# On Rings whose Simple Singular R-Modules are GP-Injective

**Zubayda M. Ibraheem**
*z.mohammed@uomosul.edu.iq*
*College of Computers Sciences and Mathematics*
*University of Mosul, Iraq*

**Received on: 25/04/2004** **Accepted on: 13/12/2004**

**ABSTRACT**

In this work we give a characterization of rings whose simple singular right R-modules are Gp-injective. We prove that if R is a quasi-duo ring whose simple singular right R-modules are Gp-injective, then any reduced right ideal of R is a direct summand. We also consider that a zero commutative ring with every simple singular left R-module is Gp-injective

**Keywords:** Gp-injective, R-modules, Quasi-duo ring, ZC-Ring

## حول الحلقات التي مقاساتها البسيطة المنفردة تكون غامرة من النمط-GP

**زبيدة محمد أبراهيم**
*كلية علوم الحاسوب والرياضيات*
*جامعة الموصل*

**تاريخ استلام البحث: 204/04/25** **تاريخ قبول البحث: 2004/12/14**

**الملخص**

في هذا البحث ندرس الحلقات التي تكون مقاساتها اليمنى البسيطة المنفردة غامرة من النمط – GP . برهنا أنه في حلقة كوازي– ديو التي تكون مقاساتها اليمنى البسيطة المنفردة غامرة من النمط–GP فأن كل مثالي أيمن مختزل يكون قابلا للجمع المباشر. كما بينا أن الحلقة شبه ألا بدالية تكون حلقة منتظمة ضعيفة مختزلة إذا كانت مقاساتها اليمنى البسيطة المنفردة غامرة من النمط– GP .

**الكلمات المفتاحية :** غامر من النمط–Gp, مقاسات, حلقات Quasi-duo, حلقات من النمط ZC

**1. Introduction:**

Throughout this paper, R denotes an associative ring with identity, and all modules are unitary right R- modules. Recall that: (1) A right R-module M is called general right principally injective (briefly right Gp-injective) if for any $0\neq a\in R$ there exists a positive integer n, such that $a^n\neq 0$ and any right R-homomorphism of $a^nR$ into M extends to one of R into M;(2) R is called reduced if R has no non-zero nilpotent elements; (3) R is right (left) quasi-duo ring if every maximal right (left) ideal of R is an ideal of R; (4) A ring R is called semi-prime if 0 is the only nilpotent ideal ; (5) for any element a in R we define a right annihilator of a by $r(a)=\{x\in R:ax=0\}$ and a left annihilator of a, l(a) is similarly defined.

**2. Rings whose simple singular modules are GP-Injective:**

In this section, we study rings whose simple singular right R-modules are Gp-injective.
We begin this section with the following result.

**Proposition 2-1:**
Let R be a qusi-duo ring, with every simple singular right R-modules is Gp-injective. Then any reduced right ideal of R is a direct summand.
**Proof:** Let I=aR be a reduced principal right ideal of R. We shall show that aR+r(a)=R. if not, there exists a maximal right ideal M of R such that $aR+r(a)\subseteq M$ .Now, M is essential right ideal of R, if not, then there exists a non-zero right ideal L of R such that $M \cap L=0$ . Then $aRL\subseteq ML \subseteq M\cap L=0$, implies that $L \subseteq r(a) \subseteq M$, so $M\cap L=L=0$, and this is a contradiction.

So M must be essential right ideal of R. Therefore R/M is Gp-injective. Then there exists a positive integer n such that any R-homomorphism of $a^nR$ into R/M extends to one of R into R/M . let $f:a^nR\rightarrow R/M$ be defined by $f(a^nr)=r+M$. f is a well-defined R-homomorphism. Indeed , let $r_1,r_2\in R$ such that $a^nr_1=a^nr_2$ . Then $a^nr_1-a^nr_2=0$, implies that $a^n(r_1-r_2)=0$, so $r_1-r_2\in r(a^n)$, since I is reduced. Therefore $r(a^n)=r(a)$ ,this implies that $r_1-r_2\in r(a)\subseteq M$. Hence, $r_1+M=r_2+M$. Now R/M is Gp-injective,so there exists $c\in R$ such that $1+M=f(a^n) =ca^n+M$. Hence , $1-ca^n\in M$, since $a^n\in M$ and R is a quasi-duo ring , then $ca^n\in M$ and so $1\in M$ . This contradicts $M\neq R$ .

Therefore aR+r(a)=R. In particular ar+c=1, for some $r\in R$ and $c\in r(a)$, whence $a^2r=a$. if we set $d=ar^2\in I$ , then $a=a^2d$. clearly $(a-ada)^2=0$, since I is reduced, thus a =ada , and hence I=eR, where e=ad is an idempotent element. Thus I is a direct summand.

**Proposition 2-2:**

Let R be a semi-prime ring with every simple singular right R-module is Gp-injective . Then every right ideal of R is an idempotent .

