

## On the Generalized Curvature

Tahir H. Ismail

tahir\_hsis@uomosul.edu.iq

College of Computer Sciences and Mathematics

University of Mosul

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Ibrahim O. Hamad

ibrahim.hamad@su.edu.krd

College of Sciences

University of Salahaddin

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### ABSTRACT

By using methods of nonstandard analysis given by **Robinson, A.**, and axiomatized by **Nelson, E.**, we try in this paper to establish the generalized curvature of a plane curve  $\gamma(t)$  at regular points and at points infinitely close to a singular point. It is known that the radius of **curvature** of a plane curve  $\gamma(t)$  is the limit of the radius of a circle circumscribed to a triangle  $ABC$ , where  $B$  and  $C$  are points of  $\gamma$  infinitely close to  $A$ . Our goal is to give a nonstandard proof of this fact. More precisely, if  $A$  is a standard point of a standard curve  $\gamma$  and  $B, C$  are points of  $\gamma$  defined by  $B = \gamma(t + \alpha)$  and  $C = \gamma(t + \beta)$  where  $\alpha$  and  $\beta$  are infinitesimals, we intend to calculate the quantity  $\frac{\tan \hat{A}}{\|BC\|}$  in the cases where  $A$  is **biregular**,

**regular, singular or singular** of order  $p$ .

**Keywords:** infinitesimals, curvature, torsion, singularity.

### حول التقعر المعمم

إبراهيم عثمان حمد

كلية العلوم

جامعة صلاح الدين

ظاهر حسن إسماعيل

كلية علوم الحاسوب والرياضيات

جامعة الموصل

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### الملخص

باستخدام بعض مفاهيم التحليل غير القياسي الذي أوجده **Robinson, A.** ووضعها **Nelson, E.** بأسلوب منطقي نحاول في هذا البحث الحصول على تقوس معمم للمنحني المستوي  $\gamma(t)$  عند نقاط نظامية وكذلك عند نقاط متناهية القرب من نقطة شاذة. المعروف أن نصف قطر التقوس للمنحني المستوي  $\gamma(t)$  هو غاية نصف قطر الدائرة التي تحيط المثلث  $ABC$  عندما يكون  $B$  و  $C$  نقطتين للمنحني  $\gamma$  متناهيتين القرب من النقطة  $A$ . هدفنا هو إعطاء برهان غير قياسي لهذه الحقيقة. بشكل أدق، إذا كانت  $A$  نقطة قياسية على المنحني القياسي  $\gamma$  و  $B, C$  نقطتين على المنحني معرفتين على النحو الآتي  $B = \gamma(t + \alpha)$  و  $C = \gamma(t + \beta)$ ، حيث

$\alpha$  و  $\beta$  كميات متناهية الصغر نريد حساب الكمية  $\frac{\tan \hat{A}}{\|BC\|}$  في حالة كون النقطة A نقطة نظامية

ثنائية أو نظامية أو شاذة أو شاذة من الرتبة  $p$ .

الكلمات المفتاحية: ما لانهاية من الصغر، انحناء، التواء، شاذ.

## 1. Introduction.

The radius of curvature is the limit of the radius of a circle circumscribed to triangle ABC, where B and C are points of  $\gamma$  infinitely close to A [8] [10]. There are many applications of curvature in geometrical study of curves [5], [10] and [26], technology of computer vision and computer graphics [19] and [24], dielectric surfaces [1], chemical interatomic surfaces [17], biological membranes [12], double babbles [18], and other applications.

This study is different from the classical studies of curvature [2], [15] and [19] which deals with the properties of curves in a plane or on a sphere [17], [22]. The following nonstandard definitions are needed throughout this paper.

