On Upwind Galerkin Finite Element Method for Incompressible Flow of Contamination, from Nuclear Waste, in Porous Media

Abbas Y. Al-Bayati   Saad A. Manaa   Ekhlass S. Al-Rawi

profabbasalbayati@yahoo.com   drekhlass-alrawi@uomosul.edu.iq
College of Computer Sciences and Mathematics
University of Mosul, Iraq

ABSTRACT

Anon-linear parabolic system is derived to describe incompressible nuclear waste disposal contamination in porous media. Galerkin method is applied for the pressure equation. For the concentration, a kind of partial upwind finite element scheme is constructed. The finite element solution satisfies discrete maximum principle and converges to the solution in norm $L^{\infty}(0,T,L^2(\Omega))$.

Keywords: Galerkin method; A kind of partial upwind finite element scheme; Non-linear parabolic system

الحل العددي لنموذج ناشئ من مخلفات الطاقة لحالة غير قابلة للانضغاط خلال مناطق مسامية باستخدام تقنية تستعمل نوعاً من العناصر المنتهية الجزئية

اخلاص سعدالله الراوي   سعد عبدالله مناع   عباس يونس البياتي
كلية علوم الحاسوب والرياضيات/جامعة الموصل/العراق

الملخص

في منظومة المعادلات اللاخطية التي تصف مخلفات الطاقة في المناطق المسامية لحالة غير قابلة للانضغاط، تم استخدام طريقة كلاركن لحل معادلة الضغط وكذلك تم استخدام تقنية جديدة تستعمل نوعاً من العناصر المنتهية الجزئية لحل معادلة التركيز وذلك بتخمين الخطأ الناتج من القيمة التقريبية لمعادلة التركيز . الحل العددي باستخدام العناصر المنتهية يحقق قاعدة الحد الأعلى للانفصال والتقارب الى الحل باستخدام المعيار $L^{\infty}(0,T,L^2(\Omega))$.

الكلمات المفتاحية: طريقة كلاركن، نوع من العناصر المنتهية الجزئية، منظومة المعادلات اللاخطية.

1.Introduction

The proposed disposal of high-level nuclear waste in underground repositories is an important environmental topic for many countries. Decisions on the feasibility and safety of the various sites and disposal methods will be based, in part, on numerical models for describing the flow of contaminated brines and groundwater through porous or fractured media under severe thermal regimes caused by the radioactive contaminants.

A fully discrete formulation is given in some detail to present key ideas that are essential in code development. The non-linear couplings between the unknowns are important in modeling the correct physics properties of flow.

In this model, we obtain a convection-diffusion equations which represent a mathematical model for a case of diffusion phenomena in which underlying flow is present; Δw *and* $b\nabla w$ correspond to the transport of w through the diffusion process and the convection effects, respectively, where ∇ *and* Δ denoted respectively the gradient operater and the Laplacian operator in the spatial coordinates.

In this paper we will consider the fluid flow in porous media using a Galerkin method for the pressure equation and a kind of partial upwind finite element scheme is constructed for the convection dominated saturation (or concentration). For more details of this subject see Douglas, 2002; Douglas, 2001 and Huang, 2000.

2.Model Equations

The model for incompressible flow and transport of contaminated brine in porous media can be described by a differential system that can be put into the following form, (see Douglas, 2002).

Fluid:

(a) $\quad \nabla.u = -q + R_s^{'}$

(b) $\quad u = -\dfrac{k(x)}{\mu(c)}\nabla P = -a(c)\nabla P$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (1)

Brine:

$$\phi\frac{\partial c}{\partial t} + u.\nabla c - \nabla.(E_c\nabla c) = g(c) \qquad\qquad (2)$$

Heat:

$$d_2\frac{\partial T}{\partial t} + c_p u.\nabla T - \nabla.(E_H\nabla T) = Q(u,T,c,p) \qquad (3)$$

Radionuclide :

$$\phi K_i\frac{\partial c_i}{\partial t} + u.\nabla c_i - \nabla.(E_c\nabla c_i) = f_i(c,c_1,....,c_N) \qquad (4)$$

With the boundary conditions

(a) $\quad u.n = 0 \qquad on \quad \Gamma$

(b) $\quad (E_c\nabla c - cu).n = 0 \qquad on \qquad \Gamma$

(c) $\quad (E_c\nabla c_i - c_i u).n = 0 \quad on \qquad \Gamma$

(d) $\quad (E_H\nabla T - c_p Tu).n = 0 \quad on \quad \Gamma$

(e) $\quad \dfrac{\partial p}{\partial n} = o \qquad ; \qquad (x,t)\in\Gamma\times(0,T]$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (5)

And the initial conditions

$$
\left.
\begin{array}{llll}
(a) & p(x,0) = p_0(x) & ; & x \in \Omega \\
(b) & c(x,0) = c_0(x) & ; & x \in \Omega \\
(c) & c_i(x,0) = c_{0i}(x) & ; & x \in \Omega \\
(d) & T(x,0) = T_0 & ; & x \in \Omega
\end{array}
\right\} \qquad (6)
$$

