On Expectation Correlate System and Chaotic Dynamics in Time-Series

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ABSTRACT

This paper suggests a new system of time-series called Expectation Correlate System (ECS) that are good at detecting the behavior of dynamical systems (both deterministic and stochastic systems) and the dependence on initial values. A new measure on sensitivity to initial values can be monitored by the newly defined Lyaponov Correlate, so ECS can be a signal to chaotic property. In a stochastic systems, small shifts in some initial value can lead to error in prediction, this property and a new measure to nonlinear systems are study by using the conditional variance of ECS. All results are computed by using Matlab.

Keywords: time-series, dynamical system, Lyaponov Correlate, chaotic, stochastic systems, prediction, Matlab.

حول نظام الارتباط المتوقع وديناميكية الجيشان في السلاسل الزمنية

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الملخص

في هذأ البحث تم اقتراح نظام جديد للسلاسل الزمنية يدعى بنظام الارتباط المتوقع الذي يمكن أن يكون مؤشراً جيداً لسلوك النظم الديناميكية (محددة كانت أم تصادفية) فضلاً عن أن خاصية الاعتماد على القيم الابتدائية. مقياس جديد مُنذر للحساسية بالنسبة إلى القيم الابتدائية باستخدام التعريف الجديد ارتباط لابنوف (Lyaponov Correlate). على هذا الأساس فان نظام الارتباط المتوقع يكون كإشارة إلى الخاصية الجيشانية. في النظم التصادفية, التغيرات الطفيفة في بعض القيم الابتدائية يمكن أن تقود إلى خطا في عملية التنبؤ, هذه الخاصية ومقياس جديد للنظم غير الخطية درست باستخدام التباين الشرطي لنظام الارتباط المتوقع. جميع النتائج تم احتسابها باستخدام برمجيات ال Matlab.

الكلمات المفتاحية: السلاسل الزمنية، النظم الديناميكية، ارتباط لابنوف، الجيشان، النظم التصادفية، التنبؤ، ماتلاب.

1-Introduction

In the past few years a large literature on chaos and non-linear science has appeared in economics and many scientists and technologists from diverse disciplines including mathematics (both pure and applied) because chaos is useful as a lens through which to view the word in epidemiology, biology and ecology, not because it helps so much in prediction but because it is suggestive of pathways to complex dynamics [1]. It is associated with complex and unpredictable behavior of phenomena over time and nonlinear science studies stochastic and deterministic dynamical systems that lead to “complex” dynamics. In [6], York who proposed the word ‘chaos’ as a label for a kind of dynamical behavior characterized by the triad: infinite number of periodic trajectories; uncountable number of no periodic trajectories; hyperbolicity (instability) of all (or the overwhelming majority of; as is proposed here) trajectories in the regime.

A simple deterministic system may be defined as follows: For a discrete time index, T={0,1,2,…}, consider a time-series $\{X_t ; t \in T\}$. Assume that $X_0 = x_0$ is an initial condition and that

$$X_t = F(X_{t-1}) \quad \ldots\ldots\ldots\ldots (1)$$

for t>1, where X_t denoted a state vector in R^d, F is a real-vector function (bounded continuous first derivatives) see [2]. In a deterministic system, it is agreed that the sensitive dependence on initial condition is a typical feature of chaotic system, which is characterized by the well-known Lyaponov exponent [4].

A discrete time stochastic system can be described by the equation

$$X_t = F(X_{t-1}, \in_t) \quad \ldots\ldots\ldots\ldots (2)$$

Where $\{\in_t\}$ is a noise process which satisfies the equation $E(\in_t | X_0, X_1, \ldots, X_{n-1}) = 0.$

If the noise is additive, equation (2) can be written as follows:

$$X_t = F(X_{t1}) + \in_t \quad \ldots\ldots\ldots\ldots (3)$$

Just as in deterministic systems, there has been no general accepted definition of chaos in stochastic systems. Stochastic chaotic system sometimes means a system with a deterministically chaotic skeleton [7].

