# On ERT and MERT-Rings

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**ABSTRACT**

The main purpose of this paper is to study ERT and MERT rings, in order to study the connection between such rings and II-regular rings.

**Keyword:** MERT-Rings, ERT-Rings and π-Regular Rings

# حول الحلقات من النوع ERT وMERT

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**الملخص**

الهدف الرئيس من البحث هو دراسة الحلقات من النمط ERT وMERT لكي ندرس العلاقة بين هذه الحلقات والحلقات المنتظمة من النمط–II .

**الكلمات المفتاحية:** حلقات من النمط-MERT , حلقات من النمط-ERT , حلقات منتظمة من النمط–π

**1- Introduction:**

Throughout this paper, *R* denotes an associative ring with identity, and all modules are unitary right R-module. Recall that; 1- An ideal I of the ring *R* is essential if I has a non-zero intersection with every non-zero ideal of *R; 2*- A ring R is said to be ∏-regular if for every *a* in *R* there exist a positive integer *n* and *b* in *R* such that $a^n = a^n b a^n$ 3- A right R-module M is said to be GP- injective if, for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism *of* $a^nR$ into *M* extends to one of R into *M*. 4- For any element *a* in *R*, *r(a)*, *I (a)* denote the right annihilator of *a* and the left annihilator of a, respectively.

**2- ERT-R1NGS:**

Following [3J, a ring *R* is said to be ERT-ring if every essential right ideal of *R* is a two-sided ideal.

**Definition 2-1:**

A ring *R* is said to be right weakly regular if for all *a* in *R* , there exists *b* in *RaR* such that $a = ab$, or equivalently every right ideal of R is idempotent.

We begin this section with the following main result:

**Theorem 2.2:**

If *R* is ERT-ring with every essential right ideal is idempotent, then *R* is weakly regular.

**Proof:**

For any $a \in R$, if *RaR* not essential, then there exists an ideal I, such that $K = RaR \oplus I$ is essential then $K = K^2$.

In order to prove that *R* is weakly regular, we need to prove $RaR = (RaR)^2$.

For $a \in K$, we have $a \in K^2$ *,that is* $a \in (RaR \oplus I)^2$

Thus $a = (rar' + i)(sas' + i')$ for some *r*, r' *s*, $s' \in R$ and i, $i' \in$ I.

This implies that $a = (rar' + i)sas' + (rar' + i)\, i'$

$$= rar'sas' + isas' + (rar^1 + i)\, i'$$

but $isas' \in I \cap RaR = 0$, also we have $(rar' + i)i' \in RaR \cap I = 0$.
Therefore $a = (rar')(sas') \in (RaR)^2$ , this implies that $RaR \subseteq (RaR)^2$ Thus $RaR = (RaR)^2$, this proves that $R$ is weakly regular ring.
Following [2], the singular submodule *of R* is
$Y(R) = \{y \in R, r(y) \text{ is essential right ideal of} R\}$.

**Theorem 2.3:**
Let *R* be a semi-prime ERT right GP-injective ring. Then *R* is a right non singular.

**Proof:**
Let *E* be an essential right ideal of *R*. Then E is a two-sided ideal, and hence $l(E)$ is a two-sided ideal *ofR.*
Now $(l(E) \cap E)^2 \subseteq (E)E = 0$.
Since *R* is semi-prime, then $l(E) \cap E = 0$, whence $l(E) = 0$. This proves that *R* is right non singular.

## 3- MERT-R1NGS:

Following [3], a ring *R* is said to be MERT-ring if every maximal essential right ideal of/? is a two-sided ideal.

**Theorem 3.1:**
Let *R* be an MERT-ring, if for any maximal right ideal A/of R, and for any *b* e *M, bR/bM* is GP-injective, then *R* is strongly Pi-regular ring.

**Proof:**
Let *b* be a non-zero element in R, we claim that $b^n r + r(b'') = R$.
If $b^n r + r(b^n) \neq R$, let M be a maximal right ideal containing $b^n r + r(b^n)$. Then *M* is essential right ideal of R.

If $bR = bM$, then b = bc, for some $c$ in M, this implies $(1-c) \in r(b) \subset r(b^n) \subset M$, therefore $1 \in M$, this contradics $M \neq R$.

Now, since $R/M \cong bR/bM$. Then $R/M$ is GP-injective.

Now, define $f : b^n R \rightarrow R/M$ by $f(b^n r)=r + M$, note that f is a well-defined R-homomorphism.

Since R/M is GP-injective, then there exists $c \in R$, such that:

$1+M=f(b'')=cb''+M$ and so $(1-c\, b^n) \in M$, since $b^n \in M$, and R is MERT-ring, this implies that M is a two-sided ideal, and hence $\in c\, b'' \in M$.

Thus $I \in M$, a contradiction.

Therefore $b^n R + r(b^n) = R$.

*In particular* $l=b^n u+v; v \in r(b^n), u \in R$.

Thus $b^n = b^{2n} u$ and therefore R is strongly $\prod$-regular ring.

**Theorem 3.2:**

If $R$ is MERT-ring with every simple singular right ideal is GP-injective, then $Y(R)=0$.

**Proof:**

If $Y(R) \neq 0$, by Lemma (7) of [6], there exists $0 \neq y \in Y(R)$ with $y^2=0$. Let $L$ be a maximal right ideal of $R$, set $L = y R + r(y)$, we claim that $L$ is essential right ideal of R. Suppose this is not true, then there exists a non-zero ideal $T$ of R such that $L \cap T = (0)$. Then $yRT \subseteq LT \subseteq L \cap T = 0$ implies $T \subseteq r(y) \subseteq L$, so

$L \cap T=(0)$. This contradiction proves that $L$ is an essential right ideal, that is $R/L$ is

simple singular and hence $R/L$ is GP-injective.

Now; Let $f;yR \longrightarrow R/L$ be defined by $f(yr)=r+L$, then f is a well-defined $R$-

homomorphism.

Since $R/L$ is GP-injective, so $\exists c \in R$, such that $l+L=f(y)=cy+L$.

Hence $l+L=cy+L$, implies that $1-cy \in L$.

Since $R$ is MERT, then *eye* $L$ and thus $1 \in L$, a contradiction. Therefore $Y(R)=[0]$.

Following [1], a ring $R$ is zero insertive ( briefly ZI) if for $a, b \in R$, $ab=0$ implies $aRb=0$.

**Theorem 3.3:**

Let $R$ be a ZT ring. If every simple singular rights-modules is GP-injective which is left self-injective, then $R$ is strongly H-regular ring.

**Proof:**

Since $R$ is simple singular GP-injective, then $R$ is semi-prime, by Lemma (4)
of [5].
Thus for any left ideal I, $L(I) \cap l = 0$.
Since $R$ is simple singular GP-injective and ZI, then $R$ is reduced and hence $r(a)=l(a)$ for any element $a$ in $R$.
Thus $l(r(a)) \cap l(a)=l(l(a)) \cap l(a)=0$.
Since $R$ is left self-injective ring, then $aR$ is a right annihilator, by Proposition (4)of[4].
Since $r(a) \subseteq r(a^n)$, then $a^nR = r(a^{fl})$.
Now, since $R= r(l(r(a)))+ r(l(a))_9$ then we have $R = r(l(r(a^n)) + r(l((a^n)) = r(a^n) +a^nR$
In particular, for some $b$ in $R$, and d in $r(a^n)$.
Thus $a'' = a^{n2} b$.
Therefore $R$ is strongly $\prod$-regular.

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