Where n is the unit outer normal to Γ, $x \in \Omega \subset R^2, t \in (0,T]$, u is the Darcy velocity, P the pressure, $\phi_1 = \phi c_w, q = q(x,t)$ is a production term, $R_s^{'} = R_s^{'}(c) = [c_s \phi K_s f_s /(1 + c_s)](1 - c)$ is a salt dissolution term, k(x) is the permeability of the rock, and $\mu(c)$ is the viscosity of the fluid, is dependent upon c , the concentration of the brine in the fluid , T is the temperature of the fluid, $d_2 = \phi c_p + (1 - \phi)\rho_R c_{pR}$,

$$
\tilde{E}_H = Dc_{pw} + K_m^{'} I, K_m^{'} = k_m / \rho_0, D = (D_{ij})
$$
$$
= (\alpha_T |u| \delta_{ij} + (\alpha_L - \alpha_T) u_i u_j /|u|),
$$

and

$$
Q(u,T,c,p) = -\{ [\nabla U_0 - c_p \nabla T_0].u + [U_0 + c_p(T - T_0)) + (p/\rho)][-q + R_s^{'}] \}
$$
$$
- q_L - qH - q_H.
$$
$$
E_c = D + D_m I, \text{and } g(c) = -c\{ [c_s \phi K_s f_s /(1 + c_s)](1 - c) \} - q_c + R_s^{'}
$$
,

c_i is the trace concentration of the i-th radionuclide , and

$$
f_i(c,c_1,c_2,...,c_N) = c_i \{ q - [c_s \phi K_s f_s /(1 + c_s)](1 - c) \} - q c_i - q_{ci} + q_{oi}
$$
$$
+ \sum_{j=1}^{N} k_{ij} \lambda_j K_j \phi c_j - \lambda_i K_i \phi c_i.
$$

The reservoir Ω will be taken to be of unit thickness and will be identified with a bounded domain in R^2. We shall omit

gravitational terms for simplicity of exposition, no significant mathematical questions arises the lower order terms are included. We define

$$L_2(\Omega) = \{v \mid v \text{ is defined on } \Omega \text{ and } \int_\Omega v^2 dx \prec \infty\}.$$

$$H^1(\Omega) = \{v \in L_2(\Omega) : \frac{\partial v}{\partial x_i} \in L_2(\Omega), i = 1,......,n\}.$$

and introduce the corresponding scalar products and norms

$$(v,w) = \int_\Omega vw \, dx, \qquad \|v\|_{L_2(\Omega)} = (\int_\Omega v^2 \, dx)^{1/2},$$

$$(v,w)_{H^1(\Omega)} = \int_\Omega [vw + \nabla v.\nabla w] \, dx, \|v\|_{H^1(\Omega)} = (\int_\Omega [v^2 + |\nabla v|^2] \, dx)^{1/2},$$

We also define $H_0^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma\}$, where Γ is the boundary of Ω and we equip $H_0^1(\Omega)$ with the same scalar product and norm as $H^1(\Omega)$.

$$H^2(\Omega) = \{v \in L_2(\Omega) : \frac{\partial v}{\partial x_i}, \frac{\partial^2 v}{\partial x_i^2} \in L_2(\Omega), i = 1,......, n\}.$$

With norm $\|v\|_{H^2(\Omega)} = (\int_\Omega [v^2 + |\nabla v|^2 + |\Delta v|^2] \, dx)^{1/2}$,

Let $W_s^k(\Omega)$ be the Sobolev space on Ω with norm

$$\|v\|_{W_s^k} = (\sum_{|\alpha| \leq k} \left\|\frac{\partial^\alpha v}{\partial x^\alpha}\right\|_{L^s(\Omega)}^s)^{1/s}$$

with the usual modification for s = ∞. When s=2, let $\|v\|_{W_2^k} = \|v\|_{H^k} = \|v\|_k$.

(see Johnson ,1987).
We assume that
(A1)

$$a(c), R_s^{'}(c), g(c), f_i(c, c_1,....., c_N), Q(u, Tc, p) \in C_0^1(R) ,$$

$$\phi(x), K_i, d_2 \in H'(\Omega), q \in L^\infty(0, T; H^1(\Omega))$$

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$$0 < c_0 \le a(c), R_s^{'}(c), g(c), \phi(x), f_i, d_2 Q \le c_1 \quad, \forall c_0, c_1 \in R, x \in \Omega$$

$$D_m > 0 \, , \, \alpha_l \ge \alpha_t > 0$$

(A2) The solutions of the problem (1-6) are regular:

$$c(x,t) \in L^2(0,T;H^2(\Omega)) \bigcap L^\infty(0,T;w_\infty^{'}(\Omega))$$

$$P(x,t) \in L^\infty(0,T;H^{r+1}(\Omega)) \, , \, (r \ge 2)$$

$$c_t, c_{tt}, c_{ttt} \in L^\infty(0,T;H^1(\Omega)) \, ; \, p_t, p_{tt} \in L^\infty(0,T;L^\infty(\Omega))$$

(A3) For any $\phi \in L^2(\Omega)$, the boundary values problem:

$$-\Delta \varphi + \varphi = \phi \quad , \quad x \in \Omega$$

$$\frac{\partial \varphi}{\partial n} = 0 \quad , \quad x \in \Gamma$$

There exists unique solution $\varphi \in H^2(\Omega)$ and a positive constant M such that $\|\varphi\|_2 \le M \|\phi\|$.(see Manaa, 2000).

3.Finite Element Spaces

Consider a regular family $\{T_h\}$ of triangulation defined over Ω, where h is the longest diameter of a triangular element with the triangular T_h, we have a set of close triangles $\{e_i\}(1 \le i \le N_e)$ and a set of nodes $\{P_i\}(1 \le i \le N_P + M_P)$ where $P_i(1 \le i \le N_P)$ are interior nodes in Ω and $P_j(N_{P+1} \le j \le N_P + M_p)$ are boundary nodes on Γ. We put h_s to be the maximum side length of triangles and k to be minimum perpendicular length of triangles for all $e \in T_h$.

