Stability of Liquid Film with Negligible Viscosity

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ABSTRACT

In this paper, we consider the stability analysis for a disturbed unsteady flow, which is two-dimensional incompressible flow in a symmetric film where the effect of viscosity can be neglected in comparison with inertia forces. The partial differential equations governing such flow are obtained from the Navier - Stokes equations and we obtain an analytic solution for those equations. The whole system is disturbed and we found the regions where the flow is stable or unstable.

Keywords: stability analysis, unsteady flow, incompressible flow, viscosity, inertia forces, partial differential equations, Navier - Stokes equations.

استقرارية الغشاء السائل ذو اللزوجة المهملة

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الملخص

تم في هذا البحث دراسة وتحليل الاستقرارية للجريان الزمني وغير القابل للانضغاط في نظام ثنائي البعد في غشاء متناظر وبإهمال قوى اللزوجة مقارنة بقوى القصور الذاتي. وقد تم الحصول على المعادلات التفاضلية الجزئية التي تحكم هذا النوع من الجريان وبالاعتماد على معادلات نافير–ستوكس. وقد استخدمت الطرائق التحليلية في ايجاد حل هذه المعادلات. وقد أزعج النظام بأكمله وحددت المناطق التي يكون فيها النظام مستقرا او غير مستقر.

الكلمات المفتاحية: تحليل الاستقرارية، الجريان الزمني، الجريان غير القابل للانضغاط، قوى اللزوجة، قوى القصور الذاتي، المعادلات التفاضلية الجزئية، معادلات نافير–ستوكس.

1- Introduction:

The dynamics of thin liquids films has been studied by G. I. Taylor (1959). The subject is of considerable scientific and technological importance.

Brown (1961) studied experimentally the general behaviour of a thin sheet of moving liquid, he found that this measured velocity distribution in the curtain compared with the prediction based on a non linear differential equation was attributed to Taylor (1959). He observed that the film is with disintegrate if the follow rate is reduced to a film minimum value. He also discussed film stability on the basis of a simple momentum balance applied to a stationary free edge resulting from the film breaking.

The principle of stability of a viscous liquid film has been investigated by S. P. Lin (1981). It is shown to be stable with respect to temporally and spatially changing varicose disturbances.

Abdulahad (1994) determines the thickness of a liquid film with negligible inertia and also he studied the similarity solution for unsteady flow for such liquid films.

Mosa (2002) considered the stability analysis for fluid flow between two infinite parallel plates.

In this paper we consider the stability analysis of a viscous liquid film when the viscous forces are very small compared with the inertia forces.

2. Stability equation:

The stability of the described basic flow with respect to two-dimensional disturbances are to be investigated.

The general form of the Navier-Stokes equation is defined by:

$$\rho\frac{DU_i}{Dt} = -\frac{\partial P}{\partial X_i} + \mu\nabla^2 U_i \qquad \ldots\ldots(1)$$

When the viscosity is very small, the Navier-Stokes equation reduces to:

$$\rho\frac{DU_i}{Dt} = -\frac{\partial P}{\partial X_i} \qquad \ldots\ldots(2)$$

Substituting the perturbed flow quantities
($U = \overline{U} + U,\ V = \overline{V} + V$) in to (2) and neglecting some of terms which have no perturbation, we have:

$$\left.\begin{aligned}
&\frac{\partial U}{\partial t} + (\overline{U} + U)\frac{\partial U}{\partial X} + U\frac{\partial \overline{U}}{\partial X} + (\overline{V} + V)\frac{\partial U}{\partial Y} + V\frac{\partial \overline{U}}{\partial Y} = -\frac{1}{\rho}\frac{\partial P}{\partial X}\\
&\frac{\partial V}{\partial t} + (\overline{U} + U)\frac{\partial V}{\partial X} + U\frac{\partial \overline{V}}{\partial X} + (\overline{V} + V)\frac{\partial V}{\partial Y} + V\frac{\partial \overline{V}}{\partial Y} = -\frac{1}{\rho}\frac{\partial P}{\partial X}\\
&\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0
\end{aligned}\right\} \quad \ldots(3)$$

