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The Hausdorff Algebra Fuzzy Distance and its Basic Properties

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ABSTRACT

In this article we recall the definition of algebra fuzzy metric space and its basic properties. In order to introduced the Hausdorff algebra fuzzy metric from fuzzy compact set to another fuzzy compact set we define the algebra fuzzy distance between two fuzzy compact sets after that basic properties of the Hausdorff algebra fuzzy metric between two fuzzy compact sets are proved. Finally the main result in this paper is proved that is if (S, m, ⊙) is a fuzzy complete algebra fuzzy metric space then (AFH(S), h, ⊙) is a fuzzy complete algebra fuzzy metric space.

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1. Introduction

Kider in 2011, [1] introduced the definition of a fuzzy normed space. Also he proved this fuzzy normed space has a completion in [2]. Also Kider in 2012, [3] introduce a new type of fuzzy normed space. Kider in 2014, [4] proved that the Hausdorff standard fuzzy metric space is complete. Kider and Kadhum in 2017, [5] introduce the fuzzy norm for a fuzzy bounded operator on a fuzzy normed space and proved its basic properties then other properties was proved by Kadhum in 2017 [6]. Ali in 2018, [7] proved basic properties of complete fuzzy normed algebra. Again Kider and Ali in 2018, [8] introduce the notion of fuzzy absolute value and study properties of finite dimensional fuzzy normed space.

The concept of general fuzzy normed space were presented by Kider and Gheeab in 2019, [9] [10] also they proved basic properties of this space and the general fuzzy normed space GFB (V, U). Kider and Kadhum in 2019, [11] introduce the notion fuzzy compact linear operator and proved its basic properties. Kider in 2020, [12] introduce the notion fuzzy soft metric space after that he investigated and proved some basic properties of this space again Kider in 2020, [13] introduce new type of fuzzy metric space called algebra fuzzy metric space after that the basic properties of this space is proved.

Here in this work the definition of algebra fuzzy metric space is used then basic properties of this space with some examples is recalled. After that the algebra fuzzy metric from a point in the universal set to a fuzzy compact set and the algebra fuzzy metric from a fuzzy compact set in the universal set to another a fuzzy compact set are defined. This will be the back ground to define the Hausdorff algebra fuzzy metric from a fuzzy compact set in the universal set to a fuzzy compact set. Then basic properties of the Hausdorff algebra fuzzy metric space is investigated and proved. The final result in this paper is that if (S, m, ⊚) is a fuzzy complete algebra fuzzy metric space then (AFH(S), h, ⊚) is a fuzzy complete algebra fuzzy metric space where AFH(S) is the set of all nonempty fuzzy compact set in S.

2. Basic Properties of Algebra fuzzy metric space

Definition 2.1: [13]

Let ⊚:I × I →I be a binary operation function then ⊚ is said to be continuous t-conorm (or simply t-conorm) if it satisfies the following conditions s, r ,z, w ∈ I where I=[0, 1]

(i)s ⊚ r= r ⊚ s

(ii) s ⊚ [r ⊚ z] = [s ⊚ r] ⊚ z

(iii) ⊚ is continuous function

(iv)s ⊚ 0 = 0

(v) (s ⊚ r) ≤ (z ⊚ w) whenever s ≤ z and r ≤ w.

Lemma 2.2:[13]

If ⊚ is a continuous t-conorm on I then

(i)1 ⊚ 1=1

(ii)0 ⊚ 1 = 1 ⊚ 0 =1

(iii)0 ⊚ 0 = 0

(iv)r ⊚ r ≥ r for all r ∈ [0, 1].

Example 2.3:[13]

The algebra product a ⊚ b = a + b − ab is a continuous t-conorm for all a, b ∈I.

Definition 2.4:[13]

Assume that S≠ ∅, a fuzzy set $\widetilde{D}$ in S is represented by $\widetilde{D}$ = {(s, $\mu_{\widetilde{D}}$(s)): s∈S, 0 ≤ $\mu_{\widetilde{D}}$(s) ≤ 1}where $\mu_{\widetilde{D}}$(x): S→ I is a membership function.

Definition 2.5:[13]

If S≠ ∅, ⊚ is a continuous t- conorm and m: S×S→I satisfying the following conditions:

(1)0 < m(s, r) ≤ 1 ;

(2) m(s, r) = 0 if and only if s = r;

(3) m(s, r) = m(r, s) ;

(4) m(s, t) ≤ m(s, r) ⊚ m(r, t)

For all s, r, t ∈ S then the triple (S, m, ⊚) is said to be the algebra fuzzy metric space

Example 2.6:[13]

If (S, d) is a metric space and t ⊚ r = t + r − tr for all t, r ∈ [0, 1]. Put m_d(s, u) = $\frac{d(s,u)}{1+d(s,u)}$ for all s, u ∈ S. Then (S, m_d, ⊚) is algebra fuzzy metric space. m_d is known as the algebra fuzzy metric comes from d.

Example 2.7:[13]

If S≠ ∅ put $m_D(s, u) = \begin{cases} 0 & if\ s = u \\ 1 & if\ s \neq u \end{cases}$

Then (S, m_D, ⊚) is algebra fuzzy metric space known as the discrete algebra fuzzy metric space.