Definition 3.1: A family T_h of triangulations is of weakly acute type, if there exists a constant $\theta_0 > 0$ independent of h such that, the internal angle θ of any triangle $e_i \in T_h$ satisfies $\theta_0 \le \theta \le \frac{\pi}{2}$. Let $\phi_i(p), (1 \le i \le M)$, be the continuous function in Ω s.t. $\phi_i(p)$, is linear on each $e \in T_h$ and $\phi_i(p_j) = \delta_{ij}$ for any nodal point p_j.

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We denote M_h as the linear span of $\phi_i, (1 \le i \le M)$, i.e., a finite dimensional subspace of $H^1(\Omega)$

$$M_h = \{z_h \mid z_h \in C(\Omega); z_h \; is \, a \, linear \, function \, on \, e, \forall e \in T_h\}.$$

And a subspace of $H_0^1(\Omega)$

$$M_{0h} = \{z_h \mid z_h \in M_h; z_h(P_k) = 0, k = M+1,......,K\}.$$

We associate the index set

$\Delta_i = \{j \ne i : P_j \; is \; adjacent \; to \; P_i\}$. Let $P_i P_j, P_k$, be three vertices of triangular element e and $\lambda_i, \lambda_j, \lambda_k$ be

barycentric coordinates.

We have the following definitions see (Hu & Tian(1992))

Definition 3.2: with each vertex P_i belonging to triangle e, the barycentric subdivision Ω_i^e is given by:

$$\Omega_i^e = \{P \mid P \in e \; ; \; \lambda_i(P) \ge \lambda_j(P) \; , \; \lambda_i(P) \ge \lambda_k(P), \forall P_j \in e\}, \; and$$

the barycentric domain Ω_i associated with vertex P_i in Ω is given by $\Omega_i = \cup \Omega_i^e \; , e \in T_h$.

Definition 3.3: with the characteristic function $\mu_i(x)$ of barycentric domain Ω_i, the mass lumping operator $\wedge : w \in C(\Omega) \to \hat{w} \in L_\infty(\Omega)$ is defined by

$$\hat{w}(p) = \sum_i^{N_P + M_P} w(p_i)\mu_i(p) \qquad .$$

Using interpolation theory in Sobolev space (see Ciarlet,1978) and inverse inequality, with step length h_c we have the relation between $\hat{w}$ and w from the following lemma

Lemma 3.1: There exists a constant C such that:

$$\|w - \hat{w}\|_{o,p} \le Mh_c |w|_{1,p} \qquad , \qquad \forall w \in M_h \;, p \ge 1 \qquad (7)$$

$$|w_h|_1 \le Mh_c^{-1} \|w_h\| \;, \qquad \forall \; w_h \in M_{0h} \qquad (8)$$

Proof: (see Hu and Tian,1992)

Lemma 3.2: There exists constants $C_1 , C_2 > 0$ such that:

$$C_1 \|w\|_{0,p} \leq \|\hat{w}\|_{0,p} \leq C_2 \|w\|_{0,p} \qquad , \qquad \forall w \in M_h \qquad (9).$$

Proof: (see Hu and Tian,1992)

Definition 3.4: Let $\{M_h\}$ be a family of finite dimensional subspaces of $C(\Omega)$, which is a piecewise polynomial space of degree less or equal to r with step length h_P and the following property: for $P \in [1,\infty]$, $r \geq 2$, there exists a constant M such that for $0 \leq q \leq 2$ and $\phi \in w_P^{r+1}(\Omega)$:

$$\inf_{x \in \{M_h\}} \|\phi - x\|_{q,p} \leq Mh^{r+1-q} |\phi|_{r+1,p}$$

Similarly, we define $\{N_h\}$ as a family of finite-dimensional subspace of $C(\Omega) \times C(\Omega)$, which is piecewise polynomial space of degree less or equal to r-1 with the similar property as M_h and $0 \leq q \leq r-1$.

We also assume that the families $\{M_h\}$ and $\{N_h\}$ satisfy inverse inequalities:

$$\|\phi\|_{L^\infty} \leq Mh_P^{-1} \|\phi\| \qquad \|\nabla \phi\|_{L^\infty} \leq Mh_P^{-1} \|\nabla \phi\| \qquad \forall \phi \in M_h$$

(see Manaa,2000).

4.Error Estimates

Let $\tau > 0$ is a time step and $N_\tau = T/\tau$. We use a Galerkin finite element method for the pressure and velocity and partial upwind finite element scheme for brine, radionuclides, and heat equation.

Let $C^0 \in M_h$ be a $L^2(\Omega)$ - projection of c^0 in M_h:

$$(C^0 - c^0, z_h) = 0 \quad \forall z_h \in M_h.$$

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We can get $P^0 \in V_h$ such that

$$\int_\Omega P^0 dx = 0 \ , \ (\frac{k(x)}{\mu(c^0)}\nabla P^0, \nabla v) = (-q^0, v) + (R_s', v) \ , \ \forall v \in V_h \ and \ U^0 \in W_h \ from$$

$$U^0 = -\frac{k(x)}{\mu(C^0)}\nabla P^0 = -a(C^0)\nabla P^0$$

If the approximate solution

$$\{P^m, U^m, C^m, C_i^m (i=1,2,\dots, N), T^m\} \in V_h \times W_h \times M_h \times M_h^N \times R_h$$

is known, we want to find

$$\{P^{m+1}, U^{m+1}, C^{m+1}, C_i^{m+1} (i=1,2,\dots, N), T^{m+1}\} \in V_h \times W_h \times M_h \times M_h^N \times R_h$$

at $t = t^{m+1}$, with three steps . Let (.,.) denote the inner product in $L^2(\Omega)$