Now, we introduce the dimensionless variables as follows:

$X = d_0 x$, $U = \bar{u}_o u$, $\bar{u}_o = Q/d_o$, $t = \dfrac{d_o}{u_o}t'$

Y = d$_o$y, $V = \bar{u}_o v$, $\bar{u}_o = Q/d_o$, $P = \rho\bar{u}_o^2$

Now equation (3) has a dimensionless form, which are:

$$\left.\begin{aligned} &u_t + (\bar{u}+u)u_x + u\bar{u}_x + (\bar{v}+v)u_y + v\bar{u}_y = -P_x \\ &v_t + (\bar{u}+u)v_x + u\bar{v}_x + (\bar{v}+v)v_y + v\bar{v}_y = -P_y \\ &u_x + v_y = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots..(4)$$

Where all subscripts denote partial differentials, (x,y) are the Cartesian coordinates in the unit of the film thickness d$_o$, $(\bar{u},\bar{v})$ and (u,v) are respectively the (x,y) components of the primary flow velocity and the velocity perturbations in the unit of $\bar{u}_o$, Q being the volumetric flow rate per unit width of the film and t is the time.

It is easily verified that $\bar{u}_x = (R/4F^2)^{1/3} U_x$

Where $R = \rho\bar{u}_o d_o/\mu \equiv$ Reynolds number

and $F = \bar{u}_o^2/gd_o \equiv$ Froude number

$U_x = O(1)$ Thus $\bar{u}_x = \delta U_x = -\bar{v}_y < 1$ if

$(R/4F^2)^{1/3} = (g^2/4\nu)^{1/3}(d_o^2/Q) = \delta < 1,\ \nu = \dfrac{\mu}{\rho}$

For the case of thin films such that δ<1, we define a slow variable ξ, such that ξ = δx

by use of the above relation and neglecting terms of o(δ) as well as the non-linear terms in perturbations, we reduce the first two equations in (4) to the forms:

$$\left.\begin{aligned} &u_t + \bar{u}(\xi)u_x = -P_x \ \text{(a)} \\ &v_t + \bar{u}(\xi)v_x = -P_y \ \text{(b)} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(5)$$

By elimination of the pressure terms by differentiate equation (a) for y and equation (b) for x, we get:

$$\frac{\partial^2 u}{\partial y \partial t} + \bar{u}\frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 v}{\partial x \partial t} - \bar{u}\frac{\partial^2 v}{\partial x^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(6)$$

or

$$[\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}](\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}) = 0$$

or

$$[\partial_t + \bar{u}\partial_x](u_y - v_x) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(7)$$

Equation (7) is satisfied by the stream function ψ related to the velocity perturbations by

$u = \Psi y$, $v = - \Psi x$ (8)

By substituting (8) in to (7), we have

$$[\partial_t + \bar{u}\partial_x](\partial_{xx} - \partial_{yy})\psi = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(9)$$

Equation (9) is the governing differential equation of the linear stability problem under consideration.

Let the free surfaces of the basic flow and the perturbed flow be:

$$y = \pm\frac{h}{2}(x) \text{ and } y = \pm\frac{h}{2}(x) + \eta(x,t) = \zeta(x,t)$$

Following Lin (1981), the boundary conditions are as follows:

1- The kinematic condition at the free surface $y = \xi$ requires that

$$v = \zeta_t + (\bar{u} + u)\zeta_x$$

2- The dynamic condition of the free surface, which is massless by definition, demands that the net force be zero at the free surface. Demanding the vanishing of the force per unit area of the free surface in the x and y directions, we have

$$[-p + \frac{2}{R}(\bar{u}+u)_x]\zeta_x - [(\bar{u}+u)_y + (\bar{v}+v)_x]/R \mp WK\zeta_x = 0\ ,$$

$$[-p + \frac{2}{R}(\bar{v}+v)_y] - [(\bar{u}+u)_y + (\bar{v}+v)_x]\zeta_x / R \mp WK = 0$$

, are respectively

Where K is the total surface curvature and W is the Weber number,

$$K = \frac{(\pm\frac{1}{2}h + \eta)_{xx}}{[1 + (\pm\frac{1}{2}h + \eta)_x^2]^{3/2}},\quad W = \frac{T}{\rho\bar{u}_o^2 d_o}$$

T is the surface tension.