Definition 2.8:[13]

If (S, m, ⊚) is algebra fuzzy metric space then fb(s, j) ={u∈S: m(s, u) <j} is known as an open fuzzy ball with center s∈S and radius j ∈(0, 1). Similarly closed fuzzy ball is defined by fb[s, j] ={u∈S: m(s, u) ≤j}.

Definition 2.9:[13]
If (S, m, ⊚) is algebra fuzzy metric space and W⊆S is known as fuzzy open if fb(w, j) ⊆W for any arbitrary w∈W and for some j ∈(0, 1).
Also D⊆S is known as fuzzy closed if D^C is fuzzy open then the fuzzy closure of D, $\overline{D}$ is defined to be the smallest fuzzy closed set contains D.
Definition 2.10:[13]
If (S, m, ⊚) is algebra fuzzy metric space then D⊆S is known as fuzzy dense in S whenever $\overline{D}$=S.
Theorem 2.11:[13]
If FB(s, j) is open fuzzy ball in algebra fuzzy metric space (S, m, ⊚) then it is a fuzzy open set.
Proposition 2.12:[13]
Suppose that (S, m, ⊚) is algebra fuzzy metric space then s_n →s if and only if m(s_n, s) →0.
Definition 2.13:[13]
In algebra fuzzy metric space (S, m, ⊚) a sequence (s_n) is fuzzy Cauchy if for any given 0<t<1 then there is N ∈ ℕ with m(s_n, s_m)< t, for each m, n ≤ N.
Definition 2.14:[13]

An algebra fuzzy metric space (S, m, ⊚) is known as fuzzy complete if (s_n) is fuzzy Cauchy sequence then s_n →s ∈ S.

Theorem 2.15:[13]
In algebra fuzzy metric space (S, m, ⊚) if s_n →s ∈ S then (s_n) is fuzzy Cauchy sequence.
Definition 2.16:[13]
Anon empty set D in algebra fuzzy metric space (S, m, ⊚) is known as fuzzy bounded whenever we can find t∈(0, 1) with D ⊂ FB(s, t) for some s ∈S. Also a sequence (d_n) in algebra fuzzy metric space (S, m, ⊚) is fuzzy bounded if we can find t∈(0, 1) with $\{d_1, d_2, \ldots, d_n, \ldots\}$ ⊆ FB(s, t) for some s ∈S.
Lemma 2.17:[13]
In algebra fuzzy metric space (S, m, ⊚) If the sequence (s_n) ∈S with s_n →s ∈S. Then (s_n) is fuzzy bounded.
Lemma 2.18:[13]
In algebra fuzzy metric space (S, m, ⊚) if (s_n) ∈ S with s_n →s ∈S and s_n → d ∈S as n→ ∞. Then s=d.
Theorem 2.19: [13]

In algebra fuzzy metric space (S, m, ⊚) when D⊂S then d∈$\overline{D}$ if and only if there is (d_n) ∈ D with d_n→ d.

Definition 2.20:[13]
If (S, m_S, ⊚) and (V, m_V, ⊚) are two algebra fuzzy metric spaces and U ⊆ S. Then a function T:S→V is called fuzzy continuous at u∈U. If for every 0 < r < 1, we can find some 0 < t < 1, with m_V[T(u), T(s)] < r as s∈U and m_S(u, s) < t.

If T is fuzzy continuous at every point of U then T is said to be fuzzy continuous on U.

Theorem 2.21:[13]

If (S, m_S, ⊚) and (V, m_V, ⊚) are two algebra fuzzy metric spaces and U ⊆ S. Then a function T:S→V is fuzzy continuous at u∈U ⇔ whenever u_n → u in U then T(u_n)→ T(u) in V.

Theorem 2.22:[13]
The function T:S→V is fuzzy continuous on S if and only if T^{-1}(D) is fuzzy open in S for all fuzzy open subset D of V where (S, m_S, ⊚) and (V, m_V, ⊚) are algebra fuzzy metric spaces.

Here we will began to introduce the basic notions and some properties of these notions that will be used later in section three.
Definition 2.23:

Suppose that (S, m, ⊚) is algebra fuzzy metric space then S is **fuzzy compact** if every fuzzy open covering Ω of U has a finite fuzzy open sub covering that is there is a finite sub collection $\{O_1, O_2, O_3, \ldots, O_k\} \subseteq \Omega$ such that $U = \cup_{j=1}^{k} O_j$.

Definition 2.24:

Assume that (S, m, ⊚) is algebra fuzzy metric space and P be a subset of S. Then P is called **totally fuzzy bounded** if for each 0 < r < 1 , there is a finite set of points $\{p_1, p_2, \ldots, p_k\} \subseteq P$ such that whenever u in S, m(u, p_j) < r for some $p_j \in \{p_1, p_2, \ldots, p_k\}$. This set of points $\{p_1, p_2, \ldots, p_k\}$ is called **fuzzy r-net**.

Proposition 2.25:

A totally bounded algebra fuzzy metric space is fuzzy bounded.