Step 1. Find C^{m+1} for $m = 0,1,\dots,N_\tau - 1$, such that

$$(\hat{\phi}D_\tau \hat{C}^m, \hat{z}_h) + (E_c \nabla C^{m+1/2}, \nabla z) + R(U^m, C^{m+1/2}, z_h) = (\hat{g}(C^{m+1/2}), \hat{z}_h) \ \forall z_h \in M_h \ (10)$$

where $D_\tau C^m = (C^{m+1} - C^m)/\tau \ , \ C^{m+1/2} = (C^{m+1} + C^m)/2$ and

$$R(U^m, C^{m+1/2}, z_h) = \sum_{i=1}^M z_i \sum_{j \in \Lambda i} \beta_{ij}^m (\alpha_{ij}^m C_i^{m+1/2} + \alpha_{ji}^m C_j^{m+1/2}) \ (11)$$

with $z_i = z_h(P_i)$, $C_i^{m+1/2} = C^{m+1/2}(P_i)$,and $\beta_{ij}^m = \int_{\Gamma_{ij}} U^m . n_{ij} \ d\Gamma$,here n_{ij}

is the unit outer normal to Γ_{ij} . The partial upwind coefficients should be required that (see hu & Tian(1992))

(a) $\alpha_{ij}^m + \alpha_{ji}^m = 1$

(b) $\max\{1/2, 1 - \rho_{ij}^{-1}\} \le \alpha_{ij} \le 1, if\beta_{ij} \ge 0,$ $\quad$ (12)

$\quad\quad \max\{1/2, 1 - \rho_{ij}^{-1}\} \le \alpha_{ji} \le 1, if\beta_{ij} < 0,$

Step2 -Find P^{m+1} such that:

$$\int_\Omega P^{m+1} dx = 0 \ , \ (\frac{k(x)}{\mu(C^{m+1})}\nabla P^{m+1}, \nabla v) = (-q^{m+1}, v) + (R_s', v) \ , \ \forall v \in V_h \ (13)$$

Step3 -Find U^{m+1} as :

$$U^{m+1} = -\frac{k(x)}{\mu(C^{m+1})}\nabla P^{m+1} = -a(C^{m+1})\nabla P^{m+1} \quad (14)$$

<u>Lemma (4.1)</u>: Let $\bar{p} \in V_h$ be the elliptic projection of $p \in H^1(\Omega)$ into V_h defined by $(a(c)\nabla\bar{p},\nabla v) = (a(c)\nabla p,\nabla v)$, $\forall v \in V_h$ then there exists a constant k_1 such that $\|\bar{p} - p\| + h_p\|\nabla\bar{p} - \nabla p\| \le k_1\|p\|_{r+1}h_p^{r+1}$

<u>Proof:</u> (see Quarteroni, 1997).

<u>Some Important Remarks</u>:
1. $U^{m+1}.n = 0$ in Γ $\hspace{4cm}$ (15)
2. if $\|U^l - u^l\| \le Mh_P$ $(0 \le l \le m)$ then $\|U^{m+1} - u^{m+1}\| \le Mh_P$.

$\hspace{1cm}$ and if $\|U^m - u^m\| \le Mh_P$, then $\|\nabla U^m\| \le M_1$ $\hspace{2cm}$ (16)

see (Manaa,2000).
3. We will make the inductive assumption that if $\|U^l\|_{L_\infty} \le k*$ $(0 \le l \le m)$, then $\|U^{m+1}\|_{L_\infty} \le k*$ $\hspace{1cm}$ (17)
4. From (<u>Quarteroni, 1997</u>) if T_h is regular triangulation of weakly acute type we have
$|w_h|_1 \le \sqrt{6}/k\|\hat{w}_h\|$, $\forall w_h \in M_h$ $\hspace{3cm}$ (18)

<u>Theorem (4.1)</u>: C satisfies discrete mass conservation law
$$\int_\Omega \hat{\phi}D_\tau\hat{C}^m d\Omega = \int_\Omega \hat{g}^{m+1/2}(C)d\Omega , m = 1,2,....,N_\tau \hspace{2cm} (19)$$

<u>Proof</u>: In (10), let $z_h = 1$, then $(Ec\nabla C^{m+1/2},\nabla 1) = 0$ and

$$R(U^m, C^{m+1/2}, 1) = \sum_{i=1}^{M} 1 \sum_{j \in \Lambda_i} \beta_{ij}^m C_{ij}^{m+1/2} =$$

$$\sum_{e \in T_h}^{M} \sum_{p_i, p_j \in e, i < j} (\beta_{ij}^m C_{ij}^{m+1/2} - \beta_{ij}^m C_{ij}^{m+1/2}) = 0$$

Then (19) holds.

Lemma (4.2): let
$\bar{c} : [0,t] \to M_h$ such that
$(Ec\nabla c - \bar{c}, \nabla z) - \lambda(c - \bar{c}, z) = 0 \ \ \forall z \in M_h$,
$t \in J$ and let $c - \bar{c} = \varsigma$ then

$$\|\varsigma\|_1 \leq Mh_c \ \ , \ \ \left\|\frac{\partial \varsigma}{\partial t}\right\| \leq Mh_c^2 \ \ , \left\|\frac{\partial \varsigma}{\partial t}\right\|_1 \leq Mh_c$$

Proof: see (Ciarlet, 1978).

Lemma (4.3): For all $z_h \in M_h$ and $\varsigma = c - \bar{c}$, $\xi = C - \bar{c}$,

$$\left| ((u^m.\nabla c^{m+1/2}), z_h) - R(U^m, C^{m+1/2}, z_h) \right| \leq M(h_c^2 + \left\|\xi^{m+1/2}\right\|^2 + \left\|\varsigma^{m+1/2}\right\|^2 + \left. \right\}$$
$$\left. \left\|\nabla \varsigma^{m+1/2}\right\|^2 + \left\|U^m - u^m\right\|^2 + \left\|z_h\right\|^2) + \varepsilon \left\|\nabla z_h\right\|^2 \right\} \quad (20)$$

where $\varepsilon > 0$ is arbitrary small constant.