The plan of the paper is as follow: Section (2) provides a brief sketch of a new system called Expectation Correlate System (ECS) and the correlates of the trajectories are studies. The concept of Lyapunov exponent has been developed to characterize the sensitivity dependence on the initial value of system (1) in section (3). Section (4) presents a quantitative description of how small noise can be amplified rapidly in a ECS if the corresponding system is chaotic. A new simple procedure is suggested to measure the non-linear property in section (5). In section (6) the method is illustrated with example.

2- The Expectation Correlate System and Deterministic Systems.

By the Expectation Correlate System (ECS) of the system (1), mean the suggested system defined as:

$$C_n^{(s)} = X_n - X_{n-s} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

n>s, s is called the step correlate. Let as consider s=1, so the ECS has the form

$$C_n^{(1)} = X_n - X_{n-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

In a chaotic system, small changes of parameters can change the dynamical behavior from stable periodic cycle or limit point into a strange attracted system [5]. The suggested system (5) will try to study the correlate behavior of the trajectories which dependence on the parameters of the system and how small change in this parameter can change the correlate of the trajectories.

By using the deterministic system (1) (let us consider the one dimension case d=1), starting at the initial points $X_o=x_o$ and $X_o = x_o + \delta$, respectively, for small $\delta > 0$, let

$$\begin{aligned}\mu_n(x_0) &= E[C_n^{(1)} \mid X_0 = x_o + \delta] - E[C_n^{(1)} \mid X_0 = x_o] \\ &= E[X_n - X_{n-1} \mid X_0 = x_o + \delta] - E[X_n - X_{n-1} \mid X_0 = x_o]\end{aligned}$$

Where $E[C_n^{(1)} \mid X_0 = x_0]$ is the conditional mean given x_0. For n = 1, we have

$$\begin{aligned}\mu_1(x_o) &= E[X_1 - X_0 \mid X_0 = x_o + \delta] - E[X_1 - X_0 \mid X_0 = x_o] \\ &= E[F(X_0) - X_0 \mid X_0 = x_o + \delta] - E[F(X_0) - X_0 \mid X_0 = x_o] \\ &= F(x_o + \delta) - F(x_o) - \delta \\ &\approx \dot{F}(x_0).\delta - \delta \quad \text{...}(6)\end{aligned}$$

Also, for n=2, we can see that

$$\mu_2(x_0) \approx \dot{F}^{(2)}(x_0).\delta - \dot{F}(x_0).\delta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Here, $F^{(2)}(.)$ means the 2-component of the function F and the over – dot denotes the differential operator. From (6) and (7), we have

$$\mu_2(x_0) = \dot{F}^{(2)}(x_0)\delta - \mu_1(x_0) - \delta$$

For n>1, the general form as:

$$\mu_n(x_0) = \dot{F}^{(n)}(x_0)\delta - \sum_{i=1}^{n-1} \mu_i(x_0) - \delta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

For the value of x_0 such that $|\mu_n(x_0)| \succ 0$, a small shift δ in the initial value can lead to a considerable divergence in the ECS. This means that the ECS depends on x_0 sensitivity when $|\mu_n(x_0)| \succ 0$.

$|\mu_n(x_0)|$ Can be a signal to the correlate of the trajectories of system (1) and can determine the bifurcation parameter in a dynamical system that leads to chaotic system.

3- Lyapunov Correlate of ECS.

Lyapunov exponent is one of the most popular measures of chaos which is defined as :

$$\dot{F}^{(n)}(x_0) \approx e^{n\lambda(x_0)} \quad \text{...}(9)$$

where

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \ln\left|\dot{F}^{(n)}(x_0)\right| \quad \text{...}(10)$$

We call $\lambda(x_0)$ the Lyapunov exponent at x_0 [3]. To modify a new measure of chaos, we use ESC as follows: from equation (8) and (9) we have

$$\mu_n(x_0) \approx e^{n\lambda(x_0)}\delta - \sum_{i=1}^{n-1} \mu_i(x_0) - \delta$$

therfor

$$\lambda_C(x_0) = \lim_{n\to\infty} \frac{1}{n} \ln\left|\frac{1}{\delta}\sum_{i=1}^{n}\mu_i(x_0) + 1\right| \quad \text{...}(11)$$

assuming that the limit exists, $\lambda_C(x_0)$ is called Lyapunov Correlate, which characterize the average rate of exponential divergence of nearby trajectories. The magnitude of the positive of the Lyapunov correlate for all n is a measure of the degree of chaos. Zero Lyapunov correlate characterize the average of the rate of convergence of trajectories.