Proof:

Suppose that (S, m, ⊚) is totally fuzzy bounded and let 0 < r < 1 is given. Then there exists a finite fuzzy r- net for S, say A. Since A is a finite set of points and 0 < n(A) < 1, where n(A) = sup{m(d, a): d, a∈A}. Now let u_1 and u_2 be any two points of S. There exists points d and a in A such that m(u_1, d) <r and m(u_2, a) < r . Now for n(A) and r there is t, where 0 < q < 1 such that r ⊚ n(A) ⊚ r ≤ q. It follows that

m(u_1, u_2) ≤ m(u_1, d) ⊚ m(d, a) ⊚ m(a, u_2) ≤ r ⊚ n(A) ⊚ r ≤ q

So, n(S) = sup{m(u_1, u_2):u_1, u_2 ∈S}≤ t . Hence S is fuzzy bounded.

Proposition 2.26:

If (S, m, ⊚) is a compact algebra fuzzy metric space then S is fuzzy totally bounded.

Proof:

Let 0 < r < 1 be given then {fb(p, r) :for p∈S} is a fuzzy open cover of S. But S is fuzzy compact this implies that {fb(p, r) :for p∈S} contains {fb(p_1, r), fb(p_2, r),…, fb(p_k, r)} such that $S = \cup_{j=1}^{k} fb(p_j, r)$. Hence for 0 < r < 1, { $p_1, p_2. \ldots, p_k$}⊆ S is a finite fuzzy r-net for S. Hence S is fuzzy totally bounded.

Proposition 2.27:

If (S, m, ⊚) is a compact algebra fuzzy metric space then (S, m, ⊚) is fuzzy complete.

Proof:

Assume that (S, m, ⊚) is a fuzzy compact algebra fuzzy metric which is not fuzzy complete. Then we can find a fuzzy Cauchy (p_k) in S does not has a limit in S. Let p∈S, since $p_k \nrightarrow p$ ∃ 0 < r < 1 such that m(p_k, p) ≥ r for k=1, 2, … but (p_k) is fuzzy Cauchy ∃ N∈ ℕ s. t. m(p_j, p_m) <t ∀ j, m ≥ N. Choose m ≥ N for which m(p_m, p) < t . So, the fuzzy open ball fb(p, t) contains { p_1 , p_2, …,p_k} where k ∈ ℕ. Now consider fuzzy { fb(p_1, t(p_1)), fb(p_2, t(p_2)),… fb(p_k, t(p_k))} where 0 < t(p_k) < 1 and $S = \cup_{j=1}^{k} fb(p_j, t(p_j))$. But each fb($p_j$, t($p_j$))} contains p_k for only a finite number of values so S, must contains p_k for only a finite number of values of k. This is a contradiction. Hence (S, m, ⊚) must be fuzzy complete.

Theorem 2.28:

If (S, m, ⊚) is totally bounded and fuzzy complete algebra fuzzy metric space then (S, m, ⊚) is fuzzy compact.

Proof:

Assume that (S, m, ⊚) is not fuzzy compact. Then we can find a fuzzy open covering $\{O_\lambda : \lambda \in \Lambda\}$ of S that does not have a finite fuzzy open sub covering. But S is fuzzy totally bounded, so it is fuzzy bounded, hence consider fb(p, r) for some 0 < r < 1 and some p∈S, clearly fb(p, r) ⊆S if S ⊆ fb(p, r) then we must have S = fb(p, r).

Put $t_k = \frac{r}{2^k}$ but S is fuzzy totally bounded this means that S can be covered by finite fuzzy open balls of radius t_1. By our assumption at least one of these fuzzy open balls, say fb(p_1, t_1)≠ $\cup_{j=1}^{k} O_{\lambda_j}$ since fb(p_1, t_1) is itself fuzzy totally bounded. ∃ p_2 ∈ fb(p_1, t_1), s. t. fb(p_2, t_2))≠ $\cup_{j=1}^{k} O_{\lambda_j}$.

After many steps we get a sequence (p_k) has the property that for each k, $fb(p_k, t_k)$)≠ $\cup_{j=1}^{k} O_{\lambda_j}$ and p_{k+1} ∈$fb(p_k,t_k)$. We next show that the sequence (p_k) is convergent. Since p_{k+1} ∈$fb(p_k,t_k)$ it follows that $m(p_k, p_{k+1}) < t_k$. Let 0 < t < 1 such that $t_k \circledcirc t_{k+1} \circledcirc \ldots\ldots \circledcirc t_m < t$. Hence

$m(p_k, p_m) \le m(p_k, p_{k+1}) \circledcirc \ldots\ldots \circledcirc m(p_{m-1}, p_m) \le t_k \circledcirc t_{k+1} \circledcirc \ldots\ldots \circledcirc t_m < t$

So (p_k) is a fuzzy Cauchy sequence in S and since S is fuzzy complete, it fuzzy converges to p∈S. Because p∈S ∃ $\lambda_0 \in \Lambda$ s. t. p∈ O_{λ_0}. Since O_{λ_0} is fuzzy open it contains fb(p, s) for some 0 < s < 1. Let N∈ ℕ, $m(p_k, p) < s$ Then, for any u∈S such that $m(u, p_k) < t_k$. It follows that

$$m(u, p) \le m(u, p_k) \circledcirc m(p_k, p) \le t_k \circledcirc s < r,$$

for some 0 < r < 1. So that $fb(p_k, t_n) \subseteq fb(p, r)$. Therefore $fb(p_k,t_k)$ has a finite fuzzy open sub covering, namely by the set O_{λ_0}. Since this contradicts $fb(p_k, t_k)$)≠ $\cup_{j=1}^{k} O_{\lambda_j}$. The proof is complete.