Every set or elements defined in the classical mathematics are called standard. [11]

Any set or formula, which does not involve new predicates such as “standard, infinitesimals, limited, unlimited...etc”, is called internal, otherwise it is called external. [14]

A real number  $x$  is called unlimited if and only if  $|x| > r$  for all positive standard real numbers  $r$ , otherwise it is called limited. [4], [3]

The set of all unlimited real numbers is denoted by  $\bar{\mathbf{R}}$ , and the set of all limited real numbers is denoted by  $\mathbf{R}$  [23]

A real number  $x$  is called infinitesimal if and only if  $|x| < r$  for all positive standard real numbers  $r$ . [9]

A real number  $x$  is called appreciable if it is neither unlimited nor infinitesimal, and the set of all positive appreciable numbers is denoted by  $A^+$ . [25]

Two real numbers  $x$  and  $y$  are said to be infinitely close if and only if  $x - y$  is infinitesimal and denoted by  $x \cong y$ . [20]

If  $x$  is a limited number in  $\mathbf{R}$ , then it is infinitely close to a unique standard real number, this unique number is called the standard part or shadow of  $x$  and it is denoted by  $st(x)$  or  ${}^0x$ . [21]

If  $f$  is a real valued function then: -

- 1-  $f$  is called **continuous** at  $x_0$  if  $f$  and  $x_0$  are standard and if  $x \cong x_0$ , which implies that  $f(x) \cong f(x_0)$  for all  $x$ .
- 2-  $f$  is called **s-continuous** at  $x_0$  if  $x \cong x_0$  which implies that  $f(x) \cong f(x_0)$  for all  $x$ .

## 2.The Curvature at a Point Near a Study Point

Today nonstandard analysis rather than it is a branch of mathematics it is a tool used to survey other branches of mathematics and sciences [6], [7], [9], [16].

Meziani [13], Velemorvic [26] and others, studied the curvature via nonstandard analysis. In this paper, the nonstandard tools will be used to present new and more developed forms of curvature.

Let  $Q$  and  $R$  be two points near to the point  $P$  on a curve  $\gamma(t)$  in a plane as shown in **Fig. 2.1**. Consider the triangle  $\Delta QPR$  such that points  $Q$  and  $R$  approach  $P$ , then the triangle  $\Delta QPR$  is reduced to the triangle  $\Delta Q^*PR^*$ . By continuing the approach process of  $Q^*$  and  $R^*$  to  $P$ , we get the infinitesimal triangle  $\Delta Q^*PR^*$  in which  $Q^*$  and  $R^*$  become infinitely close to  $P$  on  $\gamma(t)$ .

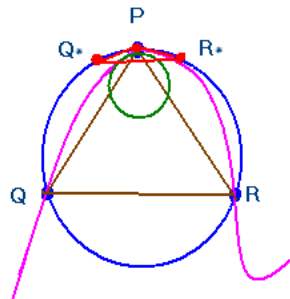


Figure 2.1

### Lemma 2.1

For some  $t_0 \in \mathbf{R}$  let  $P = \gamma(t_0)$  be a point on the curve  $\gamma$ , and let  $Q^*$  and  $R^*$  be two points described before, such that

$Q^* = \gamma(t_0 + \alpha)$  and  $R^* = \gamma(t_0 + \beta)$ , where  $\alpha \cong 0$ ,  $\beta \cong 0$ . Then

1.  $Q^* \cong P \cong R^*$ .

2. The ratio of change of the segment  $\overline{QR}$  with respect to the arc length  $QR$  is infinitely close to the tangent of  $\gamma$  at  $P$ .
3. If the order of  $\gamma$  is not less than the order of  $C^2$ , then any tangent at the point  $Q^*$  or  $R^*$  is infinitely close to the tangent of  $\gamma$  at  $P$ , as shown in **Fig. 2.1**

**Proof:**

1. Let  $P = \gamma(t_o) = x(t_o)e_1 + y(t_o)e_2$ , where  $e_1, e_2$  are basic unit coordinate vectors, then

$$Q^* = \gamma(t_o + \alpha) = x(t_o + \alpha)e_1 + y(t_o + \alpha)e_2 \text{ and}$$

$$R^* = \gamma(t_o + \beta) = x(t_o + \beta)e_1 + y(t_o + \beta)e_2$$

$$P - Q^* = (x(t_o) - x(t_o + \alpha))e_1 + (y(t_o) - y(t_o + \alpha))e_2.$$

Since  $\gamma$  is a differentiable parametric curve, therefore  $\gamma$  is continuous, and  $t_o \cong t_o + \alpha$  so we get  $x(t_o) \cong x(t_o + \alpha)$  and  $y(t_o) \cong y(t_o + \alpha)$ .

