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Existence Solution of Volterra Hammerstein Equation in L^p Space

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Abstract

In this paper we use the fixed-point theorem of Latrach, Taoudi and Zeghal under some conditions to find a solution for Volterra_Hammerstein integral equation in the Banach space $L^p([0,m],R)$. We use this fixed point theorem with new assumptions.

KeyWords: Volterra-Hammerstein equation, Fixed point theorem, Weakly compact.

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وجود حل لمعادلة فولتيرا – هامرستين في الفضاء L^p

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الملخص

استخدمت مبرهة النقطة الصامدة للاتراش وتودي وزيكال تحت بعض الشروط للحصول على حل لمعادلة فولتيرا هامرستين التكاملية في فضاء بناخ $L^p([0,m],R)$. حيث اننا استخدمنا مبرهنة النقطة الصامدة هذه مع شروط جديدة.

الكلمات الدالة: معادلة فولتيرا – هامرستين، نظرية النقطة الصامدة، المتراص بشكل ضعيف.

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1. Introduction:

This paper studies Volterra_Hammerstein integral equations in the Banach space $L^p([0,m],R)$. We get a solution for Volterra-Hammerstein integral equation:

$$u(t) = h(t) + \int_0^t k(t,s)f\big(s,u(s)\big)ds \quad , \quad t \in [0,m] \tag{1}$$

Where $m > 0$ is fixed. Here $u(t)$ takes values in the space $L^p([0,m],R)$. Note $h: [0,m] \to L^p([0,m],R)$, $f: [0,m] \times L^p([0,m],R) \to L^p([0,m],R)$ and $k: [0,m] \times [0,m] \to R$. In this study we investigate the solution of Volterra-Hammerstein integral equation in $L^p([0,m],R)$ using fixed point theorem of K. Latrach, M.A. Taoudi, A. Zeghal [1]. We assume that the integral $\int_0^m u(s)ds$ exists.

Equations of Hammerstein-type play an important role in automation and in the network theory and in the optimal control system [2].

Equation of Hammerstein-type was studied by many authors. Aref Jeribi, Bilel Krichen and Bilel Mefteh found a solution for equation (1) by using Krasnosel'skii type theorem [3]. Mustafa Nader gave conditions that ensure the existence and the uniqueness of the solution for equation (1) in the L^p space [4]. Authors in [5-8] found sequences converged to the exact solution of equation (1) under such assumptions.

2. Preliminaries:

At the beginning define the nonlinear operator

$$Nu(t) = h(t) + \int_0^t k(t,s)f\big(s,u(s)\big)ds.$$

In the sequel we need the conditions below:

H1: If $(x_n)_{n\in N}$ is a weakly convergent sequence in $L^p([0,m],R)$, then $(Nx_n)_{n\in N}$ has a strongly convergent subsequence in $L^p([0,m],R)$.

H2: $\int_0^t \|k(t,s)f(s,u(s))\|\, ds \leq t\|u(t)\| \qquad \forall\ t,s \in [0,m]$.

H3: $a(t) \leq a(t) - h_0$, where $a(t) = \int_0^t \|u(s)\| ds$ and $h_0 = \sup(\|h(t)\| : t \in [0,m])$.

H4: $\int_0^t \big\|k(s,t)\big(f\big(s,u(s)\big) - f\big(s,v(s)\big)\big)\big\| ds \leq M\|u - v\|$,

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where $M > 0$.

H5: $T(h(t)) \neq 0, T\left(\int_0^t k(t,s)f(s,u(s))\right) \neq 0$ and $T(h(t) \neq -T\left(\int_0^t k(t,s)f(s,u(s))\right)$ $\forall$ $t,s \in [0,m]$,

where T is any linear functional on $L^p([0,m],R)$.

If $u(t) = 0$ then the above conditions H1, H2, H3, H4 and H5 will be satisfied.

Theorem 1: ([1], Theorem. 2.1.) Let D be a nonempty closed convex subset of a Banach space X. Assume that $N : D \to D$ is a continuous function which satisfies $(H1)$. If $N(D)$ is relatively weakly compact, then there exists $x \in D$ such that $Nx = x$.

Theorem 2: [9] If $f : A \to R$ satisfies the Lipschitz condition, then f is uniformly continuous on A.

Theorem 3: ([10], Theorem 3, Robert C. James). A bounded closed convex subset C of a Banach space is not weakly compact if and only if there is a positive number r for which there exists a sequence $\{z_i\}$ of members of C, and a sequence $\{f_i\}$ of linear functionals with unit norms, such that $f_n(z_i) > r$ if $n \leq i$ and $f_n(z_i) = 0$ if $n > i$.

