\Delta v\Delta p + \int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} p_2(ɱ,ị,ɽ,\zeta,\lambda,\tau)\Delta\tau\Delta\lambda\Delta\zeta\},$$

by equations (10) and (11), we can write

$$\mathrm{D}(ɱ,ị,ɽ) \le \mathrm{C} + \mathrm{D}(ɱ,ị,ɽ)G_1(ɱ,ị,ɽ) + \mathrm{D}(\ell_1,\ell_2,\ell_3)G_2(ɱ,ị,ɽ,\ell_1,\ell_2,\ell_3),$$
$$\mathrm{D}(ɱ,ị,ɽ)(1 - G_1(ɱ,ị,ɽ)) \le \mathrm{C} + \mathrm{D}(\ell_1,\ell_2,\ell_3)G_2(ɱ,ị,ɽ,\ell_1,\ell_2,\ell_3),$$

or

$$\mathrm{D}(ɱ,ị,ɽ) \le \frac{\mathrm{C}}{(1-G_1(ɱ,ị,ɽ))} + \frac{\mathrm{D}(\ell_1,\ell_2,\ell_3)G_2(ɱ,ị,ɽ,\ell_1,\ell_2,\ell_3)}{(1-G_1(ɱ,ị,ɽ))}, \tag{12}$$

and in particular, $ɱ = \ell_1, ị = \ell_2, ɽ = \ell_3$

$$\mathrm{D}(\ell_1,\ell_2,\ell_3) \le \mathrm{C} + \mathrm{D}(\ell_1,\ell_2,\ell_3)G_1(\ell_1,\ell_2,\ell_3) + \mathrm{D}(\ell_1,\ell_2,\ell_3)G_2(\ell_1,\ell_2,\ell_3,\ell_1,\ell_2,\ell_3),$$

or

$$(1 - G_1(\ell_1,\ell_2,\ell_3) - G_2(\ell_1,\ell_2,\ell_3,\ell_1,\ell_2,\ell_3))\,\mathrm{D}(\ell_1,\ell_2,\ell_3) \le \mathrm{C},$$
$$\mathrm{D}(\ell_1,\ell_2,\ell_3) \le \frac{\mathrm{C}}{(1-G_1(\ell_1,\ell_2,\ell_3)-G_2(\ell_1,\ell_2,\ell_3,\ell_1,\ell_2,\ell_3))},$$

now from equation (12), we get

$$\mathrm{D}(ɱ,ị,ɽ) \le \frac{\mathrm{C}}{1-G_1(ɱ,ị,ɽ)} + \frac{\mathrm{C}\,G_2(ɱ,ị,ɽ,\ell_1,\ell_2,\ell_3)}{(1-G_1(ɱ,ị,ɽ))(1-G_1(\ell_1,\ell_2,\ell_3)-G_2(\ell_1,\ell_2,\ell_3,\ell_1,\ell_2,\ell_3))}.$$

The following theorem deals with estimating the difference between the two approximate solutions of equation (1) with the initial-boundary condition (2).

Theorem 3.2: Assume the functions F, h, and g in equation (1) satisfy the following hypotheses

$$\left|F(ɱ,ị,ɽ,q,s,b,\mathrm{d}) - F(ɱ,ị,ɽ,\bar{q},\bar{s},\bar{b},\bar{\mathrm{d}})\right| \le \mathrm{L}[|q-\bar{q}| + |s-\bar{s}|] + |b-\bar{b}| + |\mathrm{d}-\bar{\mathrm{d}}|, \tag{13}$$
$$|h(ɱ,ị,ɽ,\varpi,\varrho,\upsilon,q,s) - h(ɱ,ị,ɽ,\varpi,\varrho,\upsilon,\bar{q},\bar{s})| \le \mathrm{J}_1(ɱ,ị,ɽ,\varpi,\varrho,\upsilon)[|q-\bar{q}| + |s-\bar{s}|], \tag{14}$$
$$|g(ɱ,ị,ɽ,\varpi,\varrho,\upsilon,q,s) - g(ɱ,ị,ɽ,\varpi,\varrho,\upsilon,\bar{q},\bar{s})| \le \mathrm{J}_2(ɱ,ị,ɽ,\varpi,\varrho,\upsilon)[|q-\bar{q}| + |s-\bar{s}|], \tag{15}$$

where L is a non-negative constant such that $\mathrm{L} < 1$ and $\mathrm{J}_1(ɱ,ị,ɽ,\varpi,\varrho,\upsilon), \mathrm{J}_2(ɱ,ị,ɽ,\varpi,\varrho,\upsilon) \in \mathrm{C}_{rd}(\mathrm{A}\times\mathrm{A},\mathbb{R}_+)$, and $\mathrm{D}_i(ɱ,ị,ɽ)$ $(i = 1,2)$ are ε_i-approximate solutions of equation (1) with the following initial-boundary conditions

$$\mathrm{D}_i(ɱ,ị,z_0) = \rho_i(ɱ,ị),\ \mathrm{D}_i(ɱ,y_0,ɽ) = \delta_i(ɱ,ɽ),\ \mathrm{D}_i(x_0,ị,ɽ) = \alpha_i(ị,ɽ),\ \mathrm{D}_i(ɱ,y_0,z_0) = \beta_i(ɱ),$$
$$\mathrm{D}_i(x_0,ị,z_0) = \sigma_i(ị),\ \mathrm{D}_i(x_0,y_0,ɽ) = \xi_i(ɽ),\ \mathrm{D}_i(x_0,y_0,z_0) = 0, \tag{16}$$

where $\rho_i,\delta_i,\alpha_i \in (\mathbb{T}\times\mathbb{T},\mathbb{R}^n)$, $\beta_i,\sigma_i,\xi_i \in (\mathbb{T},\mathbb{R}^n)$,$(i = 1,2)$ and that

$$|\rho_1(ɱ,ị) - \rho_2(ɱ,ị) + \delta_1(ɱ,ɽ) - \delta_2(ɱ,ɽ) + \alpha_1(ị,ɽ) - \alpha_2(ị,ɽ) - \beta_1(ɱ) + \beta_2(ɱ) - \sigma_1(ị) + \sigma_2(ị) - \xi_1(ɽ) + \xi_2(ɽ)| \le \tau, \tag{17}$$