**Proof:**For any right ideal I of R, suppose there exists an element b in I , such that $b \notin I^2$. Then $bR \neq (bR)^2$. Since R is a semi-prime ring, then $(bR)^2$ is essential in bR. By zorn's lemma , the set of right ideals J such that $(bR)^2 \subseteq J \subset bR$ has a maximal member L. Then bR/L is a simple singular, and therefore is Gp-injective. Now, let $f:bR \to bR/L$ is the canonical homomorphism defined by f(br)=br+L for all ring R , since bR/L is Gp-injective, so there exists $c \in R$, such that f(br)=(bc+L)br. Then f(b)=(bc+L)b=b+L , which implies that b+L=bcb+L. Hence; $b-bcb \in L$, whence it follows that $b \in L$. Thus $bR \subset L$ and this is a contradiction. Therefore $I=I^2$ .

**3-Zero Commutative Rings**

In this section we introduce the notion of a zero commutative ring in order to study the connection between rings whose simple singular right R-modules are Gp-injective and other rings.

**Definition3-1:**

A ring R is called zero commutative (briefly ZC)if for $a,b \in R$ , ab=0 if ba=0.

We shall begin this section with the following result.

**Lemma 3-2:**

Let R be a ZC ring. Then RaR+l(a) is an essential left ideal of R.

**Proof:**Given $a \in R$, assume that $[RaR+l(a)] \cap I=0$ , where I is a right ideal of R. Then $aI \subseteq I \cap RaR=0$, so $I \subseteq r(a) \subseteq l(a)$. Hence, I=0; where RaR+l(a)is an essential left ideal of R

**Lemma 3-3:**

Let R be a ZC ring with every simple singular left R-module is Gp-injective, then R is reduced.

**Proof:**Let $a^2=0$ . suppose that $a \neq 0$. By lemma (3-2), l(a)is an essential left ideal of R. since $a \neq 0$, $l(a) \neq R$. Thus, there exists a maximal essential left ideal M of R containing L(a), therefore R/M is Gp-injective. So any R-homomorphism of Ra intoR/M extends to one of R into R/M. Let $f:Ra \to R/M$ be defined by f(ra)=r+M. Clearly, f is a well-defined R-homomorphism . Thus 1+M=f(a)=ac+M. Hence, 1-ac∈M and so 1∈M, which is a contradiction. Hence a=0, and so R is reduced.

**Definition3-4:**

A ring R is said to be right weakly regular if for all a in R , there exists b in RaR such that a=ab .

Now, we give the main result.

**Proposition 3-5:**

If R is ZC and every simple singular left R-module is Gp-injective, then R is a reduced weakly regular ring.

**Proof:**By Lemma (3-3) , R is a reduced ring. We shall show that RaR+l(a)=R for any a∈R. Suppose that there exists b∈R such that RbR+l(b)≠R . Then there exists a maximal left ideal M of R containing RbR+l(b). By Lemma (3-2), M must be essential in R. Therefore R/M is Gp-injective. So there exists a positive integer n such that any R-homomrphism of $Rb^n$ into R/M extends to one of R into R/M. let $f:Rb^n \to R/M$ be defined by $f(rb^n)=r+M$ . Since R is a reduced ring, f is a well- R-homomorphism. Now, R/M is Gp-injective , so there exists c∈R such that $1+M=f(b^n)=b^nc+M$ . Hence $1-b^nc \in M$ and so 1∈M, which is a contradiction. Therefore RaR+l(a)=R for any a∈R. Hence R is a left weakly regular ring. Since R is reduced, then RaR+r(a)=R, implies that R is a right weakly regular ring. Therefore R is a weakly regular ring.

**Kimand Nam in [2] proved that.** Rings whose simple right R-modules are Gp-injective are always semi-prime. But in general rings whose simple singular right R-modules are Gp-injective need not be semi-prime.

**Proposition 3-6:**

Let R be a ZC ring, and every simple singular left R-module is Gp-injective, then R is a semi-prime ring .

**Proof:** From Lemma (3-3), R is a reduced ring and then R is a semi-prime ring

## *REFERENCES*

[1] Kim N.K., Nam S.B. and Kim J.Y. (1999), "**On simple Singular Gp-injective modules**", *Comm. In Algebra, 27(5)*, 2087-2096.

[2] N.K.kim, Nam S.B. and Kim J.Y. (1995), "**On simple Gp-injective modules**", *Comm. In Algabera, 23(14),*5437-5444.

[3] SANG Bok NAM (1999), "**A Note on Simple singular Gp-injective Modules**" *kangweon-kyungki Math. Jour.7, No.2*, pp.215-218.

[4] Ming R.Y.C. (1986) "**On Semi-Prime and reduced ring**" *Riv. Mat.Univ.Parma (4) 12*,167-175..

[5] Ming R.Y.C. (1994) "**A not on regular rings,II**", *Bull.Math. Soc. Sc.38*.

[6] Ramamurthi V.S. (1973) "**Weakly regular rings**", *Canad. Math.Bull.16*,317-321.