Thus  $Q^* \cong P$ , similarly we can prove that  $P \cong R^*$ .

2. Let  $\vec{T}(t_o)$  be the tangent vector of  $\gamma$  at  $P$ , then

$$\vec{T}(t_o) = \left. \frac{d\gamma(t)}{ds} \right|_{t=t_o} = \frac{\gamma'(t_o)}{|\gamma'(t_o)|}.$$

Now  $\Delta\gamma = \|\widehat{QR}\|$  and  $\Delta s = \|\widehat{QR}\|$  then as  $\Delta s \cong 0$  we get  $Q \cong P \cong R$  and conversely.

$$\text{Thus } \frac{\Delta\gamma}{\Delta s} \cong \frac{d\gamma}{ds} = \frac{\gamma'(t_o)}{|\gamma'(t_o)|} = T(t_o).$$

3. Let  $\vec{T}_o = \vec{T}(t_o)$  be the tangent vector of  $\gamma$  at  $P$ , then

$$T_o = \frac{\gamma(t_o)}{|\gamma'(t_o)|} \text{ and } T_\alpha = \left. \frac{d\gamma(t)}{ds} \right|_{t=t_o+\alpha} = \frac{\gamma'(t_o + \alpha)}{|\gamma'(t_o + \alpha)|} \text{ since } \gamma' \text{ is continuous,}$$

therefore  $T_o \cong T_\alpha$ . Similarly we can prove  $T_o \cong T_\beta$ , thus  $T_\alpha \cong T_o \cong T_\beta$ .

### 3- Generalized Curvature

The nonstandard principles are used for obtaining a generalized curvature at points infinitely close to regular and singular points. The problem of studying the behavior of curves by curvature at or near singular

points cannot be obtained with classical differential geometry [8], [15], while with nonstandard analysis this problem can be easily treated [26].

Let  $A$  be a standard point on the curve  $\gamma$ , and let  $B$  and  $C$  be two points infinitely close to the point  $A$ , that is  $\Delta ABC$  is infinitesimal triangle as shown in **Fig. 3.1**, then we have the following cases to the point  $A$ :

- 1- If  $\gamma' \neq \mathbf{0}$ ,  $\gamma'' \neq \mathbf{0}$  and  $\gamma' \cdot \gamma'' \neq \mathbf{0}$  then the point  $A$  is called **biregular point**.
- 2- If  $\gamma' \neq \mathbf{0}$  then the point  $A$  is called **regular point**.
- 3- If  $\gamma' \neq \mathbf{0}$  and  $\gamma' \cdot \gamma'' = \mathbf{0}$  then the point  $A$  is called **only regular point**.
- 4- If  $\gamma' = \mathbf{0}$  then the point is called **singular point**. In general if  $\gamma' = \gamma'' = \dots = \gamma^{(p-1)} = \gamma^{(p)} = \mathbf{0}$  but  $\gamma^{(p+1)} \neq \mathbf{0}$  the point  $A$  is called **singular point of order  $p$** .

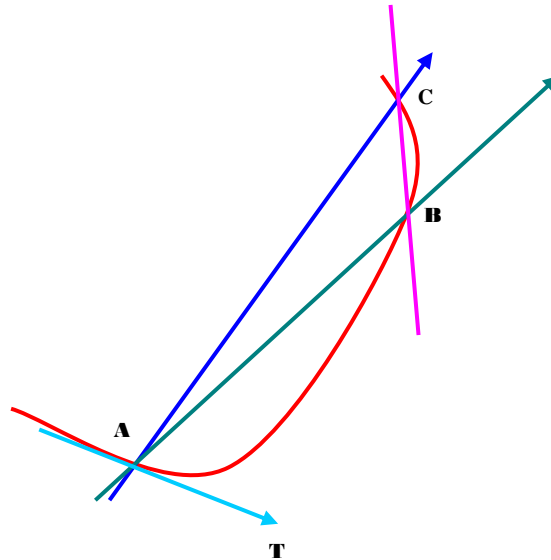


Figure 3.1

**Theorem 3.1**

Let  $A$  be a standard point on the planar curve  $\gamma$ , and let  $B$  and  $C$  be two points infinitely close to the point  $A$  as shown in **Fig. 3.1**, then

