Stability Analysis for Fluid Flow between Two Infinite Parallel Plates II

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ABSTRACT

A model of fluid flow with heat transfer by conduction , convection and radiation has been discussed for stability with respect to restricted parameters (k, α, r, T^{*}) which are proportional to: wave numbers, thermal expansion coefficient, combination of many numbers $(Re, Pr, Ec, Bo, W, \gamma)$ and the ratio of walls temperatures, respectively using analytical technique which illustrates that the stability of the system depends on these parameters and the disturbances with a larger wave number, grows faster than that with smaller wave number .A clear picture of the flow is shown in the velocity field $\langle u_2, v_2 \rangle$ tables and figures.

تحليل الاستقرارية لجريان مائع بين لوحين متوازيين لا نهائيين II

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الملخص

تمت في هذا البحث دراسة الاستقرارية لنظام يمثل مائعا مثاليا بين لوحين

متوازيين لا نهائيين يسخن من الأسفل واعتماد انتقال الحرارة بالتوصيل والحمل

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والإشعاع. وقد نوقش الموضوع تحت تأثير عدد من المعلمات ذات دلالة فيزيائية ، إذ تبين أن استقرارية النظام تعتمد على تلك المعلمات ، كما تبين أن الازعاج يزداد (ينمو) بزيادة العدد الموجي و استخدمت في أيجاد ذلك طريقة تحليلية تم من خلالها الحصول على صورة واضحة للجريان بتأثير التسخين.

الكلمات المفتاحية: تحليل الاستقرارية، جريان مائع، التوصيل، الحمل الحراري، السرعة، ألواح متوازية.

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Introduction:

The principle of stability of fluid flow with heat transfer and its applications has been investigated by many authors such as Lorenz [1] and Yorke [4] and others.

In this paper a model of heat transfer by conduction, convection and radiation in a fluid flow between two infinite parallel flat plates has been considered.

The first like model without heating from below was investigated by Logan [2].

Model and Governing Differential Equations:

Consider, an ideal fluid confined between two infinite parallel plates $Y_1=0$, $Y_1=d$ in X_1 Z_1 Y_1 space separated by a distance d, and heated from below, which is under the influence of a constant gravitation field g acting in the negative Y_1 direction as shown in the following figure:
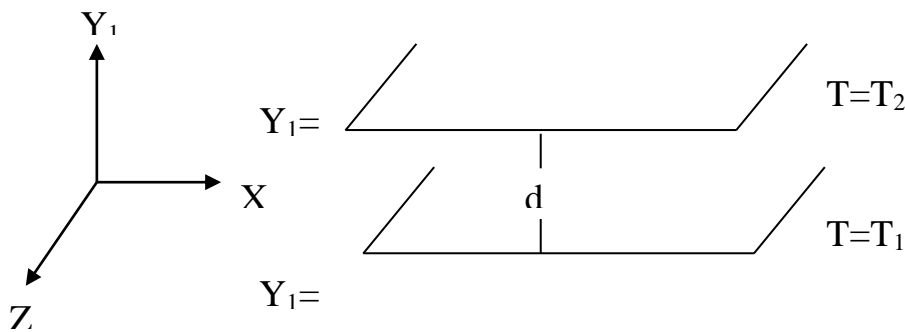
Figure(1): fluid flow between two parallel plates.

Using Boussinisque approximation and the optical thick limit and if the temperature differences between the walls and the fluid is small [3] the non- dimensional differential equations (in the new plane x z y , where $x=X_1/d$,$z=Z_1/d$, and $y=Y_1/d$) of the disturbance are :

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial w_2}{\partial z} = 0 \qquad 1$$

$$\frac{Du_2}{Dt} = -\frac{\partial p_2}{\partial x} \qquad 2$$

$$\frac{Dv_2}{Dt} = -\frac{\partial p_2}{\partial y} + \alpha\theta_2 \qquad 3 \qquad\qquad (1)$$

$$\frac{Dw_2}{Dt} = -\frac{\partial p_2}{\partial z} \qquad 4$$

$$\frac{D\theta_2}{Dt} = R\nabla^2\theta_2 \qquad 5$$

Where R= $\dfrac{\gamma}{Pr.Re} + \dfrac{16}{3}\dfrac{Ec.\gamma}{Bo.w}$, $\quad \alpha = T_1\beta$,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