Proof:

$$((u^m.\nabla c^{m+1/2}), z_h) - R(U^m, C^{m+1/2}, z_h) = ((u^m - U^m).\nabla c^{m+1/2}, z_h)$$
$$+ (U^m \nabla(c^{m+1/2} - C^{m+1/2}), z_h) + (U^m \nabla C^{m+1/2}, z_h - \hat{z}_h)$$
$$+ [(U^m.\nabla C^{m+1/2}, \hat{z}_h) - R(U^m, C^{m+1/2}, z_h)] = J1 + J2 + J3 + J4$$

With (A1) and (A2) we have :
$$J1 = ((u^m - U^m).\nabla c^{m+1/2}, z_h) \leq M \left\|u^m - U^m\right\|.\left\|z_h\right\|$$
$$\leq M(\left\|u^m - U^m\right\|^2 + \left\|z_h\right\|^2)$$

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Using (A1),(17),(18) and (9) we have:

$$J2 = (U^m . \nabla(c^{m+1/2} - C^{m+1/2}), z_h) \le M[\|\nabla(c^{m+1/2} - \bar{c}^{m+1/2})\| + \|\nabla(-(C^{m+1/2} - c^{m+1/2}))\|] . \|z_h\|$$

$$\le M[\|\nabla \varsigma^{m+1/2}\| + \|\nabla \xi^{m+1/2}\|] . \|z_h\|$$

$$\le M[\frac{\sqrt{6}}{k}\|\hat{\xi}^{m+1/2}\|^2 + \frac{\sqrt{6}}{k}\|\hat{\xi}^{m+1/2}\|^2] + M\|z_h\|^2$$

$$\le M(\|\varsigma^{m+1/2}\|^2 + \|\xi^{m+1/2}\|^2 + \|z_h\|^2)$$

From (A2),(7),(8),(17) and (18) we have

$$J3 = (U^m \nabla C^{m+1/2}, z_h - \hat{z}_h) \le M\|U^m\| \|\nabla C^{m+1/2}\| . \|z_h - \hat{z}_h\|$$

$$\le M\|\nabla \xi^{m+1/2}\| \|z_h - \hat{z}_h\| + M\|\nabla \varsigma^{m+1/2}\| \|z_h - \hat{z}_h\|$$

$$+ M\|\nabla c^{m+1/2}\| \|z_h - \hat{z}_h\| = k_1 + k_2 + k_3$$

$$k_1 \le M[h_c^2 + \|\xi^{m+1/2}\|^2] + \varepsilon \|\nabla z_h\|^2$$

$$k_2 \le M[h_c^2 + \|\nabla \varsigma^{m+1/2}\|^2 + \|z_h\|^2]$$

$$k_3 \le M[h_c^2 + \|z_h\|^2]$$

$$J_3 \le M[h_c^2 + \|\xi^{m+1/2}\|^2 + \|\nabla \varsigma^{m+1/2}\|^2 + \|z_h\|^2] + \varepsilon \|\nabla z_h\|^2$$

Using (9),(11),(16) and (17) implies

$$J4 \le \left| \sum_{i=1}^{M} z_i \sum_{j \in \Lambda i} \int_{\Gamma_{ij}} (C^{m+1/2} - C_{ij}^{m+1/2}) U^m . n_{ij} \, d\Gamma \right| + \|C^{m+1/2}\| \|\nabla U^m\| \|\hat{z}_h\|$$

Since $C_{ij}^{m+1/2} = \alpha_{ij}^m C_i^{m+1/2} + \alpha_{ji}^m C_j^{m+1/2}$ so

$$J4 \le \left| \sum_{e \in T_h} \sum_{P_i, P_j \in e} (z_i - z_j) \int_{\Gamma_{ij}^e} (C^{m+1/2} - C_{ij}^{m+1/2}) U^m . n_{ij} \, d\Gamma \right| +$$

$$M\|C^{m+1/2} - \bar{c}^{m+1/2} + \bar{c}^{m+1/2} - c^{m+1/2} + c^{m+1/2}\| \|\hat{z}_h\|$$

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$$\leq M \sum_{e \in T_h} h_e \left| \nabla(z_h|_e) \right| \sum_{P_i, P_j \in e, i<j, i<j} \int_{\Gamma_{ij}^e} \left| C^{m+1/2} - C_{ij}^{m+1/2} \right| d\Gamma$$

$$+ M[\left\| \xi^{m+1/2} \right\| + \left\| \varsigma^{m+1/2} \right\| + \left\| c^{m+1/2} \right\|] \|z_h\|$$

And in element e we have $\left| C^{m+1/2} - C_{ij}^{m+1/2} \right| \leq 2 \left| \nabla(C^{m+1/2}|e) \right| h_e$

Then

$$J4 \leq 3M2 \sum_{e \in T_h} h_e^3 \left\| \nabla(z_h|e) \right\| . \left\| \nabla(C^{m+1/2}|e) \right\|$$

$$+ M[\left\| \xi^{m+1/2} \right\|^2 + \left\| \varsigma^{m+1/2} \right\|^2 + \left\| c^{m+1/2} \right\|^2 + \left\| z_h \right\|^2]$$

$$\leq M(h_c^2 + \left\| \xi^{m+1/2} \right\|^2 + \left\| \varsigma^{m+1/2} \right\|^2 + \left\| \nabla \varsigma^{m+1/2} \right\|^2 + \left\| z_h \right\|^2) + \varepsilon \left\| \nabla z_h \right\|^2$$

Hence J1+J2+J3+J4 Implies (20).