Where $\tau \ge 0$ is a constant, then

$$|\mathrm{D}_1(ɱ,ị,ɽ) - \mathrm{D}_2(ɱ,ị,ɽ)| + \left|\mathrm{D}_1{}^{\Delta_3\Delta_2\Delta_1}(ɱ,ị,ɽ) - \mathrm{D}_2{}^{\Delta_3\Delta_2\Delta_1}(ɱ,ị,ɽ)\right| \le$$
$$\frac{\frac{\varepsilon(1+(\ell_1-x_0)(\ell_2-y_0)(\ell_3-z_0))+\tau}{1-\mathrm{L}}}{\vartheta_1(ɱ,ị,ɽ)} + \frac{\frac{\varepsilon(1+(\ell_1-x_0)(\ell_2-y_0)(\ell_3-z_0))+\tau}{1-\mathrm{L}}e(ɱ,ị,ɽ)}{\vartheta_1(ɱ,ị,ɽ)N(\ell_1,\ell_2,\ell_3)}. \tag{18}$$

For every $(ɱ,ị,ɽ) \in \mathrm{A}$, $\varepsilon = \varepsilon_1 + \varepsilon_2$,

$$\vartheta_1(ɱ,ị,ɽ) = 1 - \frac{1}{1-\mathrm{L}}[\int_{x_0}^{ɱ}\int_{y_0}^{ị}\int_{z_0}^{ɽ}\{\mathrm{L} + \int_{x_0}^{v}\int_{y_0}^{p}\int_{z_0}^{e}\mathrm{J}_1(v,p,e,\varpi,\varrho,\upsilon)\Delta\upsilon\Delta\varrho\Delta\varpi + \mathrm{J}_1(ɱ,ị,ɽ,v,p,e)\}\Delta e\Delta p\Delta v], \tag{19}$$
$$e(ɱ,ị,ɽ) = \frac{1}{1-\mathrm{L}}\int_{x_0}^{ɱ}\int_{y_0}^{ị}\int_{z_0}^{ɽ}\int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3}\mathrm{J}_2(v,p,e,\varpi,\varrho,\upsilon)\Delta\upsilon\Delta\varrho\Delta\varpi\Delta e\Delta p\Delta v + \frac{1}{1-\mathrm{L}}\int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3}\mathrm{J}_2(ɱ,ị,ɽ,\varpi,\varrho,\upsilon)\Delta\upsilon\Delta\varrho\Delta\varpi, \tag{20}$$
$$N(\ell_1,\ell_2,\ell_3) = \vartheta_1(\ell_1,\ell_2,\ell_3) - e(\ell_1,\ell_2,\ell_3). \tag{21}$$

Proof: Since $\mathrm{D}_i(ɱ,ị,ɽ)$ $(i = 1,2)$ are two ε_i-approximate solutions of equation (1) with (16), we set

$$\left|\mathrm{D}_i{}^{\Delta_3\Delta_2\Delta_1}(ɱ,ị,ɽ) - F(ɱ,ị,ɽ,\mathrm{D}_i(ɱ,ị,ɽ),\mathrm{D}_i{}^{\Delta_3\Delta_2\Delta_1}(ɱ,ị,ɽ),(H\mathrm{D}_i)(ɱ,ị,ɽ),(G\mathrm{D}_i)(ɱ,ị,ɽ))\right| \le \varepsilon_i. \tag{22}$$

From (22) we put $i = 1,2$ and in the light of inequalities

$$|ð_1| - |ð_2| \le |ð_1 - ð_2|, |ð_1 - ð_2| \le |ð_1| + |ð_2|, \tag{23}$$

we get

$$\varepsilon_1 + \varepsilon_2 \ge |\mathrm{D}_1{}^{\Delta_3\Delta_2\Delta_1}(ɱ,ị,ɽ) - F(ɱ,ị,ɽ,\mathrm{D}_1(ɱ,ị,ɽ),\mathrm{D}_1{}^{\Delta_3\Delta_2\Delta_1}(ɱ,ị,ɽ),(H\mathrm{D}_1)(ɱ,ị,ɽ),(G\mathrm{D}_1)(ɱ,ị,ɽ))| +$$
$$\left|\mathrm{D}_2{}^{\Delta_3\Delta_2\Delta_1}(ɱ,ị,ɽ) - F(ɱ,ị,ɽ,\mathrm{D}_2(ɱ,ị,ɽ),\mathrm{D}_2{}^{\Delta_3\Delta_2\Delta_1}(ɱ,ị,ɽ),(H\mathrm{D}_2)(ɱ,ị,ɽ),(G\mathrm{D}_2)(ɱ,ị,ɽ))\right|,$$

$$\varepsilon \ge |\{\mathrm{D}_1{}^{\Delta_3\Delta_2\Delta_1}(ɱ,ị,ɽ) - F(ɱ,ị,ɽ,\mathrm{D}_1(ɱ,ị,ɽ),\mathrm{D}_1{}^{\Delta_3\Delta_2\Delta_1}(ɱ,ị,ɽ),(H\mathrm{D}_1)(ɱ,ị,ɽ),(G\mathrm{D}_1)(ɱ,ị,ɽ))\} -$$

$\{\mathrm{D}_2{}^{\Delta_3\Delta_2\Delta_1}(ɱ,ᶖ,ɼ) - F(ɱ,ᶖ,ɼ,\mathrm{D}_2(ɱ,ᶖ,ɼ),\mathrm{D}_2{}^{\Delta_3\Delta_2\Delta_1}(ɱ,ᶖ,ɼ),(H\mathrm{D}_2)(ɱ,ᶖ,ɼ),(G\mathrm{D}_2)(ɱ,ᶖ,ɼ))\}|,$

or

$\varepsilon \geq \left|\mathrm{D}_1{}^{\Delta_3\Delta_2\Delta_1}(ɱ,ᶖ,ɼ) - \mathrm{D}_2{}^{\Delta_3\Delta_2\Delta_1}(ɱ,ᶖ,ɼ)\right| - |F(ɱ,ᶖ,ɼ,\mathrm{D}_1(ɱ,ᶖ,ɼ),\mathrm{D}_1{}^{\Delta_3\Delta_2\Delta_1}(ɱ,ᶖ,ɼ),(H\mathrm{D}_1)(ɱ,ᶖ,ɼ),(G\mathrm{D}_1)(ɱ,ᶖ,ɼ)) -$
$F(ɱ,ᶖ,ɼ,\mathrm{D}_2(ɱ,ᶖ,ɼ),\mathrm{D}_2{}^{\Delta_3\Delta_2\Delta_1}(ɱ,ᶖ,ɼ),(H\mathrm{D}_2)(ɱ,ᶖ,ɼ),(G\mathrm{D}_2)(ɱ,ᶖ,ɼ))|,$