Theorem 2.29:

Suppose that (S, m, ⊚) is a complete algebra fuzzy metric space and assume that U ⊆ S. Then U is fuzzy complete ⟺ U is fuzzy closed.

Proof:

Assume that U is fuzzy complete then by Theorem 2.19 for any u∈ $\overline{U}$ there is (u_k) ∈U with u_k →u but (u_k) is fuzzy Cauchy and U is fuzzy complete so u∈U so $\overline{U}$ ⊆U but U⊆ $\overline{U}$ this implies that U=$\overline{U}$. Hence U is fuzzy closed.

For the converse assume that U is fuzzy closed and let (w_k) be a fuzzy Cauchy in U. Then w_k →u∈S this implies that u∈ $\overline{U}$ but U=$\overline{U}$so u∈U. Hence U is fuzzy complete.

Proposition 2.30:

Suppose that (S, m, ⊚) is algebra fuzzy metric space and let U ⊆ S. If U is fuzzy compact then U is fuzzy closed and fuzzy bounded.

Proof:

By Theorem 2.19 for any w∈ $\overline{U}$ there is (w_k) ∈U with w_k →u but U is fuzzy compact so u∈U hence $\overline{U}$ ⊆U but U⊆ $\overline{U}$ this implies that U=$\overline{U}$. Thus U is fuzzy closed.

Now assume that U is fuzzy unbounded so any sequence (w_k) ∈U will be unbounded so any fuzzy open cover for U could not have a finite fuzzy open sub cover for U. This contradicts our assumption U is fuzzy compact. Hence U must be fuzzy bounded.

3.THE HAUSDORFF ALGEBRA FUZZY DISTANCE

In this section we used the definition of algebra fuzzy metric space and the basic properties of this space after that began to define the algebra fuzzy distance between two fuzzy compact sets this will give us the idea of the notion of Hausdorff algebra fuzzy distance between two compact sets. This notion is the key of all results in this section.

Definition 3.1:

Suppose that (S, m, ⊚) is a fuzzy complete algebra fuzzy metric space and let AFH(S)= {D : D ≠ ∅, D ⊂ S, D are fuzzy compact}.

Definition 3.2:

If (S, m, ⊚) is a fuzzy complete algebra fuzzy metric space and s ∈ S, D ∈ AFH(S). Then m(s, D) = inf{m(s, d) : d ∈ D} where m(s, D) is called the **algebra fuzzy distance from s to the set D.**

Remark 3.3:

The set of real numbers {m(z, d) : d ∈ D} is a subset of [0, 1] it has a infimum point and now we show that it contains its infimum point define L: D → [0, 1] by L(d) = m(z, d) for all d ∈ D then L is fuzzy continuous since m is fuzzy continuous . Assume that α = inf{ L(d) : d ∈ D} to show that there is $d_0 \in$ d such that m(z, d_0) = α we can find sequence (d_k) in D such that $L(d_k) - d_0 < \frac{1}{k}$. Now by fuzzy compactness of D we have (d_k) has a subsequence (d_{k_j}) such that $d_{k_j} \to d_0 \in D$. Finally we use fuzzy continuity of L to get L(d_0) = α.

Definition 3.4:

Suppose that (S, m, ⊚) is a fuzzy complete algebra fuzzy metric space. Let W and D ∈ AFH(S) define m(D, W) = max{ m(d, W) : d ∈ D} where m(D, W) is **the algebra fuzzy metric from D to W.**

Remark 3.5:
Similarly we can show that the m(D, W) is well defined. That is there exists $d_0 \in$ D and $w_0 \in$ W which implies that m(D, W)=m(d_0, w_0).
Lemma 3.6:
Suppose that (S, m, ⊚) is a fuzzy complete algebra fuzzy metric space. If D, W ∈ AFH(S) with D ⊆ W then m(z, W) ≤ m(z, D) for any z ∈ S.
Proof:
m(z, W) = min{ m(z, w) : w ∈ W−D} ≤ min{ m(z, d) : d ∈ D} = m(z, D) since m(z, w) ≤ m(z, d) for all w ∈ W−D and for all d ∈ D.
Lemma 3.7:
Suppose that (S, m, ⊚) is a fuzzy complete algebra fuzzy metric space. If D, W, Z ∈ AFH(S) with W ⊆ Z then m(D, Z) ≤ m(D, W).
Proof:
By Lemma 3.5 m(d, Z) ≤ m(d, W) for any d ∈ D and so m(D, Z) =max{ m(d, Z) : d ∈ D} ≤ max{m(d, W): d ∈ d} = m(M, B).
The poof of the following Lemma is clear hence is omitted.
Lemma 3.8:
Suppose that (S, m, ⊚) is a fuzzy complete algebra fuzzy metric space.
Assume that D and W ∈ AFH(S) then
(1)If D ⊆ W then m(D, W) =0.
(2) If D ≠ W then m(D, W) ≠0.
Lemma 3.9:
Suppose that (S, m, ⊚) is a fuzzy complete algebra fuzzy metric space. If
D, W, Z ∈ AFH(S) then
(i)m(D∪W, Z) = m(D, Z) ∨ m(W, Z)
(ii)m(D, W) ≤ m(D, Z) ⊚ m(Z, W)
Proof :
(i)m(D∪W, Z) = max{m(u, Z): u ∈ D∪W}
= max{m(u, Z): u ∈ D} ∨ max{m(z, Z): u ∈ W}
= m(D, Z) ∨ m(W, Z)