Lemma (4.4): There exists a positive constant k_2 such that:

$$\left\| \nabla P^{m+1} - \nabla \overline{P}^{m+1} \right\| \leq k_2 \left\| C^{m+1} - c^{m+1} \right\| \qquad (21)$$

Proof:

We have that:

$$(a(C^{m+1}) \nabla P^{m+1}, \nabla \upsilon) = (q^{m+1}, v) + (R_s'(C^{m+1}), v), \qquad (22)$$

$$(a(c^{m+1}) \nabla \overline{p}^{m+1}, \nabla \upsilon) = (q^{m+1}, v) + (R_s'(c^{m+1}), v), \qquad (23)$$

Subtracting (23) from (22), we get

$$(a(C^{m+1}) \nabla(P^{m+1} - \overline{P}^{m+1}), \nabla \upsilon) + ([a(C^{m+1}) - a(c^{m+1})] \nabla \overline{P}^{m+1}, \nabla \upsilon)$$

$$= (R_s'(C^{m+1}) - (R_s'(c^{m+1}), v)$$

Let $\upsilon = P^{m+1} - \overline{P}^{m+1} \in V_h$, then:

$$\left\| \nabla(P^{m+1} - \overline{P}^{m+1}) \right\|^2 \leq \left| (a(C^{m+1}) \nabla(P^{m+1} - \overline{P}^{m+1}), \nabla(P^{m+1} - \overline{P}^{m+1})) \right|$$

$$= \left| ([a(c^{m+1}) - a(C^{m+1})] \nabla \overline{P}^{m+1}, \nabla(P^{m+1} - \overline{P}^{m+1})) + (R_s'(C^{m+1}) - (R_s'(c^{m+1}), P^{m+1} - \overline{P}^{m+1}) \right|$$

Using (A1) we have

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$$\leq M\left\|\nabla \overline{P}^{m+1}\right\| \ \left\|C^{m+1}-c^{m+1}\right\| \ \left\|\nabla (P^{m+1}-\overline{P}^{m+1})\right\|$$

$$+\left\|C^{m+1}-c^{m+1}\right\| \ \left\|P^{m+1}-\overline{P}^{m+1}\right\|$$

with lemma (4.1) and (A2) if $h_P > 0$ is sufficiently small

$$\left\|\nabla \overline{P}^{m+1}\right\| \leq \left\|\nabla P^{m+1}\right\| + k_1\left\|P\right\|_{r+1} h_P^r < M \text{ , so we have (21).}$$

Lemma (4.5): There exists a positive constant k_3 such that

$$\left\|U^{m+1}-u^{m+1}\right\| = k_3(\left\|C^{m+1}-c^{m+1}\right\| + h_P^r) \tag{24}$$

Proof:

$$\left\|U^{m+1}-u^{m+1}\right\| = \left\|a(C^{m+1})\nabla P^{m+1} - a(c^{m+1})\nabla P^{m+1}\right\|$$

$$\leq \left\|a(C^{m+1})\nabla (P^{m+1}-p^{m+1})\right\| + \left\|a(C^{m+1})-a(c^{m+1})\right\| \ \left\|\nabla p^{m+1}\right\|$$

From (A1) and (A2) we have

$$\leq \text{const.}\left\|\nabla P^{m+1}-\nabla p^{m+1}\right\| + \left\|C^{m+1}-c^{m+1}\right\| \ \left\|\nabla p^{m+1}\right\|_{L_\infty}$$

we have $\left\|\nabla P^{m+1}-\nabla p^{m+1}\right\| \leq \left\|\nabla P^{m+1}-\nabla \overline{p}^{m+1}\right\| + \left\|\nabla \overline{P}^{m+1}-\nabla p^{m+1}\right\|$

using lemma (4.2) and lemma (4.4) , we get

$\left\|\nabla P^{m+1}-\nabla p^{m+1}\right\| \leq k_2\left\|C^{m+1}-c^{m+1}\right\| + k_1\left\|P^{m+1}\right\|_{r+1} h_p^r$, from (A2), (24)

holds.

Theorem (4.2): For all $m \leq l \leq N_\tau$ if $\tau \leq \tau_0$, then

$\left\|C^{l+1}-c^{l+1}\right\| \leq M(\tau + h_c + h_P^r)$, where M independent of τ and h_c.

Proof: Multiply eq.(2) by z_h and integrating by parts we obtain for

$t = (m+1/2)\tau$. Let $w_{m+1/2} = w(.,(m+1/2)\tau)$ and $w^{m+1/2} = (w^{m+1}+w^m)/2, then$

$(\phi D_\tau c^m, z_h) + (Ec\nabla c^{m+1/2}, \nabla z_h) + (u^m\nabla c^{m+1/2}, z_h) =$

$$(g(c^{m+1/2}), z_h) + (\phi(D_\tau c^m - \frac{\partial c}{\partial t}\big|_{m+1/2}), z_h) + \tag{25}$$

$$+((Ec\nabla c^{m+1/2} - Ec\nabla c_{m+1/2}), \nabla z_h) + (\ (u^m\nabla c^{m+1/2} - u_{m+1/2}\nabla c_{m+1/2}), z_h)$$