4-The Expectation Correlate System and Stochastic Systems:

In this section, the ECS measure another property of nonlinear stochastic systems by comparing the conditional variance (given the initial condition $X_0=x$)with the variance $\in_t$. Consider the ECS of system (3) when n=1, thus

$$C_1^{(1)} = X_1 - X_0 = F(x) + \in_1 - x$$

And

$$C_2^{(1)} = X_2 - X_1 = F(F(x) + \in_1) + \in_2 - F(x) - \in_1$$

By using Taylor expansion to the function *F*, we have

$$C_2^{(1)} = F^{(2)}(x) + \in_2 + \dot{F}(F(x)) \in_1 - F(x) - \in_1$$

For n=3, get

$$C_3^{(1)} = X_3 - X_2 = F(X_2) + \in_3 - F(X_1) - \in_2$$
$$= [F(F^{(2)}(x) + \dot{F}(F(x)) \in_1 + \in_2) + \in_3] - [F(F(x) - \in_1) + \in_2]$$
$$= [F^{(3)}(x) + \in_3 + \dot{F}(F^{(2)}(x))\dot{F}(F(x)) \in_1 + \dot{F}(F^{(2)}(x) \in_2]$$
$$-[F^{(2)}(x) + \dot{F}(F(x)) \in_1 + \in_2]$$

Then, the general form of ECS as follows:

$$C_n^{(1)} = F^{(n)}(x) - F^{(n-1)}(x) + \in_n - \in_{n-1} + \sum_{j=1}^{n-1}\left\{\prod_{k=j}^{n-1}\dot{F}(F^{(k)}(x))\right\}\in_j$$
$$- \sum_{j=1}^{n-2}\left\{\prod_{k=j}^{n-2}\dot{F}(F^{(k)}(x))\right\}\in_j \text{.......... ....}(12)$$

Suppose that, $\sigma_n^2(x) = Var(C_n^{(1)} \mid X_0 = x)$ and $\sigma^2 = Var(\in_t) = Var(\in_t \mid X_t)$, from equation (12), take the conditional variance given $X_0=x$ we have

$$\sigma_n^2(x) = \sigma^2 \sum_{j=1}^{n-1}\left\{\prod_{k=j}^{n-1}\dot{F}(F^{(k)}(x))\right\}^2 + \sigma^2 \sum_{j=1}^{n-2}\left\{\prod_{k=j}^{n-2}\dot{F}(F^{(k)}(x))\right\}^2 \ldots(13)$$

If $|\dot{F}(x)| > 1$ for a large of value of *x*, $\sigma_n^2(x)$ can be very large for moderate n. It is easy to see from equation (13) that $\sigma_n^2(x)$ also depends on the initial value sensitivity when $|\dot{F}(x)|$ is greater then 1, which indicates the dependence of the correlate prediction on initial value, this is a typical feature of nonlinear (but not necessarily chaotic) systems. If the system (3) is stochastically chaotic, equation (13) indicates that small noise can be amplified quickly when the system starts at some initial value, which mean that the n-step prediction based on these initial values could be unreliable even for small n.