(ii)m(d, W) = min{m(d, w): w ∈ W}
≤ min{ m(d, z) ⊚ m(z, w): w ∈ W} for all z ∈ Z.
≤ min{ m(d, z): z ∈ z} ⊚min{m(z, w): w ∈ W} for all z ∈ Z.
≤ min{ m(d, z): z ∈ Z} ⊚ max{ min{m(z, w): w ∈ W} for all z ∈ Z}}
≤ m(d, Z) ⊚ m(Z, W) ≤ max{ m(d, Z):d∈D}⊚ m(Z, W)
Hence m(D, W) ≤ m(D, Z) ⊚ m(Z, W)
Remark 3.10:
Suppose that (S, m, ⊚) is a fuzzy complete algebra fuzzy metric space. If D, W, ∈ AFH(S) then in general m(D, W) ≠ m(W, D) for example :
Let D = { d_1, d_2 , d_3 } and W = { w_1, w_2 , w_3 } with m(d_1, w_1) = 0.25, m(d_1, w_2) = 0.5, m(d_1, w_3) =0.75 m(d_2, w_1) = 0.3, m(d_2, w_2) =0.6, m(d_2, w_3) =0.4, m(d_3, w_1) = 0.16, m(d_3, w_2) = 0.26, m(d_3, w_3)=0.83.
Now
m(D, W) =max{ m(d_1, W), m(d_2, W), m(d_3, W) }
m(d_1, W) = min{ m(d_1, w_1), m(d_1, w_2), m(d_1, w_3) }= min{ 0.25, 0.5, 0.75} =0.25.
m(d_2, W) = min{ m(d_2, w_1), m(d_2, w_2), m(d_2, w_3) }=min{ 0.3, 0.6, 0.4 } = 0.3.
m(d_3, W) = min{ m(d_3, w_1), m(d_3, w_2), m(d_3, w_3) }=min{0.16 , 0.26, 0.83 } = 0.16.
Thus m(D, W) =max{0.25, 0.3, 0.16 } = 0.3.
Similarly we can calculate
m(W, D) =max{ m(w_1, D), m(w_2, D), m(w_3, D) }
m(w_1, D) = min{ m(w_1, d_1), m(w_1, d_2), m(w_1, d_3) }=min{0.25, 0.3, 0.16 }= 0.16.
m(w_2, D) = min{ m(w_2, d_1), m(w_2, d_2), m(w_2, d_3) }=min{0.5, 0.6, 0.26 } = = 0.26.
m(w_3, D) = min{ m(w_3, d_1), m(w_3, d_2), m(w_3, d_3) }=min{0.75, 0.4, 0.83 } = 0.4.

m(W, D) =max{0.16, 0.26, 0.4 } = 0.4.
Hence m(D, W) = 0.3 $\neq$ 0.4 = m(W, D).

Definition 3.11:

Let (S, m, ⊚) be a fuzzy complete fuzzy metric space. If D, W,∈ AFH(S) then we define h(D, W) = [m(D, W) ∨ m(W, D)], to be the Hausdorff algebra fuzzy distance between D and W in AFH(S).

Remark 3.12:

It is clear that [(p⊚r)∨(t⊚q]≤[(p∨t) ⊚(r∨q)] for all p, r, t, q ∈ (0, 1).

Theorem 3.13:

If (S, m, ⊚) is a fuzzy complete algebra fuzzy metric space then (AFH(S), h, ⊚) is algebra fuzzy metric space.

Proof:

1-Since m(D, W) > 0 and m(W, D) > 0 for all D, W, ∈ AFH(S) so h(D, W) > 0
2-h(D, W) = 0 if and only if [m(D,W) ∨ m(W, D)]=0 if and only if m(D,W) = 0 and m(W, D) =0 if and only if W ⊆ D and D ⊆ W if and only if D = W.
3-It is clear that h(D, W) = h(W, D)
4-By Lemma 3.9 we have
m(D, W) ≤ m(D, Z) ⊚ m(Z, W) and
m(W, D) ≤ m(W, Z) ⊚ m(Z, D). Now
h(W, D) = [m(W, D) ∨ m(D, W)]
≤ [m(W, Z) ⊚ m(Z, D)] ∨ [m(D, Z) ⊚ m(Z, W)]
≤ [m(W, Z) ∨ m(Z, W)] ⊚ [m(Z, D) ∨ m(D, Z)] [by Remark 3.12]
= h(W, Z) ⊚ h(Z, D)
Hence (AFH(S), h, ⊚) is algebra fuzzy metric space.

Definition 3.14:

Let (S, m, ⊚) be a fuzzy complete algebra fuzzy metric space and assume that W ∈ AFH(S) then we define W⊚r = { s ∈ S: m(w, s) ≤ r for some w ∈ W, r∈(0, 1)}.

Theorem 3.15:

Let (S, m, ⊚) be a fuzzy complete algebra fuzzy metric space and assume that W, D ∈ AFH(S) and , r∈(0, 1). Then h(W, D) ≤ r if and only if W ⊂ D⊚r and D ⊂ W⊚r.