(γ: Specific heat ratio, Pr: prandtl number, Re: Reynold number, Ec: Eckert number, Bo: Boltzman number, w: Bouguer number, and β: thermal expansion coefficient).

with the boundary conditions : $u_2 = v_2 = w_2 = \theta_2 = 0$, y =0,1

To investigate the system (1) (analytically) for stability or otherwise we attempt to find a solution of the form:

$$u_2 = U\ (y)\ e^{at}\ e^{i\ (k_1 x + k_2 z)}$$
$$v_2 = V\ (y)\ e^{at}\ e^{i\ (k_1 x + k_2 z)}$$
$$-\qquad -\qquad - \qquad\qquad\qquad (2)$$
$$-\qquad -\qquad -$$
$$\theta_2 = \theta\ (y)\ e^{at}\ e^{i\ (k_1 x + k_2 z)}$$

Where k_1, k_2 are the wave numbers and "a" is the speed number($a = a_1 + ia_2$).
Substituting the form (2) in system (1) and using the linearized theory we have:

$$
\left.
\begin{array}{ll}
ik_1 U + V' + ik_2 W = 0 & 1 \\
aV = -ik_1 P & 2 \\
aV = -P' + \alpha\theta & 3 \\
aW = -ik_2 P & 4 \\
\theta'' - \dfrac{(a + Rk^2)}{R}\theta - \dfrac{T^*}{R}V = 0 & 5
\end{array}
\right\} \qquad (3)
$$

Where $T^* = (T_2 - T_1)/T_1$, (T_1: temperature of the lower wall , T_2: temperature of the upper wall) and $k^2 = k_1^2 + k_2^2$.

From system (3) and by eliminating the pressure function in the third equation using equations (1), (2),(4) and (5) we get:

$$
V^{(4)} + L_1 V'' + L_2 V = 0 \qquad (4)
$$

$$
\text{Where: } L_1 = -\left(\frac{a}{R} + 2k^2\right) \quad , \quad L_2 = \left(k^4 + \frac{a}{R}k^2 - \frac{\alpha T^*}{aR}k^2\right)
$$

With the boundary conditions:
$$
V(0) = V(1) = V'(0) = V'(1) = 0 \qquad (5)
$$

In order to find a solution satisfying the boundary condition, we must investigate all cases we get from the characteristic equation.
$$
m^4 + L_1 m^2 + L_2 = 0 \qquad (6)
$$
which is algebraic equation of degree four, it has four roots .
Thus:
$$
m^2 = \frac{1}{2}\left[-L_1 \pm \sqrt{L_1^2 - 4L_2}\right] \qquad (7)
$$
For m^2 there are three cases (here "a" is real):

1. $L_1^2 - 4L_2 < 0$ 2. $L_1^2 - 4L_2 = 0$ 3. $L_1^2 - 4L > 0$
each one has many cases.

By investigating those cases and applying the boundary conditions (5), the nontrivial solution holds:

i) $L_1^2 - 4L_2 > 0, -L_1 < 0$ and: $|-L_1| = \sqrt{L_1^2 - 4L_2}$

ii) $L_1^2 - 4L_2 > 0, -L_1 < 0$ and: $|-L_1| > \sqrt{L_1^2 - 4L_2}$

In the case (I), the general solution can be written as:

$$V(y) = c_1 + c_2 y + c_3 \cos. \alpha_3 y + c_4 \sin \alpha_3 y , \tag{8}$$

where the roots of the characteristic equation (6) are:

$\lambda_1 = \lambda_2 = 0$, $\lambda_3 = i\alpha_3$, and $\lambda_4 = -i\alpha_3$

The boundary conditions (5) give the particular solution:

$$V_n = C_n(1 - \cos(2n\pi)y), \quad where \; \alpha_3 = 2n\pi \tag{9}$$

In this paper we illustrate the case (ii) only, where the general solution can be written as:

$$V = c_1 \cos\alpha_1 y + c_2 \sin\alpha_1 y + c_3 \cos\alpha_2 y + c_4 \sin\alpha_2 y \tag{10}$$