by using hypotheses (13), (14), and (15), we get

$\varepsilon \geq \left|\mathrm{D}_1{}^{\Delta_3\Delta_2\Delta_1}(ɱ,ᶖ,ɼ) - \mathrm{D}_2{}^{\Delta_3\Delta_2\Delta_1}(ɱ,ᶖ,ɼ)\right| - \mathrm{L}[|\mathrm{D}_1(ɱ,ᶖ,ɼ) - \mathrm{D}_2(ɱ,ᶖ,ɼ)| + |\mathrm{D}_1{}^{\Delta_3\Delta_2\Delta_1}(ɱ,ᶖ,ɼ) - \mathrm{D}_2{}^{\Delta_3\Delta_2\Delta_1}(ɱ,ᶖ,ɼ)| +$
$\int_{x_0}^{ɱ}\int_{y_0}^{ᶖ}\int_{z_0}^{ɼ} \mathrm{J}_1(ɱ,ᶖ,ɼ,\varpi,\varrho,\upsilon)\left[|\mathrm{D}_1(\varpi,\varrho,\upsilon) - \mathrm{D}_2(\varpi,\varrho,\upsilon)| + \left|\mathrm{D}_1{}^{\Delta_3\Delta_2\Delta_1}(\varpi,\varrho,\upsilon) - \mathrm{D}_2{}^{\Delta_3\Delta_2\Delta_1}(\varpi,\varrho,\upsilon)\right|\right]\Delta\upsilon\Delta\varrho\Delta\varpi +$
$\int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} \mathrm{J}_2\,(ɱ,ᶖ,ɼ,\varpi,\varrho,\upsilon)\left[|\mathrm{D}_1(\varpi,\varrho,\upsilon) - \mathrm{D}_2(\varpi,\varrho,\upsilon)| + \left|\mathrm{D}_1{}^{\Delta_3\Delta_2\Delta_1}(\varpi,\varrho,\upsilon) - \mathrm{D}_2{}^{\Delta_3\Delta_2\Delta_1}(\varpi,\varrho,\upsilon)\right|\right]\Delta\upsilon\Delta\varrho\Delta\varpi.$ (24)

By delta integrating both sides of (22) concerning $(ɱ,ᶖ,ɼ)$ we find that

$\varepsilon_i(ɱ - x_0)(ᶖ - y_0)(ɼ - z_0) \geq \int_{x_0}^{ɱ}\int_{y_0}^{ᶖ}\int_{z_0}^{ɼ} |\mathrm{D}_i{}^{\Delta_3\Delta_2\Delta_1}(v,p,e) -$
$F(v,p,e,\mathrm{D}_i(v,p,e),\mathrm{D}_i{}^{\Delta_3\Delta_2\Delta_1}(v,p,e),(H\mathrm{D}_i)(v,p,e),(G\mathrm{D}_i)(v,p,e))|\Delta e\Delta p\Delta v,$

$\varepsilon_i(ɱ - x_0)(ᶖ - y_0)(ɼ - z_0) \geq |\int_{x_0}^{ɱ}\int_{y_0}^{ᶖ}\int_{z_0}^{ɼ}\{\mathrm{D}_i{}^{\Delta_3\Delta_2\Delta_1}(v,p,e) -$
$F(v,p,e,\mathrm{D}_i(v,p,e),\mathrm{D}_i{}^{\Delta_3\Delta_2\Delta_1}(v,p,e),(H\mathrm{D}_i)(v,p,e),(G\mathrm{D}_i)(v,p,e))\}\Delta e\Delta p\Delta v|,$
$= |\mathrm{D}_i(ɱ,ᶖ,ɼ) - \rho_i(ɱ,ᶖ) - \delta_i(ɱ,ɼ) - \alpha_i(ᶖ,ɼ) + \beta_i(ɱ) + \sigma_i(ᶖ) + \xi_i(ɼ) -$
$\int_{x_0}^{ɱ}\int_{y_0}^{ᶖ}\int_{z_0}^{ɼ} F(v,p,e,\mathrm{D}_i(v,p,e),\mathrm{D}_i{}^{\Delta_3\Delta_2\Delta_1}(v,p,e),(H\mathrm{D}_i)(v,p,e),(G\mathrm{D}_i)(v,p,e))\Delta e\Delta p\Delta v|.$ (25)

From (25) and we put $i = 1,2$ and according to the inequalities (23), we get

$(\varepsilon_1 + \varepsilon_2)(ɱ - x_0)(ᶖ - y_0)(ɼ - z_0) \geq |\mathrm{D}_1(ɱ,ᶖ,ɼ) - \rho_1(ɱ,ᶖ) - \delta_1(ɱ,ɼ) - \alpha_1(ᶖ,ɼ) + \beta_1(ɱ) + \sigma_1(ᶖ) + \xi_1(ɼ) -$
$\int_{x_0}^{ɱ}\int_{y_0}^{ᶖ}\int_{z_0}^{ɼ} F(v,p,e,\mathrm{D}_1(v,p,e),\mathrm{D}_1{}^{\Delta_3\Delta_2\Delta_1}(v,p,e),(H\mathrm{D}_1)(v,p,e),(G\mathrm{D}_1)(v,p,e))\Delta e\Delta p\Delta v| +$
$|\mathrm{D}_2(ɱ,ᶖ,ɼ) - \rho_2(ɱ,ᶖ) - \delta_2(ɱ,ɼ) - \alpha_2(ᶖ,ɼ) + \beta_2(ɱ) + \sigma_2(ᶖ) + \xi_2(ɼ) -$
$\int_{x_0}^{ɱ}\int_{y_0}^{ᶖ}\int_{z_0}^{ɼ} F(v,p,e,\mathrm{D}_2(v,p,e),\mathrm{D}_2{}^{\Delta_3\Delta_2\Delta_1}(v,p,e),(H\mathrm{D}_2)(v,p,e),(G\mathrm{D}_2)(v,p,e))\,\Delta e\Delta p\Delta v|,$