3. Main Result:

We now in a position to give the following result:

Theorem 4: Under conditions H1, H2 , H3 , H4 and H5 the equation (1) has at least one solution.

Proof: the proof is based on theorem 1.

Define

$C = \{u(s) \in L^p([0,m]): \|u(s)\| \leq a(s)$ and if $u \neq 0$ $then$ $T(u) \neq 0\}$,

Where T is linear functional in $L^p([0,m],R)$. We can see the set C is nonempty because $0 \in C$.

C is closed bounded convex set. Indeed, since

$\|u(s)\| \leq a(s) \leftrightarrow -a(s) \leq u(s) \leq a(s)$

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$$\leftrightarrow -\infty < -\int_0^s u(t)\,dt \le u(s) \le \int_0^s u(t)dt < \infty$$

This mean that C is closed and bounded, let $u_1 \,\&\, u_2 \in C$, now we have to prove that for any $\alpha \in (0,1)$ and for every pair $u_1 \,\&\, u_2 \in C$, $\alpha u_1 + (1-\alpha)u_2 \in C$. Without loss of generality we suppose that $u_1 \le u_2$ which implies that both α & $1-\alpha$ are positive

$$\alpha u_1 + (1-\alpha)u_2 \le \alpha u_2 + (1-\alpha)u_2 = u_2,$$

and

$$\alpha u_1 + (1-\alpha)u_2 \ge \alpha u_1 + (1-\alpha)u_1 = u_1.$$

Therefore $\alpha u_1 + (1-\alpha)u_2 \in C$, hence C is convex set.

If one prove that $N: C \to C$, continuous satisfy H1, and N(C) is relatively weakly compact then Theorem 1 guarantees equation (1) has at least one solution.

We start to show that $N: C \to C$. by using H2 and H3:

$$\|N(u(t))\| = \left\| h(t) + \int_0^t k(t,s)f\big(s,u(s)\big)ds \right\|$$

$$\le \|h(s)\| + \int_0^t \|k(t,s)f(s,u(s))\|ds$$

$$\le h_0 + t\|u(t)\| \le h_0 + ta(t)$$

$$\le h_0 + a(t) - h_0 = a(t).$$

Now by H5

$$T\big(N(u)\big) = T\big(h(t)\big) + T\left(\int_0^t k(t,s)f\big(s,u(s)\big)ds\right) \ne 0$$

Therefore $N: C \to C$.

Now we prove $N: C \to C$ is continuous by H4:

$$\|Nu - Nv\| = \left\| h(t) + \int_0^t k(t,s)f\big(s,u(s)\big)ds - h(t) - \int_0^t k(t,s)f\big(s,v(s)\big)ds \right\|$$

$$\le \int_0^t \|k(t,s)(f\big(s,u(s)\big) - f\big(s,v(s)\big))\|ds$$

$$\le M\|u - v\|.$$

By theorem 2 $N: C \to C$ is uniformly continuous, therefore continuous.

Now we show that $N(C)$ is relatively weakly compact. Since $T(N(C)) \neq 0$ for all T is a linear functional, therefore by theorem 3 $N(C)$ is weakly compact and so it is relatively weakly compact.

4. Conclusions:

Equation of Hammerstein type has a solution under such assumptions. We get the same result that obtained by G. Emmanuele [11] but in different assumptions. The author assumed in [11] that f is a Caratheodory function such that F maps $L^1(D, X)$ into $L^1(D, Y)$ (where $(Fu)(\cdot) = f(\cdot, u(\cdot))$ and D is a compact subset of R^n), continuously, and k is a measurable function such that the functions $s \to k(t, s)$ belong to L^∞ and K (where $(Kv)(\cdot) = \int_D k(t, s)v(s)ds$) is a linear, continuous operator from $L^1(D, Y)$ into $L^1(D, X)$, where X, Y are finite dimensional Banach spaces and use the Schauder fixed point Theorem with conditions in [11]. But we assume that $h: [0, m] \to L^p([0, m], R)$, $u: [0, m] \to L^p([0, m], R)$, $f: [0, m] \times L^p([0, m], R) \to L^p([0, m], R)$ and $k: [0, m] \times [0, m] \to R$ and we use of Latrach, Taoudi and Zeghal fixed point theorem with conditions (H1-H5).

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