Solution (10) satisfies the four boundary conditions when
$\alpha_1 = 2n\pi$ and $\alpha_2 = 2m\pi$ (where $m \neq n$, m=1,2,…
and n=1,2,…).and then the solution takes the form:

$$V_n = c_1(\cos(2n\pi) - \cos(2m\pi)y) + c_2\left(\sin(2n\pi)y - \frac{n}{m}\sin(2m\pi)y\right) \tag{11}$$

where c_1 and c_2 are arbitraries such that c_2 depends on c_1.
V_n is called the eigenfunction.
The other eigenfunctions can be found by using the relations in system (3), so the functions: v_2, u_2, w_2, p_2, and θ_2 have the following forms:

$$v_2 = V_n e^{at} e^{i(k_1 x + k_2 z)} \tag{1}$$

$$u_2 = i\frac{k_1}{k^2}[V_n']e^{at} e^{i(k_1 x + k_2 z)} \tag{2}$$

$$w_2 = i \frac{k_2}{k^2} [V_n'] e^{at} e^{i(k_1 x + k_2 z)} \qquad (3) \qquad (12)$$

$$P_2 = -\frac{a}{k_2} [V_n'] e^{at} e^{i(k_1 x + k_2 z)} \qquad (4)$$

$$\theta_2 = -\frac{a}{\alpha} \left[\frac{V_n''}{k^2} + V_n \right] e^{at} e^{i(k_1 x + k_2 z)} \qquad (5)$$

We get a clear picture of the flow pattern by examining a two dimensional disturbance, and the case n=1, m=2, k=1, (x,y) plane with functions (12).where u_2 and v_2 are as follows:

$$u_2 = i\left(-2\pi \sin 2\pi y + 4\pi \sin 4\pi y + 2\pi \cos 2\pi y - 2\pi \cos 4\pi y \right) e^{ix} e^{at}$$

$$v_2 = \left(\cos 2\pi y - \cos 4\pi y + \sin 2\pi y - \frac{1}{2} \sin 4\pi y \right) e^{ix} e^{at}$$

To obtain the real solution, we take the real parts of the previous equations thus:

$$u_2 = -\left(-2\pi \sin 2\pi y + 4\pi \sin 4\pi y + 2\pi \cos 2\pi y - 2\pi \cos 4\pi y \right) e^{at} \sin x$$

$$v_2 = \left(\cos 2\pi y - \cos 4\pi y + \sin 2\pi y - \frac{1}{2} \sin 4\pi y \right) e^{at} \cos x$$

The following table contains the information of the sings of the functions u_2 and v_2 in the intervals $0 < x < 2\pi$ and $0 < y < 1$:

u_2	v_2
$u_2 < 0$ when: 1) $0 < x < \pi$ and $0 < y < 0.25$, $0.55 < y < 0.85$ 2) $\pi < x < 2\pi$ and : $0.25 < y < 0.55$, $0.85 < y < 1$	$v_2 < 0$ when: 1) $0 < x < \frac{\pi}{2}, \frac{3\pi}{2} < x < 2\pi$ and $0.3 < y < 0.7$ 2) $\frac{\pi}{2} < x < \frac{3\pi}{2}$ and: $0 < y < 0.3$, $0.7 < y < 1$
$u_2 = 0$ when : x= 0, π, 2π or y=0.25, 0.55, 0.85,1	$v_2 = 0$ when : x= $\frac{\pi}{2}, \frac{3\pi}{2}$, or y=0, 0.3, 0.7, 1

$u_2 > 0$ when	$v_2 > 0$ when
1) $0 < x < \pi$ and $0.25 < y < 0.55$, $0.85 < y < 1$ 2) $\pi < x < 2\pi$ and $0 < y < 0.25$, $0.55 < y < 0.85$	1) $0 < x < \dfrac{\pi}{2}, \dfrac{3\pi}{2} < x < 2\pi$ and $0 < y < 0.3$, $0.7 < y < 1$ 2) $\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$ and $0.3 < y < 0.7$

Table (1): Information about the signs of the functions u_2 and v_2 in the intervals $0 < x < 2\pi$ and $0 < y < 1$.

