On Generalized Simple P-injective Rings

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Received on: 20/06/2003 **Accepted on: 09/09/2003**

ABSTRACT

In this work, we study generalized simple P-injective rings. New properties of such rings are given, and a characterization of division rings and strongly regular rings in terms of GSP rings is obtained.

Keywords: simple p_ injective ,strongly regular, division rings

حول الحلقات البسيطة المعممة الغامرة من النمط P

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تاريخ الاستلام: 2003/06/20 **تاريخ القبول: 2003/09/09**

الملخص

في هذا البحث, درسنا الحلقات البسيطة الغامرة من النمط–P المعممة.وحصلنا على نتائج وصفات جديدة لمثل هذه الحلقات. وأخيرا" أعطينا تميزا" للحلقات المقسومة والحلقات المنتظمة القوية بدلالة الحلقات ضمن النمط GSP .

الكلمات المفتاحية: الحلقات البسيطة الغامرة من النمط p، المنتظمة، حلقات القسمة.

1. Introduction:

Throughout this paper, R denotes an associative ring with identity, and all modules are unitary right R-modules.

A right R-module M is said to be P-injective if, for any principal right ideal P of R, any right R-homomorphism $f: P \rightarrow M$, there exists y in M such that f(b)=yb for all $b \in P$. This concept was introduced by Ming [3].

Recall that: (1) R is called strongly regular if for every $a \in R$ there exists $b \in R$ such that $a=a^2b$, (2) a ring R is said to be zero commutative (briefly ZC) if for a, $b \in R$, ab=0 implies that ba=0 [2]; (3) For any element a in R, r (a) and $\ell(a)$ denote the right and left annihilator of a respectively; (4) A ring R is called reduced if, R contains no non-zero nilpotent element, (5) R is said to be right uniform, if every right ideal of R is essential [4], (6) J(R) will state for the Jacobson radical.

2. GSP-Rings:

<u>Definition 2.1:</u> A ring R is called a GSP-ring (generalized simple P-injective) if, for any maximal right ideal *M* of R, any $b \in M$, bR/bM is p-injective.

Recall the following lemma which are due to Skin [5].

<u>Lemma 2.2:</u> For any ring R, the following statements are equivalent:

(1) R is a ZC ring.

(2) For each $a \in R$, r(a) (equivalently $\ell(a)$) is a two-sided ideal of R

We shall begin this section with the following results.

<u>Theorem 2.3:</u> Let R be a ZC, GSP-ring. Then:

(1) J(R)={0}

(2) R is a reduced ring.

<u>Proof (1).</u> Let $a \in J(R)$, if $aR+r(a) \neq R$. Then there exists a maximal right ideal M containing aR+r(a). Suppose that aR=aM, then a=ac for some c in M and this implies that a(1-c)=0, so $1-c \in r(a) \subseteq M$, whence $1 \in M$ contradicting $M \neq R$. If $aR \neq aM$, the right R-homomorphism $g: R/M \rightarrow aR/aM$ defined by g(b+M)=ab+aM, for all b in R implies that $R/M \cong aR/aM$. Define $f: aR \rightarrow R/M$ as a right R- homomorphism by f(ax)=x+M, for all x in R, then f is a well-defined right R-homomorphism. Indeed; let x_1, x_2 be any two elements in R with $ax_1=ax_2$, implies that (x_1-x_2)

$\in$r(a) $\subseteq$M, thus x_1+M=x_2+M. Hence, f(ax_1)=x_1+M=x_2+M=f(ax_2). Since R/M is P-injective, then there exists an element c in R such that f(ac)=(c+M) ax =cax+M, yields 1+M=f(a)=da+M, for some d in R, whence 1$\in$M, again there is a contradiction.

Therefore aR+r(a)=R. In particular ay+d=1 fore some y$\in$R, d$\in$r(a), thus we have a^2y=a. Now, since a$\in$J (R), then there exists an invertible element u in R such that (1-ay)u=1 and this implies that (a-a^2y)u=a, whence a=0. Therefore J (R)=0.

Next, we consider the connection between GSP-ring and division rings.

Proof (2). Let a be a non-zero element of R such that a^2=0. Then there exists a maximal right ideal M of R containing r(a). If aR=aM, then a=ac for some c in M, which implies that 1-c $\in$r(a)$\subseteq$M, whence 1$\in$M, contradicting M $\neq$ R. If aR $\neq$aM the right R-homomorphism g: R/M$\rightarrow$aR/aM defined by g(r+M)=ar+aM for all r$\in$R implies that R/M$\cong$ aR/aM. Then R/M is P-injective. Consider the canonical mapping $f : aR \rightarrow R/M$, then there exists an element b$\in$R such that 1+M=f(a)=ba+M. Now r(a) is a two sided ideal of R, so ba$\in$r(a) $\subseteq$M, whence 1$\in$M, a contradiction. Therefore a=0, whence R is reduced ring.