Note that $h_x = O(\delta)$. Since $Q = (\bar{u}_o \bar{u})\,(d_o h)$ = constant and $\bar{u}_x = O(\delta)$.

Neglecting terms of $O(\delta)$, balancing out purely primary flow quantities, and expanding the remaining primary flow quantities in Taylor's series about $y = \pm\frac{1}{2}h$, and then retaining only linear terms, we reduce the above boundary conditions at $y = \zeta$ to the following to be applied at

$y = \pm\frac{1}{2}h$:

$$\eta_t + \bar{u}\eta_x + \psi_x = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(10)$$

$$\psi_{yy} - \psi_{xx} = 0 \qquad \text{……………………………………………(11)}$$

$$\pm W\eta_{xx} + p + 2\psi_{xx} / R = 0 \qquad \text{……………………………(12)}$$

where p can be obtained from (5) in terms of ψ .

Now, Substituting (8) into (6), we have

$$\frac{\partial^3 \Psi}{\partial t \partial y^2} + \bar{u}\frac{\partial^3 \Psi}{\partial x \partial y^2} + \frac{\partial^3 \Psi}{\partial t \partial x^2} + \bar{u}\frac{\partial^3 \Psi}{\partial x^3} = 0 \qquad \text{………………..(13)}$$

We consider for our solution a normal mode of travelling disturbances:

$\Psi = \phi(y)e^{i\alpha(x-ct)}$ ……………………………….………………(14)

Where $\alpha = 2\pi d_o/\lambda$, λ is the wave length, and c is the wave speed in the unit of $\bar{u}_0$.

Substituting (14) into (13), we get:

$$[i\alpha(c - \bar{u})](d^2 - \alpha^2)\phi = 0 \,, \quad \text{Where} \quad d^2 = \frac{d^2}{dy^2}$$

or

$$\frac{d^2\phi}{dy^2} - \alpha^2 \phi(y) = 0$$

The general solution of this equation is:

$$\phi(y) = A\sinh(\alpha y) + B\cosh(\alpha y) \text{ ……………………………..(15)}$$

Where A and B are integration constants.

Since the governing differential system is linear and homogeneous, we may consider the odd solution for ϕ separately.

3. Varicose waves:

The odd solution for ϕ corresponds to the anti-symmetric disturbance, which displaces each of the free surfaces in opposite directions. Now we take the odd solution from (15), which is:

$$\phi(y) = A\sinh(\alpha y) \quad (16)$$

Substituting (16) into (14), we get:

$$\psi = [A\sinh(\alpha y)]e^{i\alpha(x-ct)} \quad (17)$$

Substituting (17) into (5) and (8) and solve for p yields:

$$p = \alpha c' A\cosh(\alpha y)e^{i\alpha(x-ct)} \qquad \text{…………………………….(18)}$$

where $c' = c - \bar{u}$, $\quad y = \pm\frac{1}{2}h$

Equation (10) has a solution of the form:

$$\eta = \pm c'^{-1}[A\sinh(\frac{1}{2}\alpha h)]e^{i\alpha(x-ct)} \text{ ……………………………..(19)}$$

Substituting (17) into (11) and (17), (18), (19) into (12), yields:

$$A[2\alpha^2 \sinh(\frac{1}{2}\alpha h)] = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20)$$

and

$$A[c' \cosh(\frac{1}{2}\alpha h) - Wc'^{-1}\alpha \sinh(\frac{1}{2}\alpha h) + 2R^{-1} i\alpha \cosh(\frac{1}{2}\alpha h)] = 0 \; \ldots..(21)$$