5-Non-linear ECS

By using the idea developed in section (4), suppose that

$$\Lambda_n(x) = \sum_{j=1}^{n-1}\left\{\prod_{k=j}^{n-1}\dot{F}(F^{(k)}(x))\right\}^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

Equation (13) takes the form

$$\sigma_n^2(x) = \sigma^2\Lambda_n(x) + \sigma^2\Lambda_{n-1}(x)$$

By a simple change in equation (14), it is easy to see that for n>1

$$\sigma_n^2(x) = \left\{\left(\dot{F}(F^{(n-1)}(x))\right)^2 + 1\right\}\ \sigma^2\Lambda_{n-1}(x) \quad \ldots\ldots\ldots\ldots(15)$$

We now turn the general form of ECS which is consider in equation (4), by using the idea in equation (12), we get for n>s

$$C_n^{(s)} = F^{(n)}(x) - F^{(n-s)}(x) + \in_n - \in_{n-s} + \sum_{j=1}^{n-1}\left\{\prod_{k=j}^{n-1}\dot{F}(F^{(k)}(x))\right\}\in_j$$

$$-\sum_{j=1}^{n-s-1}\left\{\prod_{k=j}^{n-s-1}\dot{F}(F^{(k)}(x))\right\}\in_j \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

Using the conditional variance of equation (16), we have

$$\sigma_{n_s}^2(x) = \sigma^2\Lambda_n(x) + \sigma^2\Lambda_{n-s}(x)$$

$$= \left\{\prod_{k=1}^{s}\ \left(\dot{F}(F^{(n-s)}(x))\right)^2 + 1\right\}\sigma^2\Lambda_{n-s}(x)$$

Where, $\sigma_{n_s}^2(x) = Var(C_n^{(s)} \mid X_0 = x)$ and $\sigma_{n_1}^2(x) = \sigma_n^2(x)$. The dependence of $\sigma_{n_s}^2(x)$ to initial condition means that, the prediction depends on initial condition, which is a typical feature of non-linear but not necessarily chaotic systems. When *F*(.) is linear, $\dot{F}(x)$ is a constant, $\sigma_{n_s}^2(x)$ does not depend on *x* and is monotonically increasing as n increases.

6-Example

Consider one dimensional stochastic system

$$X_t = rX_{t-1} - rX_{t-1}^2 + \in_t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17)$$

Where $\in_t, t \geq 1$ are independent random variable with the same distribution as a normal random variable with mean 0 and variance 0.1^2. r is the parameter of system (17). The skeleton of system (17) is transformed logistic map with parameter r. To show the behavior of $|\mu_n(x_0)|$, take the deterministic part. By using Matlab language, we write a program to presentation of the way in which the behavior of our iteration depends on the value of the growth parameter r. For each value of r in the input interval $1 \leq r \leq 4$ in the horizontal axis of figure (2 a-b). We see a single limiting population until $r \approx 3$ then cycle with period 2, then a cycle of period 4, then one of period 8, the corresponding $|\mu_n(x_0)|$ of these trajectories is approach to zero. If $|\mu_n(x_0)|$increasing this mean there are no correlation between the trajectories as show in the range $3.6 \leq r \leq 4$, and $|\mu_n(x_0)|$has maximum value when r=4, science Logistic map has a strong attractors in this value.

We conclude that if $|\mu_n(x_0)| \longrightarrow 0$, this means that there are correlaions between the trajectories and the system has a limit point or cycle of finite periodic, and if $|\mu_n(x_0)|$>0 there are un correlated trajectories and the strong chaos depends on the value of $|\mu_n(x_0)|$.

Table (1) shows the resulting of $|\mu_n(x_0)|$ in a different value of r, which can be measured when the trajectories of logistic map are un correlated (complex trajectories), and table (2) shows that $|\mu_n(x_0)|$ has sensitivity to a small shift in initial value compared to table (2). The bifurcation parameter can be also determined by this tanle which has $|\mu_n(x_0)| \neq 0 \quad as \quad n \to \infty$.

Table (3) shows the conditional variance and the divergently resulting from a small shift in initial values, also the amplification of noise is shown.