$(\varepsilon_1 + \varepsilon_2)(ɱ - x_0)(ᶖ - y_0)(ɼ - z_0) \geq |\{\mathrm{D}_1(ɱ,ᶖ,ɼ) - \rho_1(ɱ,ᶖ) - \delta_1(ɱ,ɼ) - \alpha_1(ᶖ,ɼ) + \beta_1(ɱ) + \sigma_1(ᶖ) + \xi_1(ɼ) -$
$\int_{x_0}^{ɱ}\int_{y_0}^{ᶖ}\int_{z_0}^{ɼ} F(v,p,e,\mathrm{D}_1(v,p,e),\mathrm{D}_1{}^{\Delta_3\Delta_2\Delta_1}(v,p,e),(H\mathrm{D}_1)(v,p,e),(G\mathrm{D}_1)(v,p,e))\,\Delta e\Delta p\Delta v\} -$
$\{\mathrm{D}_2(ɱ,ᶖ,ɼ) - \rho_2(ɱ,ᶖ) - \delta_2(ɱ,ɼ) - \alpha_2(ᶖ,ɼ) + \beta_2(ɱ) + \sigma_2(ᶖ) + \xi_2(ɼ) -$
$\int_{x_0}^{ɱ}\int_{y_0}^{ᶖ}\int_{z_0}^{ɼ} F(v,p,e,\mathrm{D}_2(v,p,e),\mathrm{D}_2{}^{\Delta_3\Delta_2\Delta_1}(v,p,e),(H\mathrm{D}_2)(v,p,e),(G\mathrm{D}_2)(v,p,e))\,\Delta e\Delta p\Delta v\}|,$

or we can write

$(\varepsilon_1 + \varepsilon_2)(ɱ - x_0)(ᶖ - y_0)(ɼ - z_0) \geq |\mathrm{D}_1(ɱ,ᶖ,ɼ) - \mathrm{D}_2(ɱ,ᶖ,ɼ)| - |\rho_1(ɱ,ᶖ) - \rho_2(ɱ,ᶖ) + \delta_1(ɱ,ɼ) - \delta_2(ɱ,ɼ) + \alpha_1(ᶖ,ɼ) -$
$\alpha_2(ᶖ,ɼ) - \beta_1(ɱ) + \beta_2(ɱ) - \sigma_1(ᶖ) + \sigma_2(ᶖ) - \xi_1(ɼ) + \xi_2(ɼ)| - \int_{x_0}^{ɱ}\int_{y_0}^{ᶖ}\int_{z_0}^{ɼ} |F(v,p,e,\mathrm{D}_1(v,p,e),\mathrm{D}_1{}^{\Delta_3\Delta_2\Delta_1}(v,p,e),$
$(H\mathrm{D}_1)(v,p,e),(G\mathrm{D}_1)(v,p,e))\,\Delta e\Delta p\Delta v - F(v,p,e,\mathrm{D}_2(v,p,e),\mathrm{D}_2{}^{\Delta_3\Delta_2\Delta_1}(v,p,e),(H\mathrm{D}_2)(v,p,e),(G\mathrm{D}_2)(v,p,e))|\,\Delta e\Delta p\Delta v,$

by using hypotheses (13), (14), (15), and (17), we find

$\varepsilon(ɱ - x_0)(ᶖ - y_0)(ɼ - z_0) \geq |\mathrm{D}_1(ɱ,ᶖ,ɼ) - \mathrm{D}_2(ɱ,ᶖ,ɼ)| - \tau -$
$|\int_{x_0}^{ɱ}\int_{y_0}^{ᶖ}\int_{z_0}^{ɼ}\{\mathrm{L}[\,|\mathrm{D}_1(v,p,e) - \mathrm{D}_2(v,p,e)| + |\mathrm{D}_1{}^{\Delta_3\Delta_2\Delta_1}(v,p,e) - \mathrm{D}_2{}^{\Delta_3\Delta_2\Delta_1}(v,p,e)|] +$
$\int_{x_0}^{v}\int_{y_0}^{p}\int_{z_0}^{e} \mathrm{J}_1\,(v,p,e,\varpi,\varrho,\upsilon)[|\mathrm{D}_1(\varpi,\varrho,\upsilon) - \mathrm{D}_2(\varpi,\varrho,\upsilon)| + |\mathrm{D}_1{}^{\Delta_3\Delta_2\Delta_1}(\varpi,\varrho,\upsilon) - \mathrm{D}_2{}^{\Delta_3\Delta_2\Delta_1}(\varpi,\varrho,\upsilon)|]\Delta\upsilon\Delta\varrho\Delta\varpi +$
$\int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} \mathrm{J}_2\,(v,p,e,\varpi,\varrho,\upsilon)[|\mathrm{D}_1(\varpi,\varrho,\upsilon) - \mathrm{D}_2(\varpi,\varrho,\upsilon)| + |\mathrm{D}_1{}^{\Delta_3\Delta_2\Delta_1}(\varpi,\varrho,\upsilon) - \mathrm{D}_2{}^{\Delta_3\Delta_2\Delta_1}(\varpi,\varrho,\upsilon)|]\Delta\upsilon\Delta\varrho\Delta\varpi\}\Delta e\Delta p\Delta v|,$ (26)

Let

$$\mathrm{B}(ɱ,ᶖ,ɼ) = \left[|\mathrm{D}_1(ɱ,ᶖ,ɼ) - \mathrm{D}_2(ɱ,ᶖ,ɼ)| + \left|\mathrm{D}_1{}^{\Delta_3\Delta_2\Delta_1}(ɱ,ᶖ,ɼ) - \mathrm{D}_2{}^{\Delta_3\Delta_2\Delta_1}(ɱ,ᶖ,ɼ)\right|\right],$$