$$1. \frac{\tan \hat{A}}{\|\overrightarrow{BC}\|} = \frac{\overrightarrow{AC} \times \overrightarrow{AB}}{\frac{1}{2} \|\overrightarrow{BC}\| (\|\overrightarrow{AC}\|^2 + \|\overrightarrow{AB}\|^2 - \|\overrightarrow{BC}\|^2)} \cong \frac{\overrightarrow{AC} \times \overrightarrow{AB}}{\|\overrightarrow{BC}\|^3} \cong \frac{\overrightarrow{BC} \times \overrightarrow{AB}}{\|\overrightarrow{BC}\|^3} .$$

$$2. \overrightarrow{BC} = \varepsilon \gamma'(t) \Big|_{t=t_o+\alpha}$$

$$3. \overrightarrow{AC} = \beta \gamma'(t) \Big|_{t=t_o}$$

$$4. \overrightarrow{AB} = \alpha \gamma'(t) \Big|_{t=t_o}$$

Assuming that the derivatives of orders up to  $p$  of  $\gamma$  at  $A$  exist:

**Proof:**

$$1- \tan \hat{A} = \frac{\overrightarrow{AC} \times \overrightarrow{AB}}{\overrightarrow{AC} \cdot \overrightarrow{AB}} \text{ and } \|\overrightarrow{BC}\|^2 = \|\overrightarrow{AC}\|^2 + \|\overrightarrow{AB}\|^2 - 2(\overrightarrow{AC} \cdot \overrightarrow{AB})$$

Since  $\Delta ABC$  is an infinitesimal triangle then the quantities  $\|\overrightarrow{AC}\|$ ,  $\|\overrightarrow{AB}\|$  and  $\|\overrightarrow{BC}\|$  are mutually infinitely close. Therefore

$$\|\overrightarrow{BC}\|^2 = \overrightarrow{BC} \cdot \overrightarrow{BC} \cong \overrightarrow{AC} \cdot \overrightarrow{AB} \text{ and}$$

$$\frac{\tan \hat{A}}{\|\overrightarrow{BC}\|} = \frac{\overrightarrow{AC} \times \overrightarrow{AB}}{\frac{1}{2} \|\overrightarrow{BC}\| (\|\overrightarrow{AC}\|^2 + \|\overrightarrow{AB}\|^2 - \|\overrightarrow{BC}\|^2)} \cong \frac{\overrightarrow{AC} \times \overrightarrow{AB}}{\|\overrightarrow{BC}\|^3} \cong \frac{\overrightarrow{BC} \times \overrightarrow{AB}}{\|\overrightarrow{BC}\|^3}$$

$$2- \text{Let } A = \gamma(t_o) = x(t_o)e_1 + y(t_o)e_2 ,$$

$$B = \gamma(t_o + \alpha) = x(t_o + \alpha)e_1 + y(t_o + \alpha)e_2 ,$$

$$C = \gamma(t_o + \beta) = x(t_o + \beta)e_1 + y(t_o + \beta)e_2$$

Since  $A$ ,  $B$  and  $C$  are not identically equal we may assume  $\beta > \alpha$ , then  $\beta = \alpha + \varepsilon$  for some infinitesimal  $\varepsilon$ , therefore

$$\overrightarrow{BC} = \varepsilon \left( \frac{x(t_o + \alpha + \varepsilon) - x(t_o + \alpha)}{\varepsilon} \right) e_1 + \varepsilon \left( \frac{y(t_o + \alpha + \varepsilon) - y(t_o + \alpha)}{\varepsilon} \right) e_2 = \varepsilon \gamma'(t) \Big|_{t=t_o+\alpha}$$

In the same way we can prove the results of 3 and 4.