Proof:

To prove that m(W, D) ≤ r if and only if W ⊂ D⊚r. Suppose that m(W, D) ≤ r then max{m(w, D): w ∈ W}≤ r implies m(w, D) ≤ r for all w ∈ W⊆ S. Hence for each w ∈ W we have w ∈ D⊚r.
Thus W ⊂ D⊚r.
Now suppose that W ⊂ D⊚r let w ∈ W so there is d ∈ D with m(w, d) ≤ r for all w ∈ W. Hence m(w, D) ≤ r this is true for all w ∈ W thus m(W, D) ≤ r.
In Similar way we can show that m(D, W) ≤ r if and only if D ⊂ W⊚r.
Now h(W, D) ≤ r if and only if [m(W, D) ∨ m(D, W)] ≤ r if and only if m(W, D) ≤ r and m(D, W) ≤ r if and only if W ⊂ D⊚r and D ⊂ W⊚r.
The next result is clear by using Theorem 3.15

Proposition 3.16:

Suppose that (W_k) is a fuzzy Cauchy sequence in AFH(S) then for any r∈(0, 1) there is N∈ ℕ such that h(W_m, W_k)≤ r or W_m ⊆ W_k ⊚r and W_k ⊆ W_m ⊚r for all m, k ≥ N.

Theorem 3.17:

Suppose that (S, m, ⊚) is a fuzzy complete algebra fuzzy metric space and assume that (w_k) is a fuzzy Cauchy sequence in (AFH(S), h, ⊚). Let (k_n) be an increasing sequence with 0< k_1 < k_2 < …< k_n < …
Assume that (w_{k_j}) ∈ W_{k_j} is a fuzzy Cauchy in S then there is a fuzzy Cauchy ($\widehat{w}_k$)∈ W_k with $\widehat{w}_{k_j}$ = w_{k_j} for each j ∈ ℕ.

Proof:

The sequence ($\widehat{w}_k$) ∈ W_k is constructed as follows for k ∈ {1, 2, …, k_1} choose $\widehat{w}_k$ ∈ { w ∈ W_k: m(w, w_{k_j}) = m(w_{k_j}, W_k) } then $\widehat{w}_k$ is exists since W_k is fuzzy compact. Similarly for j ∈ {2, 3, … } and each k ∈ { k_j+1, k_j+2, …, k_{j+1} } choose $\widehat{w}_k$ ∈ { w ∈ W_k: m(w, w_{k_j}) = m(w_{k_j}, W_k)}.

Clearly $\widehat{w}_{k_j} = w_{k_j}$ by our construction. Since $(w_{k_j}) \in W_{k_j}$ is a fuzzy Cauchy sequence in S let t∈(0, 1) be given then there is $k_j, k_n \geq N_1$ with $m(w_{k_j}, w_{k_n}) \leq t$. Also since (W_k) is a fuzzy Cauchy sequence in AFH(S) there is N_2 such that $h(W_j, W_n) \leq t$ for all k, n $\geq N_2$.
Now put $N = N_1 \wedge N_2$ and for i, n ≥ N we have
$m(\widehat{w}_i, \widehat{w}_n) \leq m(\widehat{w}_i, w_{k_j}) ⊚ m(w_{k_j}, w_{k_n}) ⊚ m(w_{k_n}, \widehat{w}_n)$ where i ∈ { k_{j-1}+1, k_{j-1}+2, …, k_j }and n ∈ { k_{n-1}+1, k_{j-1}+2, …, k_n }.
But $h(W_m, W_{k_j}) \leq t$ the there exists $\widehat{w}_m \in W_m \cap [(w_{k_j}) ⊚ t]$ so that $m(\widehat{w}_m, w_{n_j}) \leq t$.
Similarly we can show that $m(w_{k_n}, \widehat{w}_k) \leq t$.
Hence $m(\widehat{w}_i, \widehat{w}_n) \leq t ⊚ t ⊚ t$. So we can find r∈(0, 1) such that t ⊚ t ⊚ t < r.
Hence $m(\widehat{w}_m, \widehat{w}_n) < r$ for all m, n ≥ N. Thus $(\widehat{w}_k) \in W_k$ is a fuzzy Cauchy sequence.
We will need the following Lemmas in the next main result.

Lemma 3.18:

Suppose that (S, m, ⊚) is a fuzzy complete fuzzy metric space and assume that (W_k) is a fuzzy Cauchy sequence in (AFH(S), h, ⊚) with $W_k \to W \in$ AFH(S) where W = {s ∈ S: there is a Cauchy sequence $(w_k) \in W_k$ such that $w_k \to s$ }. Then $W \neq \emptyset$.