from (24), (26), we get

$\mathrm{B}(ɱ,ᶖ,ɼ) \leq \varepsilon\big(1 + (ɱ - x_0)(ᶖ - y_0)(ɼ - z_0)\big) + \tau + \int_{x_0}^{ɱ}\int_{y_0}^{ᶖ}\int_{z_0}^{ɼ}\{\mathrm{L}\,\mathrm{B}(v,p,e) +$
$\int_{x_0}^{v}\int_{y_0}^{p}\int_{z_0}^{e} \mathrm{J}_1\,(v,p,e,\varpi,\varrho,\upsilon)\mathrm{B}(\varpi,\varrho,\upsilon)\Delta\upsilon\Delta\varrho\Delta\varpi + \int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} \mathrm{J}_2\,(v,p,e,\varpi,\varrho,\upsilon)\mathrm{B}(\varpi,\varrho,\upsilon)\Delta\upsilon\Delta\varrho\Delta\varpi\}\Delta e\Delta p\Delta v +$
$\mathrm{L}\mathrm{B}(ɱ,ᶖ,ɼ) + \int_{x_0}^{ɱ}\int_{y_0}^{ᶖ}\int_{z_0}^{ɼ} \mathrm{J}_1(ɱ,ᶖ,ɼ,\varpi,\varrho,\upsilon)\mathrm{B}(\varpi,\varrho,\upsilon)\Delta\upsilon\Delta\varrho\Delta\varpi + \int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} \mathrm{J}_2\,(ɱ,ᶖ,ɼ,\varpi,\varrho,\upsilon)\mathrm{B}(\varpi,\varrho,\upsilon)\Delta\upsilon\Delta\varrho\Delta\varpi,$

$\mathrm{B}(ɱ,ᶖ,ɼ)(1 - \mathrm{L}) \leq \varepsilon\big(1 + (ɱ - x_0)(ᶖ - y_0)(ɼ - z_0)\big) + \tau + \int_{x_0}^{ɱ}\int_{y_0}^{ᶖ}\int_{z_0}^{ɼ}\{\mathrm{L}\,\mathrm{B}(v,p,e) +$
$\int_{x_0}^{v}\int_{y_0}^{p}\int_{z_0}^{e} \mathrm{J}_1\,(v,p,e,\varpi,\varrho,\upsilon)\mathrm{B}(\varpi,\varrho,\upsilon)\Delta\upsilon\Delta\varrho\Delta\varpi + \int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} \mathrm{J}_2\,(v,p,e,\varpi,\varrho,\upsilon)\mathrm{B}(\varpi,\varrho,\upsilon)\Delta\upsilon\Delta\varrho\Delta\varpi\}\Delta e\Delta p\Delta v +$
$\int_{x_0}^{ɱ}\int_{y_0}^{ᶖ}\int_{z_0}^{ɼ} \mathrm{J}_1(ɱ,ᶖ,ɼ,\varpi,\varrho,\upsilon)\mathrm{B}(\varpi,\varrho,\upsilon)\Delta\upsilon\Delta\varrho\Delta\varpi + \int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} \mathrm{J}_2\,(ɱ,ᶖ,ɼ,\varpi,\varrho,\upsilon)\mathrm{B}(\varpi,\varrho,\upsilon)\Delta\upsilon\Delta\varrho\Delta\varpi,$ (27)

from relation (27) we get

$\mathrm{B}(ɱ, ꞁ, ꞅ) \leq \frac{\varepsilon(1+(\ell_1-x_0)(\ell_2-y_0)(\ell_3-z_0))+\tau}{1-\mathrm{L}} + \frac{1}{1-\mathrm{L}}[\int_{x_0}^{ɱ}\int_{y_0}^{ꞁ}\int_{z_0}^{ꞅ}\{\mathrm{L}\,\mathrm{B}(v,p,e) + \int_{x_0}^{v}\int_{y_0}^{p}\int_{z_0}^{e}\mathrm{J}_1(v,p,e,\varpi,\varrho,\upsilon)\mathrm{B}(\varpi,\varrho,\upsilon)\Delta\upsilon\Delta\varrho\Delta\varpi +$
$\mathrm{J}_1(ɱ,ꞁ,ꞅ,v,p,e)\mathrm{B}(v,p,e) + \int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3}\mathrm{J}_2(v,p,e,\varpi,\varrho,\upsilon)\mathrm{B}(\varpi,\varrho,\upsilon)\Delta\upsilon\Delta\varrho\Delta\varpi\}\Delta e\Delta p\Delta v] +$
$\frac{1}{1-\mathrm{L}}\int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3}\mathrm{J}_2(ɱ,ꞁ,ꞅ,\varpi,\varrho,\upsilon)\mathrm{B}(\varpi,\varrho,\upsilon)\Delta\upsilon\Delta\varrho\Delta\varpi.$ (28)

Appropriately applying the Lemma 3.1 over (28) yields (18). ■

Remark 3.3: If $\mathrm{D}_1(ɱ,ꞁ,ꞅ)$ is a solution (1) with (2) then $\varepsilon_1 = 0$, we note that $\mathrm{D}_2(ɱ,ꞁ,ꞅ) \to \mathrm{D}_1(ɱ,ꞁ,ꞅ)$ when $\tau \to 0$ and $\varepsilon_2 \to 0$, and if we put $\varepsilon_1 = \varepsilon_2 = 0$, and $\rho_1(ɱ,ꞁ) = \rho_2(ɱ,ꞁ)$, $\delta_1(ɱ,ꞅ) = \delta_2(ɱ,ꞅ)$, $\alpha_1(ꞁ,ꞅ) = \alpha_2(ꞁ,ꞅ)$, $\beta_1(ɱ) = \beta_2(ɱ)$, $\sigma_1(ꞁ) = \sigma_2(ꞁ)$, $\xi_1(ꞅ) = \xi_2(ꞅ)$, in equation (18) then we get the unique solution of equation (1) with the initial-boundary conditions (2).

Now we put the initial-boundary values Problem (1), (2) together with the following initial-boundary values Problem

$\mathrm{D}^{\Delta_3\Delta_2\Delta_1}(ɱ,ꞁ,ꞅ) = \bar{F}(ɱ,ꞁ,ꞅ,\mathrm{D}(ɱ,ꞁ,ꞅ),\mathrm{D}^{\Delta_3\Delta_2\Delta_1}(ɱ,ꞁ,ꞅ),(H\mathrm{D})(ɱ,ꞁ,ꞅ),(G\mathrm{D})(ɱ,ꞁ,ꞅ)),$ (29)

with the following initial-boundary conditions

$\mathrm{D}(ɱ,ꞁ,z_0) = \bar{\rho}(ɱ,ꞁ)$, $\mathrm{D}(ɱ,y_0,ꞅ) = \bar{\delta}(ɱ,ꞅ)$, $\mathrm{D}(x_0,ꞁ,ꞅ) = \bar{\alpha}(ꞁ,ꞅ)$, $\mathrm{D}(ɱ,y_0,z_0) = \bar{\beta}(ɱ)$,
$\mathrm{D}(x_0,ꞁ,z_0) = \bar{\sigma}(ꞁ)$, $\mathrm{D}(x_0,y_0,ꞅ) = \bar{\xi}(ꞅ)$, $\mathrm{D}(x_0,y_0,z_0) = 0$, (30)