### Theorem 3.2

Let  $A$  be a standard biregular point on the curve  $\gamma$ , and let  $B$  and  $C$  be two points infinitely close to the point  $A$ , then the usual curvature of the curve  $\gamma$  at the point  $A$  is given by:  $\left( \frac{\tan \hat{A}}{\|\overrightarrow{BC}\|} \right)$

**Proof:**

From the previous theorem we have  $\overrightarrow{BC} = \varepsilon\gamma'(t)|_{t=t_o+\alpha}$ ,  
 $\overrightarrow{AC} = \beta\gamma'(t)|_{t=t_o}$  and  $\overrightarrow{AB} = \alpha\gamma'(t)|_{t=t_o}$

Since  $\gamma$  is continuous

$$x(t_o + \alpha) - x(t_o) \cong x(t_o + 2\alpha) - x(t_o) \cong \alpha^2 \left( \frac{x(t_o + 2\alpha) - 2x(t_o + \alpha) - x(t_o)}{\alpha^2} \right) = \alpha^2 x''(t_o)^T$$

$$\text{herefore } \overrightarrow{AB} \cong \alpha^2 \gamma'' \Rightarrow \frac{\tan \hat{A}}{\|\overrightarrow{BC}\|} = \frac{\overrightarrow{AC} \times \overrightarrow{AB}}{\|\overrightarrow{BC}\|^3} = \frac{\beta\gamma' \times \alpha^2 \gamma''}{\|\varepsilon\gamma'\|^3} \cong \frac{\gamma' \times \gamma''}{\|\gamma'\|^3} = \kappa$$

$$\text{Thus } \frac{\tan \hat{A}}{\|\overrightarrow{BC}\|} \cong \frac{\gamma' \times \gamma''}{\|\gamma'\|^3} = \kappa$$

**Remark:**

If  $A$  is neither biregular nor regular point then the classical definition of the curvature of  $\gamma$  at the point  $A$  breaks down, and so the quantity  $\frac{\tan \hat{A}}{\|\overrightarrow{BC}\|}$  plays a mean role for determining the behaviors of the curvature vector of  $\gamma$  at a point infinitely close to  $A$ . Also if  $A$  is the only regular point of the curve  $\gamma$ , then the quantity  $\frac{\tan \hat{A}}{\|\overrightarrow{BC}\|}$  is infinitesimal, but the order of largeness

of this infinitesimal with respect to  $\|\overrightarrow{BC}\|$  depends on the orders of the first two vector derivatives of  $\gamma$  not collinear. Therefore, we can go beyond to the usual region of classical curvature and generalizing the notion of curvature to include all possible cases given at the beginning of this section. For this purpose, we start with the quantity  $\frac{\tan \hat{A}}{\|\overrightarrow{BC}\|}$  for defining generalized curvature with respect to the orders of the first two vector derivatives of  $\gamma$  not collinear, and denoting it by  $\kappa_G$ .

**Theorem 3.3**

Let  $\gamma$  be a standard curve of type at least that  $C^2$ , and let  $A$  be a standard biregular point on it; and let  $B$  and  $C$  two points infinitely close to

the point  $A$ . Then 
$$\left( \frac{|\tan \hat{A}|}{\|\overrightarrow{BC}\|} \right) \cong \frac{|x'y'' - x''y'|}{2(x'^2 + y'^2)^{\frac{3}{2}}} = \frac{|\gamma' \times \gamma''|}{2\|\gamma'\|^3}$$

**Proof:**

Let 
$$A = \gamma(t_o) = x(t_o)e_1 + y(t_o)e_2,$$

$$B = \gamma(t_o + \alpha) = x(t_o + \alpha)e_1 + y(t_o + \alpha)e_2,$$
and 
$$C = \gamma(t_o + \beta) = x(t_o + \beta)e_1 + y(t_o + \beta)e_2$$

Expanding each of  $B$  and  $C$  by Taylor development of the second order, we get

$$B = \gamma(t_o + \alpha) = \gamma(t_o) + \alpha\gamma'(t_o) + \frac{\alpha^2}{2}\gamma''(t_o) + \delta_1\alpha^2,$$
and 
$$C = \gamma(t_o + \beta) = \gamma(t_o) + \beta\gamma'(t_o) + \frac{\beta^2}{2}\gamma''(t_o) + \delta_2\beta^2.$$