Proof:

Suppose that (N_k) is a sequence where $N_k \in \mathbb{N}$ with $N_1 < N_2 < \ldots < N_m < \ldots$ such that $h(W_j, W_n) \leq (\frac{1}{2^k})$ for all j, n $\geq N_k$. Choose $w_{N_1} \in W_{N_1}$ but $h(W_{N_1}, W_{N_2}) \leq (\frac{1}{2})$ so we can find $w_{N_2} \in W_{N_2}$ such that $m(w_{N_1}, w_{N_2}) \leq (\frac{1}{2})$. In this way we can select a finite sequence $w_{N_j} \in W_{N_j}$ with j=1, 2, …, k such that $m(w_{N_{j-1}}, w_{N_j}) \leq (\frac{1}{2^{j-1}})$. Then since $h(W_{N_k}, W_{N_{k-1}}) \leq (\frac{1}{2^k})$ and $w_{N_k} \in W_{N_k}$ we can find $w_{N_{k+1}} \in W_{N_{k+1}}$ such that $m(w_{N_k}, w_{N_{k-1}}) \leq (\frac{1}{2^k})$. For example let $w_{N_{k-1}}$ be the point in $W_{N_{k+1}}$ that closest to w_{N_k}. By induction we can find a sequence $(w_{N_j}) \in W_{N_j}$ such that $m(w_{N_j}, w_{N_{j-1}}) \leq (\frac{1}{2^j})$. Now we show that (w_{N_j}) is a Fuzzy Cauchy sequence in S let $0 < \beta < 1$ and choose N_β such that
$(\frac{1}{2}) ⊚ (\frac{1}{2^2}) ⊚ (\frac{1}{2^3}) ⊚ \ldots ⊚ (\frac{1}{2^{N_j - N_n}}) < \beta$. Then for j, n $> N_\beta$ we have

$$m(w_{N_j}, w_{N_n}) \leq m(w_{N_j}, w_{N_{j+1}}) ⊚ m(w_{N_{j+1}}, w_{N_{j+2}}) ⊚ \ldots ⊚ m(w_{N_{n-1}}, w_{N_n})$$
$$\leq (\frac{1}{2}) ⊚ (\frac{1}{2^2}) ⊚ (\frac{1}{2^3}) ⊚ \ldots ⊚ (\frac{1}{2^{N_j - N_n}}) < \beta.$$

Hence by Theorem 3.17 there exists a convergent subsequence $(d_i) \in W_i$ for which $d_{N_i} = w_{N_i}$. Then lim d_i exists and is in W. Thus $W \neq \emptyset$.

Lemma 3.19:

Suppose that (S, m, ⊚) is a fuzzy complete fuzzy metric space and assume that (W_n) is a fuzzy Cauchy sequence in (AFH(S), h, ⊚) with $W_k \to W \in$ AFH(S) where W = {s ∈ S: there is a fuzzy Cauchy sequence $(w_k) \in W_k$ such that $w_k \to s$ }. Then W is fuzzy complete.

Proof:

Suppose $(w_i) \in W_i$ with $w_i \to w$ we show that w ∈ W. For each i there exists a sequence $(w_{i,k}) \in W_k$ with $w_{i,k} \to w_i$. There exists an increasing sequence (N_i) with $N_i \in \mathbb{N}$ such that $m(w_{N_i}, w) \leq (\frac{1}{i})$.
Moreover there is a sequence (n_i) with such $n_i \in \mathbb{N}$ that $m(w_{N_i, n_i}, w_{N_i}) \leq (\frac{1}{i})$. Thus
$m(w_{N_i, n_i}, w) \leq m(w_{N_i, n_i}, w_{N_i}) ⊚ m(w_{N_i}, w) \leq (\frac{1}{i}) ⊚ (\frac{1}{i})$. Put $y_{n_i} = w_{N_i, n_i}$
we see that $y_{n_i} \in W_{n_i}$ and $y_{n_i} \to w$.
Now by Theorem 3.16 (y_{n_i}) can be extended to a convergent sequence $(z_i) \in W_i$ so w ∈ W thus W is fuzzy closed. Hence W is fuzzy complete since S is fuzzy complete.

Lemma 3.20:

Suppose that (S, m, ⊚) is a fuzzy complete algebra fuzzy metric space and assume that (w_k) is a fuzzy Cauchy sequence in (AFH(S), h, ⊚) with $W_k \to W \in$ AFH(S) where W = {s ∈ S: there is a Cauchy sequence $(w_k) \in W_k$ such that $w_k \to s$ }. Then for every r∈(0, 1) there is N∈ ℕ such that $W \subseteq W_k ⊚ r$ for all k ≥ N.

Proof:
Let r∈(0, 1) then there is N ∈ ℕ such that $h(W_k, W_n) \leq r$ for all k, n ≥ N. Now for k ≥ n ≥ N, $W_k \subseteq W_n$ ⊚r. To prove that W⊆ W_n ⊚r let w ∈ W then there is a sequence $(w_i) \in W_i$ such that $w_i \to w$. Now for k ≥ N, $m(w_k, w) < r$. Then $w_k \in W_n$ ⊚r by using fuzzy compactness of W_n we can show that W_n ⊚r is fuzzy closed. Then $w_k \in W_n$ ⊚r for all k ≥ N so w must be in W_n ⊚r. This shows that W⊆ W_n ⊚r for all n ≥ N.

Lemma 3.21:
Suppose that (S, m, ⊚) is a fuzzy complete algebra fuzzy metric space and assume that (W_k) is a fuzzy Cauchy sequence in (AFH(S), h, ⊚) with $W_k \to W \in$ AFH(S) where W = {s ∈ S: there is a fuzzy Cauchy sequence $(w_k) \in W_k$ such that $w_k \to s$ }. Then W is fuzzy compact.