Definition 2.4: Let R be a ring such that every maximal right ideal is a two sided ideal. Then R is called a quasi-duo ring.

Theorem 2.5: Let R be a quasi-duo, GSP-ring. Then any non-zero divisor of R is a right and left invertible.

Proof. Let y be a non-zero element of R. If yR$\neq$R, let M be a maximal right ideal of R containing yR. Suppose that yR=yM. Then y=yc for some c in M. which implies that 1-c$\in$r(y), whence 1=c$\in$M, contradicting M$\neq$R. now, if yR$\neq$yM the right R-homomorphism g:R/M$\rightarrow$yR/yM defined by g(r+M)=yr+yM, for all r in R implies that R/M$\cong$ yR/yM. Since yR/yM is P-injective then R/M is P-injective. Consider the canonical mapping f:yR$\rightarrow$R/M, then there exists an element b in R such that f(y)=1+M=by+M. Hence 1-by$\in$M, since y$\in$Mand R is a quasi-duo ring, implies that by$\in$M. thus 1$\in$M, again there is contradiction. Therefore yR=R, in particular yr=1, for some r$\in$R, and then we have yry=y, implies that ry$\in$r(y)=0, thus ry=1. This proves that y is a right and left invertible.

Corollary 2.6: Let R be a quasi-duo, GSP-ring without zero divisors. Then R is a division ring.

The next result considers other conditions for GSP-ring to be a division ring.

Theorem 2.7: Let R be a ZC, right uniform GSP-ring. Then R is a division ring.

Proof. Let $0 \neq y \in R$, if r(1-y)=0, then $\ell(1-y)=0$ (R is ZI). Then by Theorem (2.5) 1-y is an invertible element and hence $y \in J(R)$, so by Theorem (2.3) y=0, a contradiction. Therefore $r(1-y) \neq 0$, let $0 \neq x \in r(1-y)$, then x=yx. We claim that $xR \cap r(y)=0$, if not, let $z \in xR \cap r(y)$, then z=xr for some $r \in R$ and yz=0, this implies yxr=0, yields xr=0, whence z=0. Therefore $xR \cap r(y)=0$. Since R is a right uniform ring, and $xR \neq 0$ then r(y) must be zero. Since R is ZC, then $\ell(y)=0$. Then by Corollary (2.6) R is a division ring.

Lemma 2.8: If R is a reduced ring, then R/r(a) is a reduced ring.

Proof. See [1]

Theorem 2.9: Let R be a reduced ring, such that for every $a \in R$, R/r(a) is GSP-ring. Then R is strongly regular.

Proof. Let a be a non-zero element in R, and let d=a+ r(a) $\in R/r(a)$. Clearly, $d \neq 0$ because otherwise if d=0, then a+r(a)=r(a), and this yields $a^2=0$, gives a=0 (since R is reduced). Let $d\bar{x}=0$, we shall prove that $\bar{x}=0$. Observe that (a+r(a))(x+r(a))=r(a), implies that ax+r(a)=r(a) and hence $ax \in r(a)$, so $a^2x=0$. Thus $x \in r(a^2)$ (since R is reduced). Therefore, $x \in r(a)$ implies that ax=0. Hence $\bar{x}=0$, this means that d is a right non-zero divisor. In a similar way we can prove that d is a left non-zero divisor. Since R/r(a) is GSP-ring, then by Theorem (2.5) d is an invertible element. then there exists

$0 \neq \bar{y} = y + r(a) \in R/r(a)$ such that $d\bar{y}=1$, then (a+r(a))(y+r(a))=1+r(a), so $ay-1 \in r(a)$ and a(ay-1)=0. Thus $a^2y=a$. this proves that R is a strongly regular ring.

Theorem 2.10: Let R be a GSP-ring, and let I be a reduced right ideal of R. Then I is a strongly regular ring.

Proof. Since I is reduced, for any $b \in I$, $L(b) \subseteq r(b)$. If $bR+r(b) \neq R$, let M be a maximal right ideal containing bR+r(b). If bR=bM, then b=bc for some $c \in M$ which implies that $1-c \in r(b) \subseteq M$, whence $1 \in M$, contradicting $M \neq R$. If $bM \neq bR$, the right R- homomorphism $g: R/M \to bR/bM$ defined by g(a+M)=ba+bM for all $a \in R$ implies that $R/M \cong bR/bM$, since bR/bM is P-injective, then R/M is P-injective. Consider the canonical mapping $f : bR \to R/M$, then there exists an element a in R such that f(b)=1+M=ab+M, whence $1 \in M$, again there is a contradiction. Thus bR+r(b)=R for any $b \in I$ from which $b=b^2u$, $u \in R$.

Now $b=bbu=b(b^2u)u=b^2v$, where $v=bu^2 \in I$, and $(b-bvb)^2=0$ implies that b=bvb which proves that I is a strongly regular ring.

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