Let $e = c - C = (c - \bar{c}) - (C - \bar{c}) = \varsigma - \xi$, and subtract (10) from (25), we obtain:

$$(\hat{\phi}D_\tau \hat{e}^m, \hat{z}_h) + (Ec\nabla e^{m+1/2}, \nabla z_h) = (R(U^m, C^{m+1/2}, z_h)$$

$$- (u^m \nabla c^{m+1/2}, z_h)) + ((\hat{\phi}D_\tau \hat{c}^m, \hat{z}_h) - (\phi D_\tau c^m, z_h))$$

$$+ ((g(c^{m+1/2}), z_h) - (\hat{g}(C^{m+1/2}), z_h)) + (\phi(D_\tau c^m - \frac{\partial c}{\partial t}\big|_{m+1/2}), z_h)$$

$$+ ((Ec\nabla c^{m+1/2} - Ec\nabla c_{m+1/2}), \nabla z_h) + ((u^m \nabla c^{m+1/2} - u_{m+1/2} \nabla c_{m+1/2}), z_h)$$

Hence

$$(\hat{\phi}D_\tau \hat{\xi}^m, \hat{z}_h) + (Ec\nabla \xi^{m+1/2}, \nabla z_h) = (\hat{\phi}D_\tau \hat{\varsigma}^m, \hat{z}_h) + (Ec\nabla \varsigma^{m+1/2}, \nabla z_h) +$$

$$((u^m \nabla c^{m+1/2}, z_h) - R(U^m, C^{m+1/2}, z_h)) + ((\hat{g}(C^{m+1/2}), \hat{z}_h) - (g(c^{m+1/2}), z_h)) +$$

$$((\phi D_\tau c^m, z_h) - (\hat{\phi}D_\tau \hat{c}^m, \hat{z}_h)) \ - ((\phi(D_\tau c^m - \frac{\partial c}{\partial t}\big|_{m+1/2}), z_h) + \qquad (26)$$

$$((Ec\nabla c^{m+1/2} - Ec\nabla c_{m+1/2}), \nabla z_h) + ((u^m \nabla c^{m+1/2} - u_{m+1/2} \nabla c_{m+1/2}), z_h)$$

$$= I1 + I2 + I3 + I4 + I5 + I6$$

In (26), let $z_h = \xi^{m+1/2} \in M_h$, and using (A1) the left – hand side is

$$\geq \frac{\hat{\phi}\|\hat{\xi}^{m+1}\|^2 - \|\hat{\xi}^m\|^2}{2\tau} + c_0 \|\nabla \xi^{m+1/2}\|^2$$

From (A1), we have:

$$I1 = (\hat{\phi}D_\tau \hat{\varsigma}^m, \hat{z}_h) \leq M(\|D_\tau \hat{\varsigma}^m\|^2 + \|\hat{\xi}^{m+1/2}\|^2 \ )$$

Using (7) and (8), we have

$$\|D_\tau \hat{\varsigma}^m\| \leq \|D_\tau \varsigma^m\| + \|D_\tau \varsigma^m - D_\tau \hat{\varsigma}^m\| \leq \|D_\tau \varsigma^m\| + Mh_c |D_\tau \varsigma^m|_1$$

From (9) we have

$$\|\hat{\xi}^{m+1/2}\| \leq M\|\xi^{m+1/2}\| \leq M(\|\xi^{m+1}\| + \|\xi^m\|)$$

$$I1 \leq M(\|\xi^{m+1}\|^2 + \|\xi^m\|^2 + \|D_\tau \varsigma^m\|^2 + h_c^2 \|D_\tau \varsigma^m\|_1^2)$$

Using (A1), we have: $I2 \leq M\|\nabla \varsigma^{m+1/2}\|^2 + \varepsilon \|\nabla \xi^{m+1/2}\|^2$

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From lemma (4.3)

$$I3 \le M(h_c^2 + \left\|\xi^{m+1/2}\right\|^2 + \left\|\varsigma^{m+1/2}\right\|^2 +$$

$$\left\|\nabla \varsigma^{m+1/2}\right\|^2 + \left\|U^m - u^m\right\|^2) + \varepsilon \left\|\nabla \xi^{m+1/2}\right\|^2$$

let $\theta = \phi \dfrac{c^{m+1} - c^m}{\tau} = \phi(\dfrac{\partial c}{\partial t}|_{m+1/2} + 1/24 \dfrac{\partial^3 c}{\partial t^3}|\tau^2)$, using (A1),(A2) and

(Raviort,1979) we get

$$I4 \le Mh_c(1+\tau^2)|z_h|_1 + Mh_c(1+\tau^2)\|z_h\|$$

$$\le Mh_c\|z_h\| + Mh_c|z_h|_1 + Mh_c\tau^2\|z_h\| + Mh_c\tau^2|z_h|_1$$

$$\le M(h_c^2 + h_c^2\tau^4) + \varepsilon\left\|\nabla \xi^{m+1/2}\right\|^2$$

$$I5 \le M(h_c^2 + \left\|\xi^{m+1/2}\right\|^2 + \left\|\varsigma^{m+1/2}\right\|^2) + \varepsilon\left\|\nabla \xi^{m+1/2}\right\|^2$$

Let I6=K1+K2+K3, so

$$K1 \le M(\tau^4 + \left\|\xi^{m+1}\right\|^2 + \left\|\xi^m\right\|^2), K2 \le M\tau^2 + \varepsilon\left\|\xi^{m+1/2}\right\|^2,$$

$$K3 \le M(\tau^2 + \left\|\xi^{m+1}\right\|^2 + \left\|\xi^m\right\|^2)$$

Then we get

$$I6 \le M(\tau^2 + \left\|\xi^{m+1}\right\|^2 + \left\|\xi^m\right\|^2) + \varepsilon\left\|\nabla \xi^{m+1/2}\right\|$$