Equation (20) give a trivial solution and so we neglect it.
From equation (21), A ≠0 and therefore:

$$c' \cosh(\frac{1}{2}\alpha h) - Wc'^{-1}\alpha \sinh(\frac{1}{2}\alpha h) + 2R^{-1} i\alpha \cosh(\frac{1}{2}\alpha h) = 0$$

or

$$c'^2 + (2i\alpha R^{-1})c' - W\alpha \tanh(\frac{1}{2}\alpha h) = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(22)$$

For both temporally and spatially growing disturbance of long wave lengths, $\alpha \to 0$ near the neutral stability curve and the secular equation (22) can be expanded in powers of α as

$$c'^2 + (2i\alpha R^{-1})c' - W\alpha[\frac{1}{2}\alpha h - \frac{1}{24}\alpha^3 h^3)] = 0$$

or

$$(c - \bar{u})^2 + 2i\alpha R^{-1}(c - \bar{u}) - \frac{1}{2} Wh\alpha^2 + O(\alpha^3) = 0 \qquad \ldots\ldots\ldots..(23)$$

For the temporal case α is real and c is complex. The solution of (23) for c gives.

$$c = \bar{u} - i\alpha / R \mp \alpha[\frac{1}{2} Wh - (1/R)^2]^{1/2}$$

if $\quad [\frac{1}{2} Wh - (1/R)^2] > 0, \quad$ Then

$$c_I = -\alpha / R \quad ,$$

$$c_R = \bar{u} \mp \alpha[\frac{1}{2} Wh - (1/R)^2]^{1/2}$$

But if $[\frac{1}{2} Wh - (1/R)^2] < 0, \quad$ Then

$$c_I = -\alpha / R \mp \alpha[\frac{1}{2}(1/R)^2 - \frac{1}{2} Wh]^{1/2} \quad ,$$

$$c_R = \bar{u}$$

It follows that $c_I < 0$ regardless of whether the wave speed relative to the fluid particle is zero or not. Therefore, the temporally changing varicose disturbances are damped with a dimensional damping rate given by:

$$\frac{1}{R}\alpha^2\frac{\bar{u}_o}{d_o}=\frac{4\pi^2 v}{\lambda^2}$$

To investigate the spatially growing disturbances of long wave length, we multiply (23) by α^2 and identify αc with ω to have:

$$\alpha^2(c-\bar{u})^2+2i\alpha^3R^{-1}(c-\bar{u})-\frac{1}{2}Wh\alpha^4+O(\alpha^5)=0$$

or

$$(\omega-\alpha\bar{u})^2+2i\alpha^2R^{-1}(\omega-\alpha\bar{u})-\frac{1}{2}Wh\alpha^4+O(\alpha^5)=0 \qquad \ldots(24)$$

The solution of equation (24) in power of small ω gives the following complex wave number

$$\alpha=\frac{\omega}{\bar{u}}+\frac{2i}{R}\frac{\omega^2}{\bar{u}^3}+O(\omega^3)\ , \qquad \text{Where}$$

$$\alpha_R=\frac{\omega}{\bar{u}}=\frac{\alpha c}{\bar{u}}\ ,$$

$$\alpha_I=\frac{2\omega^2}{R\bar{u}^3}=\frac{2\alpha^2}{R\bar{u}}>0$$

Thus the spatially varying disturbance are also damped travelling waves.

4- Conclusion:

We consider both cases of temporally and spatially growing disturbances. For the formal case α is real but $c = c_R+ic_I$ is complex. For the latter case $\alpha=\alpha_R+i\alpha_I$ is the complex wave number but $\alpha c = \omega$ is the real wave frequency. Thus, temporally changing disturbances are stable or unstable depending on if $c_I < 0$ or $c_I > 0$, and spatially changing disturbances are stable or unstable depending on if $\alpha_I > 0$ or $\alpha_I < 0$.

We have two notes here that in this paper we neglect the effect of viscosity and we obtain a second order non-linear differential equation, while if we take the effect of viscosity, we obtain the fourth order non-linear differential equation as it is given by Lin (1981). We note here that there is some differences between the above two cases.

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