for all $\bar{F} \in \mathrm{C}_{rd}(\mathrm{A}\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)$, $\mathrm{A} = \mathbb{T}_1\times\mathbb{T}_2\times\mathbb{T}_3$,
$\rho,\delta,\alpha,\bar{\rho},\bar{\delta},\bar{\alpha} \in (\mathbb{T}\times\mathbb{T},\mathbb{R}^n)$, $\beta,\sigma,\xi,\bar{\beta},\bar{\sigma},\bar{\xi} \in (\mathbb{T},\mathbb{R}^n)$ and $(H\mathrm{D}),(G\mathrm{D})$ as in (3), (4).

The following theorem is related to the conditions of the close of solutions for problems (1), (2) with problems (29), (30).

Theorem 3.4: If we assume that F, h, and g in eqn. (1) satisfy the following condition (13), (14), (15) and $\bar{\varepsilon},\bar{\tau} \geq 0$ are non-negative constants, such that

$|F(ɱ,ꞁ,ꞅ,q,s,b) - \bar{F}(ɱ,ꞁ,ꞅ,q,s,b)| \leq \bar{\varepsilon},$ (31)

$|\rho(ɱ,ꞁ) - \bar{\rho}(ɱ,ꞁ) + \delta(ɱ,ꞅ) - \bar{\delta}(ɱ,ꞅ) + \alpha(ꞁ,ꞅ) - \bar{\alpha}(ꞁ,ꞅ) - \beta(ɱ) + \bar{\beta}(ɱ) - \sigma(ꞁ) + \bar{\sigma}(ꞁ) - \xi(ꞅ) + \bar{\xi}(ꞅ)| \leq \bar{\tau},$ (32)

if $\mathrm{D}_1(ɱ,ꞁ,ꞅ)$, $\mathrm{D}_2(ɱ,ꞁ,ꞅ)$ are any two solutions to problems (1), (2) and (29), (30), then

$$|\mathrm{D}_1(ɱ,ꞁ,ꞅ) - \mathrm{D}_2(ɱ,ꞁ,ꞅ)| + |{\mathrm{D}_1}^{\Delta_3\Delta_2\Delta_1}(ɱ,ꞁ,ꞅ) - {\mathrm{D}_2}^{\Delta_3\Delta_2\Delta_1}(ɱ,ꞁ,ꞅ)| \leq \frac{\frac{\bar{\varepsilon}(1+(\ell_1-x_0)(\ell_2-y_0)(\ell_3-z_0))+\bar{\tau}}{1-\mathrm{L}}}{\bar{\vartheta}_1(ɱ,ꞁ,ꞅ)} + \frac{\frac{\bar{\varepsilon}(1+(\ell_1-x_0)(\ell_2-y_0)(\ell_3-z_0))+\bar{\tau}}{1-\mathrm{L}}}{\bar{\vartheta}_1(ɱ,ꞁ,ꞅ)\bar{N}(\ell_1,\ell_2,\ell_3)}. \quad (33)$$

Where $(ɱ,ꞁ,ꞅ) \in \mathrm{A}$, $\bar{F} \in \mathrm{C}_{rd}(\mathrm{A}\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)$, $\bar{\vartheta}_1(ɱ,ꞁ,ꞅ)$, $\bar{N}(\ell_1,\ell_2,\ell_3)$ as in (19), (21).

Proof: Let

$$\mathrm{B}(ɱ,ꞁ,ꞅ) = [|\mathrm{D}_1(ɱ,ꞁ,ꞅ) - \mathrm{D}_2(ɱ,ꞁ,ꞅ)| + |{\mathrm{D}_1}^{\Delta_3\Delta_2\Delta_1}(ɱ,ꞁ,ꞅ) - {\mathrm{D}_2}^{\Delta_3\Delta_2\Delta_1}(ɱ,ꞁ,ꞅ)|],$$

for $(ɱ,ꞁ,ꞅ) \in \mathrm{A}$, since $\mathrm{D}_1(ɱ,ꞁ,ꞅ)$ and $\mathrm{D}_2(ɱ,ꞁ,ꞅ)$ are solutions of the problems (1), (2) and (29), (30) respectively, then

$\mathrm{B}(ɱ,ꞁ,ꞅ) \leq |\rho(ɱ,ꞁ) - \bar{\rho}(ɱ,ꞁ) + \delta(ɱ,ꞅ) - \bar{\delta}(ɱ,ꞅ) + \alpha(ꞁ,ꞅ) - \bar{\alpha}(ꞁ,ꞅ) - \beta(ɱ) + \bar{\beta}(ɱ) - \sigma(ꞁ) + \bar{\sigma}(ꞁ) - \xi(ꞅ) + \bar{\xi}(ꞅ)| +$
$\int_{x_0}^{ɱ}\int_{y_0}^{ꞁ}\int_{z_0}^{ꞅ}|F(v,p,e,\mathrm{D}_1(v,p,e),{\mathrm{D}_1}^{\Delta_3\Delta_2\Delta_1}(v,p,e),(H\mathrm{D}_1)(v,p,e),(G\mathrm{D}_1)(v,p,e)) -$
$\bar{F}(v,p,e,\mathrm{D}_2(v,p,e),{\mathrm{D}_2}^{\Delta_3\Delta_2\Delta_1}(v,p,e),(H\mathrm{D}_2)(v,p,e),(G\mathrm{D}_2)(v,p,e))|\Delta e\Delta p\Delta v +$
$|F(ɱ,ꞁ,ꞅ,\mathrm{D}_1(ɱ,ꞁ,ꞅ),{\mathrm{D}_1}^{\Delta_3\Delta_2\Delta_1}(ɱ,ꞁ,ꞅ),(H\mathrm{D}_1)(ɱ,ꞁ,ꞅ),(G\mathrm{D}_1)(ɱ,ꞁ,ꞅ)) -$
$F(ɱ,ꞁ,ꞅ,\mathrm{D}_2(ɱ,ꞁ,ꞅ),{\mathrm{D}_2}^{\Delta_3\Delta_2\Delta_1}(ɱ,ꞁ,ꞅ),(H\mathrm{D}_2)(ɱ,ꞁ,ꞅ),(G\mathrm{D}_2)(ɱ,ꞁ,ꞅ))|,$ (34)