Equation (26) can be written now

$$\frac{\hat{\phi}\left\|\hat{\xi}^{m+1}\right\|^2 - \left\|\hat{\xi}^m\right\|^2}{2\tau} + c_0\left\|\nabla \xi^{m+1/2}\right\| \le M(\left\|\xi^{m+1}\right\|^2 + \left\|\xi^m\right\|^2$$

$$+ \left\|D_\tau \varsigma^m\right\|^2 + h_c^2\left\|D_\tau \varsigma^m\right\|_1^2 + \left\|\nabla \varsigma^{m+1}\right\|^2 + \left\|\nabla \varsigma^{m+1/2}\right\|^2$$

$$+ \left\|\varsigma^{m+1/2}\right\|^2 + \left\|U^m - u^m\right\|^2 + h_c^2 + \tau^2) + \varepsilon\left\|\nabla \xi^{m+1/2}\right\|^2$$

Take the summation from 0 to l, where

$m \le l \le N_\tau$, and $C^0 = \bar{c}^0$ so $\xi^0 = 0$

$$\hat{\phi}\left\|\hat{\xi}^{l+1}\right\|^2 \le M_1\sum_{m=0}^{l+1} \left\|\xi^m\right\|^2 \tau + M_2\left\|D_\tau \varsigma\right\|^2 L_\Delta^2(L^2) + M_2 h_c^2\left\|D_\tau \varsigma\right\|_{L_\Delta^2(H^1)}^2$$

$$+ M_2\sum_{m=0}^{l} \left\|\varsigma\right\|_1^2 \tau + M_2(h_c^2 + \tau^2) + M_3\sum_{m=0}^{l} \left\|U^m - u^m\right\|^2 \tau$$

Where $\left\|w\right\|_{L_{\Delta(x)}^2}^2 = \sum_{m=0}^{N_\tau}\left\|w^m\right\|_x^2 \tau$, using (7) and (A1), we get

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$$\left\|\xi^{l+1}\right\| \le \left\|\hat{\xi}^{l+1}\right\| + Mh_c\left|\xi^{l+1}\right|_1 \le \left\|\hat{\xi}^{l+1}\right\| + Mh_c\frac{\sqrt{6}}{k}\left\|\hat{\xi}^{l+1}\right\| \le M\left\|\hat{\xi}^{l+1}\right\|$$

$$\hat{\phi}\left\|\hat{\xi}^{l+1}\right\| \ge M_0\left\|\hat{\xi}^{l+1}\right\|, \text{ So}$$

$$\left\|\xi^{l+1}\right\|^2 \le M_1\sum_{m=0}^{l+1}\left\|\xi^m\right\|^2\tau + M_3\sum_{m=0}^{l}\left\|U^m - u^m\right\|^2\tau + M_2(\left\|D_\tau\varsigma\right\|^2_{L^2_\Delta(L^2)} + h_c^2\left\|D_\tau\varsigma\right\|^2_{L^2_\Delta(H^1)}$$

$$+ \left\|\varsigma\right\|^2_{L^2_\Delta(H^1)} + h_c^2 + \tau^2) \qquad (27)$$

From lemma (4.6) we have

$$\sum_{m=0}^{l}\left\|U^m - u^m\right\|^2\tau \le M(\sum_{m=0}^{l}\left\|\xi^m\right\|^2\tau + \left\|\varsigma\right\|^2_{L^2_\Delta(L^2)} + h_p^{2r}) \qquad (28)$$

From (27) and (28), we have

$$\left\|\xi^{l+1}\right\|^2 \le M_1\sum_{m=0}^{l+1}\left\|\xi^m\right\|^2\tau + M_2(h_c^2 + \tau^2 + h_p^{2r})$$

From Gronwall inequality , we get:

$$\left\|\xi^{l+1}\right\| \le M(h_c + \tau + h_P^r)$$

it is $$\left\|C^{l+1} - c^{l+1}\right\| \le M(h_c + \tau + h_P^r)$$

Theorem 4.3 with the assumption (A1)~ (A3), if $h_c = O(h_P), \tau = O(h_P)$,

Then $$\left\|C - c\right\|_{L^\infty(l^2)} + \left\|U - u\right\|_{L^\infty(l^2)} \le M(\tau + h_c + h_P^r)$$

Proof:
We will prove first the inductive assumption (17) . We have

$$\left\|U^{m+1}\right\|_{L_\infty} \le \left\|a(C^m)\right\|_{L_\infty}\left\|\nabla P^{m+1}\right\|_{L_\infty} \le k_4\left\|\nabla P^{m+1}\right\|_{L_\infty}$$

But

$$\left\|\nabla P^{m+1}\right\|_{L_\infty} \le \left\|\nabla P^{m+1} - \nabla\overline{P}^{m+1}\right\|_{L_\infty} + \left\|\nabla\overline{P}^{m+1}\right\|_{L_\infty}$$

$$\le Mh_P^{-1}\left\|\nabla P^{m+1} - \nabla\overline{P}^{m+1}\right\| + k_5 \le 2k_5$$

When $h_c = o(h_P)$, $\tau = o(h_P)$ $(r \ge 2)$, $\left\|\nabla P^{m+1}\right\|_{L_\infty} \le k*$,

$$\left\| U^{m+1} - u^{m+1} \right\| \le k_3 (\left\| C^{m+1} - c^{m+1} \right\| + h_P^r) \quad , \quad \forall m = 0,1,..., N_\tau \quad (29)$$

And from theorem (4.2)

$$\left\| C^{m+1} - c^{m+1} \right\| \le M(h_c + \tau + h_P^r) \quad , \quad \forall m = 0,1,..., N_\tau \quad (30)$$

Equations (29) and (30) implies

$$\left\| C^{m+1} - c^{m+1} \right\| + \left\| U^{m+1} - u^{m+1} \right\| \le M(\tau + h_P^r + h_c) \quad , \quad \forall m = 0,1,..., N_\tau$$

Which is complete the proof.

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