This information and indications given in the above satiated table have been summarized by an arrow showing the direction of the velocity field $< u_2, v_2 >$ in the following figure:
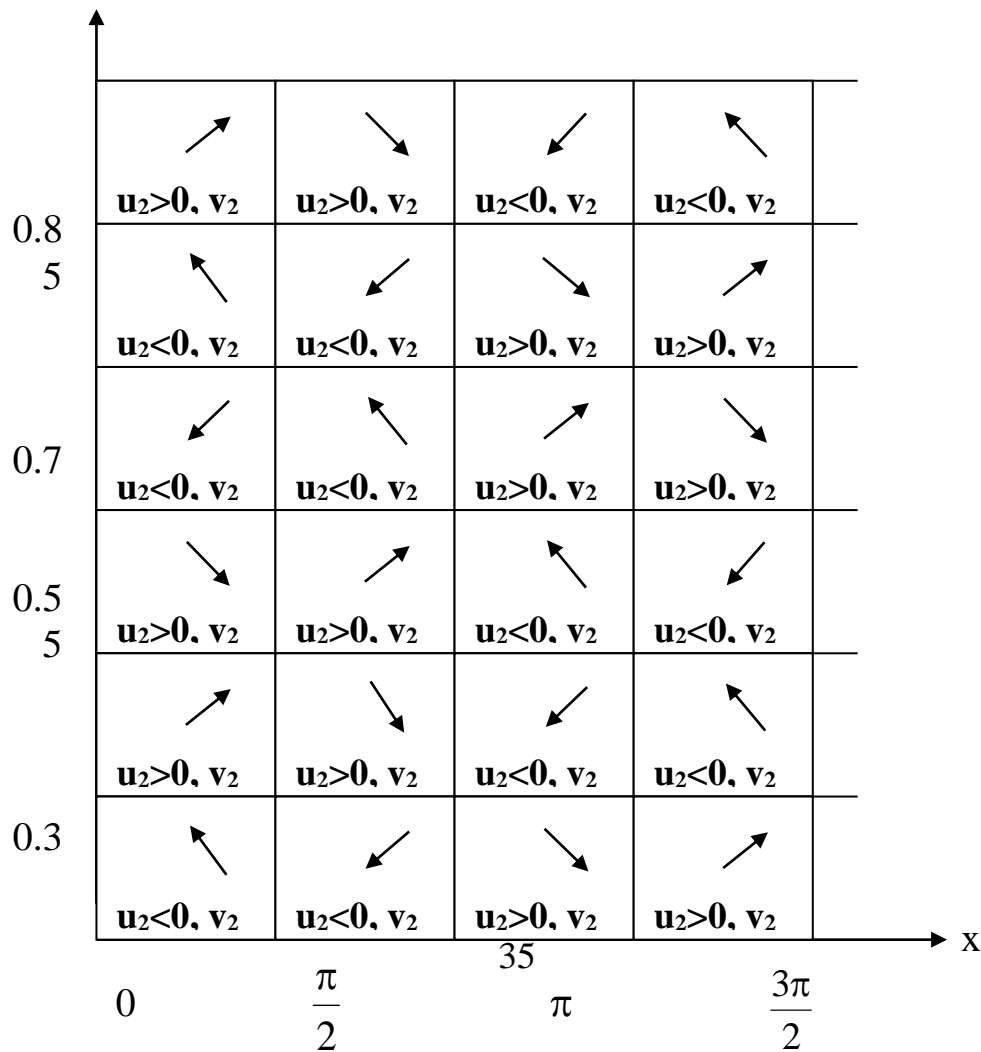
Figure (1) shows the directions of the velocity field $<u_2, v_2>$.