Then we have

$$\|\overrightarrow{BC}\| = |\beta - \alpha| \left[ \left( x'(t_o) + \frac{\beta + \alpha}{2}x''(t_o) + \delta_3(\beta - \alpha) \right)^2 + \left( y'(t_o) + \frac{\beta + \alpha}{2}y''(t_o) + \delta_3(\beta - \alpha) \right)^2 \right]^{\frac{1}{2}},$$

where  $\delta_2\beta^2 - \delta_1\alpha^2 = \delta_3(\beta - \alpha)^2$ . Therefore to find  $|\tan \hat{A}|$  we have to find the angular coefficient of each of the lines  $\ell_1$  and  $\ell_2$  as shown in the **Fig. 3.2**.

The angular coefficient of

$$\ell_1 = \tan(\angle DAB) = \frac{dy/dt}{dx/dt} = \frac{y'(t_o) + \frac{\alpha}{2}y''(t_o) + \delta_1\alpha}{x'(t_o) + \frac{\alpha}{2}x''(t_o) + \delta_1\alpha}$$

and the angular coefficient of

$$\ell_2 = \tan(\angle DAC) = \frac{dy/dt}{dx/dt} = \frac{y'(t_o) + \frac{\beta}{2}y''(t_o) + \delta_2\beta}{x'(t_o) + \frac{\beta}{2}x''(t_o) + \delta_2\beta}$$

Then

$$|\tan(\hat{A})| = \left| \frac{\tan(\angle DAC) - \tan(\angle DAB)}{1 + \tan(\angle DAC)\tan(\angle DAB)} \right|$$



$$= \frac{\left(x' + \frac{\alpha}{2}x'' + \delta_1\alpha\right)\left(y' + \frac{\beta}{2}y'' + \delta_1\beta\right) - \left(x' + \frac{\beta}{2}x'' + \delta_1\beta\right)\left(y' + \frac{\alpha}{2}y'' + \delta_1\alpha\right)}{\left(x' + \frac{\alpha}{2}x'' + \delta_1\alpha\right)\left(y' + \frac{\beta}{2}y'' + \delta_1\beta\right) + \left(x' + \frac{\beta}{2}x'' + \delta_1\beta\right)\left(y' + \frac{\alpha}{2}y'' + \delta_1\alpha\right)}$$

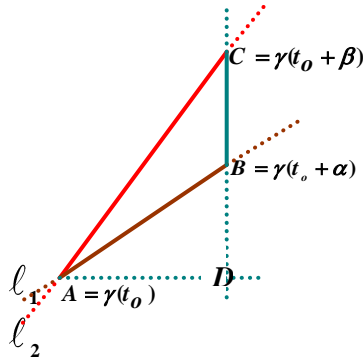


Figure 3.2

Therefore after simplifying the ratio  $\frac{|\tan \hat{A}|}{\|\overrightarrow{BC}\|}$  we get:

$$\frac{|\tan \hat{A}|}{\|\overrightarrow{BC}\|} = \frac{\frac{|\beta - \alpha|}{2} |(x'y'' - x''y') + i.s|}{|\beta - \alpha| \left( (x'^2 + i.s) + (y'^2 + i.s) \right) \left( (x' + i.s)^2 + (y' + i.s)^2 \right)}$$

where

$$i.s \text{ represent different infinitesimals. Thus } \frac{|\tan \hat{A}|}{\|\overrightarrow{BC}\|} \cong \frac{|x'y'' - x''y'|}{2(x'^2 + y'^2)^{\frac{3}{2}}}$$

From the **Theorem 3.3** we deduce that the usual curvature  $K(t)$  is infinitely

close to twice that of the quantity  $\frac{|\tan \hat{A}|}{\|\overrightarrow{BC}\|}$

The following theorem gives the notion of curvature at a standard only regular point.