Table (1)

N	r=1.9 $\delta = 0.01$	r=2.7 $\delta = 0.01$	r=3 $\delta = 0.01$	r=3.4 $\delta = 0.01$	r =3.7 $\delta = 0.01$	r=4 $\delta = 0.01$
1	0.0100	0.0100	0.0100	0.0100	0.0100	0.0100
2	0.0130	0.0095	0.0087	0.0067	0.0056	0.0044
3	0.0040	0.0146	0.0188	0.0332	0.0448	0.0590
4	0.0022	0.0077	0.0157	0.0448	0.0586	0.0489
5	0.0006	0.0056	0.0140	0.0646	0.1048	0.3275
6	0.0001	0.0036	0.0120	0.0837	0.0369	0.0638
7	0.0000	0.0026	0.0108	0.1214	0.1572	0.2265
8	0.0000	0.0018	0.0094	0.1413	0.0328	0.5253
9	0.0000	0.0013	0.0084	0.1921	0.4863	0.5217
10	0.0000	0.0009	0.0074	0.1730	0.5033	0.2155
11	0.0000	0.0006	0.0067	0.1446	0.5439	1.4807
12	0.0000	0.0004	0.0059	0.0894	0.5365	1.0873
13	0.0000	0.0003	0.0053	0.0345	0.4936	0.3062
14	0.0000	0.0002	0.0047	0.0168	0.5995	0.1522
15	0.0000	0.0001	0.0043	0.0245	0.8153	1.5032
16	0.0000	0.0001	0.0038	0.0114	0.8392	0.9061
17	0.0000	0.0001	0.0035	0.0191	0.5623	0.9014
18	0.0000	0.0001	0.0031	0.0091	0.8523	0.0039
19	0.0000	0.0000	0.0028	0.0140	0.7957	0.8674
20	0.0000	0.0000	0.0025	0.0066	0.3831	0.0124
21	0.0000	0.0000	0.0022	0.0108	0.5896	0.4020
:	:	:	:	:	:	:
98	0.0000	0.0000	0.0000	0.0000	0.0160	0.1189
99	0.0000	0.0000	0.0000	0.0000	0.0032	0.2751
100	0.0000	0.0000	0.0000	0.0000	0.0306	0.0953

Table (2)

n	r=1.9 $\delta = 0.001$	r=2.7 $\delta = 0.001$	r=3 $\delta = 0.001$	r=3.4 $\delta = 0.001$	r =3.7 $\delta = 0.001$	r=4 $\delta = 0.001$
1	0.000100	0.000100	0.0001000	0.0001	0.0001	0.0001
2	0.000132	0.000092	0.0000840	0.0001	0.0001	0.0023
3	0.000034	0.000147	0.0001089	0.0003	0.0005	0.0006
4	0.000020	0.000073	0.000154	0.0005	0.0006	0.0056
5	0.000008	0.000053	0.000137	0.0007	0.0011	0.0006
6	0.000002	0.000034	0.000116	0.0009	0.0006	0.0015
7	0.000009	0.000024	0.000104	0.0013	0.0008	0.0045
8	0.0000	0.000016	0.000089	0.0016	0.0004	0.0030
9	0.0000	0.000011	0.000081	0.0024	0.0028	0.0373
10	0.0000	0.000008	0.000070	0.0027	0.0048	0.0193
11	0.0000	0.000005	0.000063	0.0037	0.0080	0.0424
12	0.0000	0.000004	0.000056	0.0030	0.0140	0.0943
13	0.0000	0.000002	0.000050	0.0019	0.0227	0.2796
14	0.0000	0.000002	0.000045	0.0008	0.0408	0.0804
15	0.0000	0.000001	0.000040	0.0017	0.0592	1.4417
16	0.0000	0.000001	0.000036	0.0009	0.1103	1.1306
17	0.0000	0.000007	0.000032	0.0011	0.0906	0.4056
18	0.0000	0.000005	0.000029	0.0005	0.0503	0.3902
19	0.0000	0.000003	0.000026	0.0009	0.0240	0.8667
20	0.0000	0.000000	0.000023	0.0005	0.1740	0.2732
21	0.0000	0.000000	0.000021	0.0006	0.2790	0.5621
22	0.0000	0.000000	0.000019	0.0003	0.4921	0.7238
23	0.0000	0.000000	0.000017	0.0005	0.6705	0.4603
24	0.0000	0.000000	0.000015	0.0002	0.9873	0.1578
25	0.0000	0.000000	0.000013	0.0004	1.0291	0.4712
26	0.0000	0.000000	0.000012	0.0002	0.4434	0.2487
27	0.0000	0.000000	0.000011	0.0003	0.1739	1.5465
28	0.0000	0.000000	0.000010	0.0001	0.2245	0.7780
29	0.0000	0.000000	0.000009	0.0002	0.3880	0.3079
30	0.0000	0.000000	0.000000	0.0001	0.1629	0.1667
31	0.0000	0.000000	0.000000	0.0000	0.4167	0.9128