      The typical streamlines are shown in the following figure, viewed from the positive z direction:
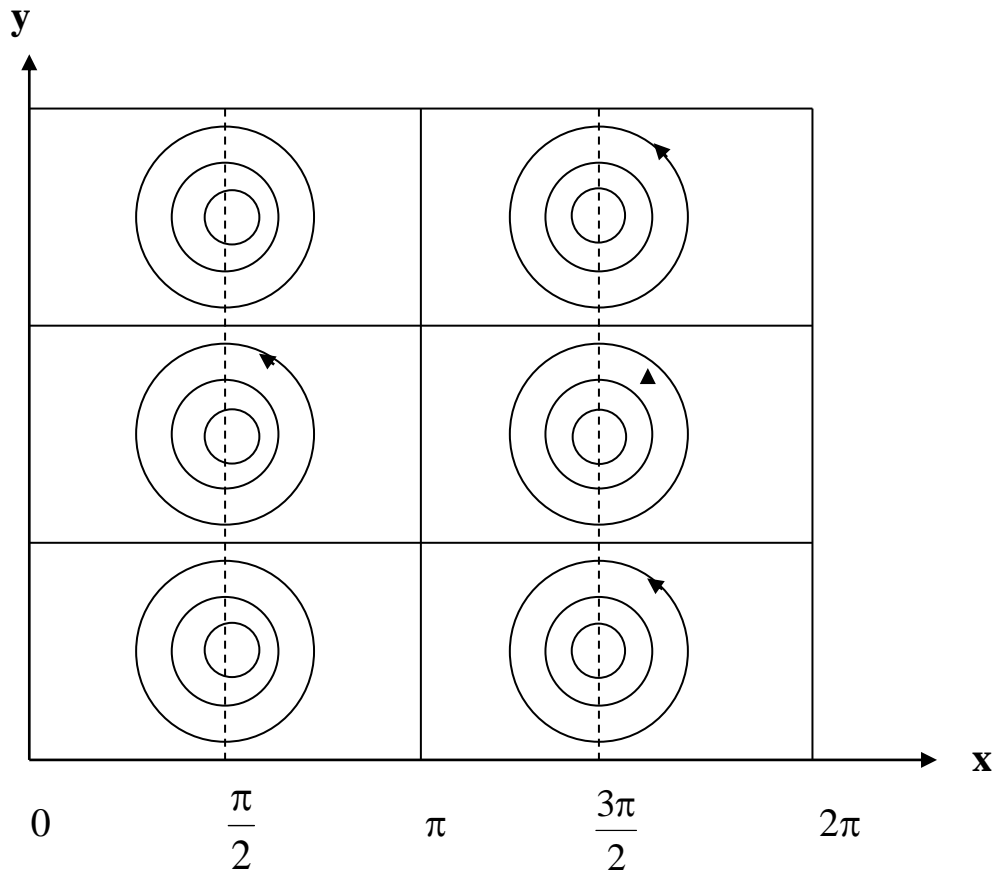

Figure (2) shows the typical streamlines

From solution (9) we obtain $\alpha_3 = 2n\pi$, which gives us:

$$a_1 = \frac{1}{3}S_1^{1/3} + 4S_2$$

$$a_2 = -\frac{1}{6}S_1^{1/3} - 2S_2 + \frac{1}{2}i\sqrt{3}\left[\frac{1}{3}S_1 - 4\frac{n^2\pi^2R^2}{S_1^{1/3}}\right]$$

$$a_3 = -\frac{1}{6}S_1^{1/3} - 2S_2 - \frac{1}{2}i\sqrt{3}\left[\frac{1}{3}S_1 - 4\frac{n^2\pi^2R^2}{S_1}\right]\dots\dots\dots(13)$$

Where

$$S_1 = -54rTK^2w + 6\left[-48n^6p^6r^6 + 81r^2T^2k^4w^2\right]^{1/2}$$

And $\quad S_2 = \dfrac{n^2p^2r^2}{S_1}$

The formulas in (13) give the values of the speed number where the real parts are different in signs so that it is unstable.

Conclusion:
* This analytic method gives a clear picture of the flow, as shown in figures (1) and (2) which shows that the cell pattern repeats itself every 2π units in the x-direction.
* From the eiginfunctions in (9) and (12), for arbitrary k and arbitrary ["n" in (9) or "n,m" in (12)] the cell pattern still appears but with additional cells .
* The growth rate of disturbance with large wave numbers k_1 and k_2 is faster than that with small wave numbers as formula (13) indicates .

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