Proof:
We will prove that W is totally fuzzy bounded. Assume that W is not totally fuzzy bounded so for r ∈ (0, 1) does not exists a finite r-fuzzy net. Then there is (w_i) in W has the property $m(w_i, w_j) \geq r$ for i ≠j. This Will gives a contradiction. Hence there is n ≥ N so that W ⊂ W_n ⊚r by Lemma 3.20. For these w_i there exists $y_i \in W_n$ with the property $m(w_i, y_i) \leq \beta$ where 0 < β < 1 with β < r. But W_n is fuzzy compact some (y_{n_i}) of (y_i) fuzzy converges. Thus there exists points in (y_{n_i}) are close together as we want. In special cases there are two points y_{n_i} and y_{n_j} has the property $m(y_{n_i}, y_{n_j}) \leq \alpha$ where $0 < \alpha < 1$ with $\alpha < r$. Now

$m(w_{n_i}, w_{n_j}) \leq m(w_{n_i}, y_{n_i}) \circledcirc m(y_{n_i}, y_{n_j}) \circledcirc m(y_{n_j}, w_{n_j}) \leq \beta \circledcirc \alpha \circledcirc \beta < r$.

Thus W is totally fuzzy bounded this implies that W is fuzzy compact by Theorem 2.28.

Here we reached to the position to give the main result in this section

Theorem 3.22:
Suppose that (S, m, ⊚) is a complete fuzzy metric space. Then (AFH(S), h, ⊚) is a complete algebra fuzzy metric space.

Proof:
Assume that (W_k) is a fuzzy Cauchy sequence in (AFH(S), h, ⊚) with $W_k \to W \in$ AFH(S) where W = {s ∈ S: there is a fuzzy Cauchy sequence $(w_k) \in W_k$ such that $w_k \to s$ }. Now by Lemma 3.17 W ≠ ∅ and by Lemma 3.18 and W is fuzzy complete. Also for every r∈(0, 1) there is N∈ ℕ such that W ⊆ W_k ⊚r for all k ≥ N by Lemma 3.18 finally W is fuzzy compact by Lemma 3.19. Now we will show that $W_k \to W$ it is enough to show that for 0 < r < 1 there exists N such that $W_k \subseteq W$⊚r for all k ≥ N. But (W_k) is a fuzzy Cauchy so for given 0 < r < 1 there exists N∈ ℕ with $h(W_k, W_n) < r$ for all k, n ≥ N. Thus for k, n ≥ N, $W_k \subseteq W_n$ ⊚r. Suppose that n ≥ N to prove that $W_n \subseteq W$⊚r. Assume that y ∈ W_n so it can be found (N_i) with $n < N_1 < N_2 < \ldots < N_k < ..$ and for k, n ≥ N_j, $W_k \subseteq W_n \circledcirc \frac{1}{2^{j-1}}$ note that $W_n \subset W_{N_i} \circledcirc \frac{1}{2}$. Since y ∈ W_n there is $w_{N_i} \in W_{N_i}$ with $m(y, w_{N_i}) \leq (\frac{1}{2})$. Since $w_{N_1} \in W_{N_1}$ there is $w_{N_2} \in W_{N_2}$ with $m(w_{N_1}, w_{N_2}) \leq (\frac{1}{2^2})$.

Similarly by induction there exists (w_{N_j}) with $w_{N_j} \in W_{N_j}$ and $m(w_{N_j}, w_{N_{j-1}}) \leq (\frac{1}{2^{j+1}})$. Using the fuzzy triangle inequality a number of times we can get $m(y, w_{N_j}) \leq (\frac{1}{2})$for all j and also show that (w_{N_j}) is a fuzzy Cauchy sequence. Now each $W_{N_j} \subset W_n \circledcirc \frac{1}{2}$ and (w_{N_j}) converges to a point a and since $W_n \circledcirc \frac{1}{2}$ is fuzzy closed a ∈ $W_n \circledcirc \frac{1}{2}$ Moreover $m(y, d_{N_j}) \leq (\frac{1}{2})$ implies

$m(y, x) \leq m(y, w_{N_j}) \circledcirc m(w_{N_j}, x) \leq (\frac{1}{2}) \circledcirc (\frac{1}{2})$. Let $(\frac{1}{2}) \circledcirc (\frac{1}{2}) < r$

for some r, 0 < r < 1 so $m(y, x) \leq (1-r)$.

Thus $W_n \subset W$ ⊚r for all n ≥ N. Hence $W_n \to W$ consequently (AFH(S), h, ⊚) is a fuzzy complete algebra fuzzy metric space.

4. Conclusions

The definition of algebra fuzzy metric space is used in this study to introduce the notion of the algebra fuzzy distance from a point in the universal set S to a fuzzy compact set in S also the algebra fuzzymetric between two fuzzy compact sets is introduced. As in the ordinary case here the algebra fuzzy metric from a fuzzy compact set A to a fuzzy compact set B is not equal to algebra fuzzy metric from a fuzzy compact set B to a fuzzy compact set A this make us to introduce the notion of the Hausdorff algebra fuzzy metric between two fuzzy compact sets. The basic results of the algebra fuzzy metric are investigated. Finally the main result in this paper is proved that is if (S, m, ⊚) is a fuzzy complete algebra fuzzy metric space then (AFH(S), h, ⊚) is a fuzzy complete algebra fuzzy metric space. Here we may suggest for future work to study this space (AFH(S), h, ⊚).

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