hence equation (34) can be written in the following form

$\mathrm{B}(ɱ,ꞁ,ꞅ) \leq |\rho(ɱ,ꞁ) - \bar{\rho}(ɱ,ꞁ) + \delta(ɱ,ꞅ) - \bar{\delta}(ɱ,ꞅ) + \alpha(ꞁ,ꞅ) - \bar{\alpha}(ꞁ,ꞅ) - \beta(ɱ) + \bar{\beta}(ɱ) - \sigma(ꞁ) + \bar{\sigma}(ꞁ) - \xi(ꞅ) + \bar{\xi}(ꞅ)| +$
$\int_{x_0}^{ɱ}\int_{y_0}^{ꞁ}\int_{z_0}^{ꞅ}|F(v,p,e,\mathrm{D}_1(v,p,e),{\mathrm{D}_1}^{\Delta_3\Delta_2\Delta_1}(v,p,e),(H\mathrm{D}_1)(v,p,e),(G\mathrm{D}_1)(v,p,e)) -$
$F(v,p,e,\mathrm{D}_2(v,p,e),{\mathrm{D}_2}^{\Delta_3\Delta_2\Delta_1}(v,p,e),(H\mathrm{D}_2)(v,p,e),(G\mathrm{D}_2)(v,p,e))|\Delta e\Delta p\Delta v +$
$\int_{x_0}^{ɱ}\int_{y_0}^{ꞁ}\int_{z_0}^{ꞅ}|F(v,p,e,\mathrm{D}_2(v,p,e),{\mathrm{D}_2}^{\Delta_3\Delta_2\Delta_1}(v,p,e),(H\mathrm{D}_2)(v,p,e),(G\mathrm{D}_2)(v,p,e)) -$
$\bar{F}(v,p,e,\mathrm{D}_2(v,p,e),{\mathrm{D}_2}^{\Delta_3\Delta_2\Delta_1}(v,p,e),(H\mathrm{D}_2)(v,p,e),(G\mathrm{D}_2)(v,p,e))|\Delta e\Delta p\Delta v +$
$|F(ɱ,ꞁ,ꞅ,\mathrm{D}_1(ɱ,ꞁ,ꞅ),{\mathrm{D}_1}^{\Delta_3\Delta_2\Delta_1}(ɱ,ꞁ,ꞅ),(H\mathrm{D}_1)(ɱ,ꞁ,ꞅ),(G\mathrm{D}_2)(ɱ,ꞁ,ꞅ)) -$
$F(ɱ,ꞁ,ꞅ,\mathrm{D}_2(ɱ,ꞁ,ꞅ),{\mathrm{D}_2}^{\Delta_3\Delta_2\Delta_1}(ɱ,ꞁ,ꞅ),(H\mathrm{D}_2)(ɱ,ꞁ,ꞅ),(G\mathrm{D}_2)(ɱ,ꞁ,ꞅ))| +$
$|F(ɱ,ꞁ,ꞅ,\mathrm{D}_2(ɱ,ꞁ,ꞅ),{\mathrm{D}_2}^{\Delta_3\Delta_2\Delta_1}(ɱ,ꞁ,ꞅ),(H\mathrm{D}_2)(ɱ,ꞁ,ꞅ),(G\mathrm{D}_2)(ɱ,ꞁ,ꞅ)) -$
$\bar{F}(ɱ,ꞁ,ꞅ,\mathrm{D}_2(ɱ,ꞁ,ꞅ),{\mathrm{D}_2}^{\Delta_3\Delta_2\Delta_1}(ɱ,ꞁ,ꞅ),(H\mathrm{D}_2)(ɱ,ꞁ,ꞅ),(G\mathrm{D}_2)(ɱ,ꞁ,ꞅ))|.$

Using hypotheses (13), (14), and (15), and through hypothesis (32), we have

$\mathrm{B}(ɱ,ꞁ,ꞅ) \leq \bar{\tau} + \int_{x_0}^{ɱ}\int_{y_0}^{ꞁ}\int_{z_0}^{ꞅ}\{\mathrm{L}[|\mathrm{D}_1(v,p,e) - \mathrm{D}_2(v,p,e)| + |{\mathrm{D}_1}^{\Delta_3\Delta_2\Delta_1}(v,p,e) - {\mathrm{D}_2}^{\Delta_3\Delta_2\Delta_1}(v,p,e)|] +$
$\int_{x_0}^{v}\int_{y_0}^{p}\int_{z_0}^{e}\mathrm{J}_1(v,p,e,\zeta,\lambda,\tau)[|\mathrm{D}_1(\zeta,\lambda,\tau) - \mathrm{D}_2(\zeta,\lambda,\tau)| + |{\mathrm{D}_1}^{\Delta_3\Delta_2\Delta_1}(\zeta,\lambda,\tau) - {\mathrm{D}_2}^{\Delta_3\Delta_2\Delta_1}(\zeta,\lambda,\tau)|]\Delta\tau\Delta\lambda\Delta\zeta +$