Table (3a)

n	$\sigma_n^2(x)$ $r=3, x_0=0.4$	$\sigma_n^2(x)$ $r=4, x_0=0.4$
2	1.6498 e-001	2.1552 e-001
3	1.6081 e-001	1.8293 e+000
4	1.3760 e-001	1.2168 e+002
5	9.4139 e-002	3.3636 e+002
6	7.4985 e-002	6.2536 e+003
7	5.3312 e-002	2.3578 e+006
8	4.4795 e-002	8.2847 e+009
9	3.4394 e-002	2.0201 e+013
10	3.0050 e-002	3.4465 e+016
11	2.4145 e-002	1.6978 e+020
12	2.1554 e-002	1.1865 e+024
13	1.7829 e-002	3.5202 e+027
14	1.6138 e-002	3.3947 e+031
15	1.3635 e-002	3.4302 e+031
16	1.2465 e-002	4.1079 e+029
17	1.0707 e-002	1.1676 e+029
18	9.8635 e-003	6.8458 e+028
19	8.5856 e-003	7.9180 e+027
20	7.9580 e-003	6.2190 e+028
21	7.0042 e-003	1.1221 e+031
22	6.5250 e-003	1.2883 e+034
23	5.7971 e-003	1.0417 e+037
24	5.4237 e-003	6.2552 e+040
25	4.8576 e-003	5.0153 e+043
26	4.5615 e-003	3.0802 e+046
27	4.1142 e-003	1.6723 e+050
28	3.8759 e-003	7.6873 e+053
29	3.5175 e-003	2.1088 e+057
30	3.0324 e-003	2.1803 e+058

Table (3b)

n	r=4	
	$\sigma_n^2(x)$	$\sigma_n^2(x)$
	$x_0 = 0.3$	$x_0 = 0.31$
2	5.8672 e-001	5.3808 e-001
3	1.1541 e+001	1.0378 e+001
4	1.4453 e+001	1.8782 e+001
5	2.0262 e+001	1.3244 e+002
6	1.2653 e+003	2.5426 e+004
7	1.2185 e+006	6.0232 e+007
8	8.5390 e+009	5.6168 e+011
9	8.5232 e+012	9.7841 e+013
10	2.4040 e+016	2.0321 e+017
11	2.1665 e+020	1.9891 e+021
12	4.1357 e+022	6.5377 e+021
13	8.1155 e+025	1.5746 e+023
14	2.8845 e+029	5.7871 e+025
15	2.3724 e+033	5.8720 e+028
16	2.3911 e+033	3.2980 e+032
17	2.0033 e+031	3.2993 e+035
18	1.0398 e+030	1.1333 e+038
19	3.2630 e+028	3.5552 e+041
20	3.6054 e+028	8.2941 e+044
21	1.0151 e+029	2.3966 e+048
22	2.6141 e+030	1.5881 e+052
23	5.4074 e+032	3.0865 e+055
24	9.2186 e+034	1.2584 e+059
25	1.8355 e+038	1.0003 e+063
26	1.4899 e+042	4.4830 e+065
27	1.6612 e+045	1.1413 e+069
28	1.2163 e+049	1.8207 e+072

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