$\int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} J_2(v,p,e,\zeta,\lambda,\tau)[|D_1(\zeta,\lambda,\tau)-D_2(\zeta,\lambda,\tau)|+|D_1^{\Delta_3\Delta_2\Delta_1}(\zeta,\lambda,\tau)-D_2^{\Delta_3\Delta_2\Delta_1}(\zeta,\lambda,\tau)|]\Delta\tau\Delta\lambda\Delta\zeta\}\Delta e\Delta p\Delta v+$
$\int_{x_0}^{m}\int_{y_0}^{i}\int_{z_0}^{r}\bar{\varepsilon}\,\Delta e\Delta p\Delta v+L[|D_1(m,i,r)-D_2(m,i,r)|+|D_1^{\Delta_3\Delta_2\Delta_1}(m,i,r)-D_2^{\Delta_3\Delta_2\Delta_1}(m,i,r)|]+$
$\int_{x_0}^{m}\int_{y_0}^{i}\int_{z_0}^{r} J_1(m,i,r,\zeta,\lambda,\tau)\left[|D_1(\zeta,\lambda,\tau)-D_2(\zeta,\lambda,\tau)|+\left|D_1^{\Delta_3\Delta_2\Delta_1}(\zeta,\lambda,\tau)-D_2^{\Delta_3\Delta_2\Delta_1}(\zeta,\lambda,\tau)\right|\right]\Delta\tau\Delta\lambda\Delta\zeta+$
$\int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} J_2(m,i,r,\zeta,\lambda,\tau)\left[|D_1(\zeta,\lambda,\tau)-D_2(\zeta,\lambda,\tau)|+\left|D_1^{\Delta_3\Delta_2\Delta_1}(\zeta,\lambda,\tau)-D_2^{\Delta_1\Delta_2\Delta_3}(\zeta,\lambda,\tau)\right|\right]\Delta\tau\Delta\lambda\Delta\zeta+\bar{\varepsilon}$
$=\bar{\tau}+\bar{\varepsilon}(1+(m-x_0)(i-y_0)(r-z_0))+\int_{x_0}^{m}\int_{y_0}^{i}\int_{z_0}^{r}\{LB(v,p,e)+\int_{x_0}^{v}\int_{y_0}^{p}\int_{z_0}^{e} J_1(v,p,e,\zeta,\lambda,\tau)B(\zeta,\lambda,\tau)\Delta\tau\Delta\lambda\Delta\zeta+$
$\int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} J_2(v,p,e,\zeta,\lambda,\tau)B(\zeta,\lambda,\tau)\Delta\tau\Delta\lambda\Delta\zeta\}\Delta e\Delta p\Delta v+LB(m,i,r)+\int_{x_0}^{m}\int_{y_0}^{i}\int_{z_0}^{r} J_1(m,i,r,\zeta,\lambda,\tau)B(\zeta,\lambda,\tau)\Delta\tau\Delta\lambda\Delta\zeta+$
$\int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} J_2(m,i,r,\zeta,\lambda,\tau)B(\zeta,\lambda,\tau)\Delta\tau\Delta\lambda\Delta\zeta,$
$B(m,i,r)(1-L)\le\bar{\tau}+\bar{\varepsilon}\big(1+(m-x_0)(i-y_0)(r-z_0)\big)+\int_{x_0}^{m}\int_{y_0}^{i}\int_{z_0}^{r}\{LB(v,p,e)+$
$\int_{x_0}^{v}\int_{y_0}^{p}\int_{z_0}^{e} J_1(v,p,e,\zeta,\lambda,\tau)B(\zeta,\lambda,\tau)\Delta\tau\Delta\lambda\Delta\zeta+\int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} J_2(v,p,e,\zeta,\lambda,\tau)B(\zeta,\lambda,\tau)\Delta\tau\Delta\lambda\Delta\zeta\}\Delta e\Delta p\Delta v+$
$\int_{x_0}^{m}\int_{y_0}^{i}\int_{z_0}^{r} J_1(m,i,r,\zeta,\lambda,\tau)B(\zeta,\lambda,\tau)\Delta\tau\Delta\lambda\Delta\zeta+\int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} J_2(m,i,r,\zeta,\lambda,\tau)B(\zeta,\lambda,\tau)\Delta\tau\Delta\lambda\Delta\zeta,$

$$B(m,i,r)\le\frac{\bar{\varepsilon}\big(1+(\ell_1-x_0)(\ell_2-y_0)(\ell_3-z_0)\big)+\bar{\tau}}{1-L}+\frac{1}{1-L}\int_{x_0}^{m}\int_{y_0}^{i}\int_{z_0}^{r}\{LB(v,p,e)+\int_{x_0}^{v}\int_{y_0}^{p}\int_{z_0}^{e} J_1(v,p,e,\zeta,\lambda,\tau)B(\zeta,\lambda,\tau)\Delta\tau\Delta\lambda\Delta\zeta+ J_1(m,i,r,v,p,e)B(v,p,e)+\int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} J_2(v,p,e,\zeta,\lambda,\tau)B(\zeta,\lambda,\tau)\Delta\tau\Delta\lambda\Delta\zeta\}\Delta e\Delta p\Delta v+ \frac{1}{1-L}\int_{x_0}^{\ell_1}\int_{y_0}^{\ell_2}\int_{z_0}^{\ell_3} J_2(m,i,r,\zeta,\lambda,\tau)B(\zeta,\lambda,\tau)\Delta\tau\Delta\lambda\Delta\zeta. \quad (35)$$

A suitable application of Lemma 3.1 to equation (35) we get (33).■

Remark 3.5: The results given by theorem 3.4 are related to the solutions of two initial- boundary values problems (1), (2) and (29), (30) in the meaning if F is close to $\bar{F}$, and $\rho,\delta,\alpha,\beta,\sigma,\xi$ are close to $\bar{\rho},\bar{\delta}\,\bar{\alpha},\bar{\beta},\bar{\sigma},\bar{\xi}$ respectively, then the solution to problems (1), (2), and problems (29), and (30) are also close to each other.

4. Conclusions

In this work, we conclude the following points

1. The explicit estimation of integral inequality in three variables on time scales has been derived.
2. The estimating of the difference between two ε-approximate solutions for the integro-differential equation is found.
3. We present the conditions that set the functions in the integro-differential equation leading to the closeness of the solutions.

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حول معادلة تكاملية تفاضلية جزئية على مقاييس الزمن

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الخلاصة

يهدف هذا البحث لدراسة اشتقاق تخمين لمتباينة تكاملية في ثلاثة متغيرات على مقاييس زمنية. تم استخدام نتيجة هذا التخمين لدراسة بعض الخصائص لحل معادلات تكاملية تفاضلية جزئية مع شروط الحدودية الابتدائية على المقاييس الزمنية، مثل تقدير الفرق بين حلين تقريبيين، والتقارب بين الحلول. هذا الاختلاف له تطبيقات عديدة في المجالات العلمية المختلفة في بعض فروع الرياضيات والفيزياء والاقتصاد والكهرباء، والأحياء. في هذا البحث قمنا بدراسة المسألة من خلال عرض نقطتين، أولاً نتعامل مع تقدير الفرق بين حلين تقريبيين (ابسلونيين) لمعادلة تكاملية تفاضلية غير خطية خاصة مع شروط الحدودية الابتدائية، التي من خلالها يمكن الحصول على تقدير مناسب للحلول التقريبية، وثانيًا لتوفير شروط التقارب لحل المسألة قيد الدراسة.