**Theorem 3.4**

Let  $\gamma$  be a standard curve of order  $C^n$  and  $A$  be a standard only regular point on it, and let  $B$  and  $C$  be two points infinitely close to the point

$A$ , then  $\left(\frac{|\tan \hat{A}|}{\|\overrightarrow{BC}\|^{q-1}}\right) \cong \frac{|x'y^{(q)} - x^{(q)}y'|}{q!(x'^2 + y'^2)^{\frac{q+1}{2}}} = \frac{|\gamma' \times \gamma^{(q)}|}{q!\|\gamma'\|^{q+1}}$  where  $q$  is the order of the first vector derivative not collinear with  $\gamma'$

**Proof:**

Expand the curve  $\gamma$  using Taylor development up to the order  $q$  at each of the points  $B = \gamma(t_o + \alpha)$  and  $C = \gamma(t_o + \beta)$ . Then

$$\|\overrightarrow{BC}\| = |\beta - \alpha| \left[ \left( \begin{aligned} &x'(t_o) + \frac{\beta + \alpha}{2} x''(t_o) + \frac{\sum_{k=1}^q \alpha^{q-k} \beta^{k-1}}{q!} x^{(q)} + \delta_3(\beta - \alpha)^{q-1} \\ &x'(t_o) + \frac{\beta + \alpha}{2} x''(t_o) + \frac{\sum_{k=1}^q \alpha^{q-k} \beta^{k-1}}{q!} x^{(q)} + \delta_3(\beta - \alpha)^{q-1} \end{aligned} \right)^2 + \left( \begin{aligned} &y'(t_o) + \frac{\beta + \alpha}{2} y''(t_o) + \frac{\sum_{k=1}^q \alpha^{q-k} \beta^{k-1}}{q!} y^{(q)} + \delta_3(\beta - \alpha)^{q-1} \\ &y'(t_o) + \frac{\beta + \alpha}{2} y''(t_o) + \frac{\sum_{k=1}^q \alpha^{q-k} \beta^{k-1}}{q!} y^{(q)} + \delta_3(\beta - \alpha)^{q-1} \end{aligned} \right)^2 \right]^{\frac{1}{2}}$$

Thus  $\|\overrightarrow{BC}\| \cong |\beta - \alpha|(x'^2 + y'^2)^{\frac{1}{2}}$

Therefore  $\|\overrightarrow{BC}\|^{q-1} \cong |\beta - \alpha|(x'^2 + y'^2)^{\frac{q-1}{2}}$ .

Hence

$$\frac{|\tan \hat{A}|}{\|\overrightarrow{BC}\|^{q-1}} = \frac{\left( \left( \sum_{k=1}^q \frac{\alpha^{k-1} x^{(k)}}{k!} + \delta_1 \alpha^{q-1} \right) \left( \sum_{k=1}^q \frac{\beta^{k-1} y^{(k)}}{k!} + \delta_2 \beta^{q-1} \right) - \left( \sum_{k=1}^q \frac{\beta^{k-1} x^{(k)}}{k!} + \delta_2 \beta^{q-1} \right) \left( \sum_{k=1}^q \frac{\alpha^{k-1} y^{(k)}}{k!} + \delta_1 \alpha^{q-1} \right) \right)}{|\beta - \alpha| \left( \left( \sum_{k=1}^q \frac{\alpha^{k-1} x^{(k)}}{k!} + \delta_1 \alpha^{q-1} \right) \left( \sum_{k=1}^q \frac{\beta^{k-1} y^{(k)}}{k!} + \delta_2 \beta^{q-1} \right) + \left( \sum_{k=1}^q \frac{\beta^{k-1} x^{(k)}}{k!} + \delta_2 \beta^{q-1} \right) \left( \sum_{k=1}^q \frac{\alpha^{k-1} y^{(k)}}{k!} + \delta_1 \alpha^{q-1} \right) \right) (x'^2 + y'^2)^{\frac{q-1}{2}}}$$

...(3.4.1)

Now since  $A$  is only regular, then  $\gamma' \neq \mathbf{0}$ ; and since the first vector derivative which is not collinear with  $\gamma'$  is  $\gamma^{(q)}$ , then all the terms containing

$\gamma'\gamma'', \gamma'\gamma''', \dots, \gamma'\gamma^{(q-1)}$  on the numerator and denominator of the right hand side of equation (3.4.1) vanish.

$$\text{Therefore } \frac{|\tan \hat{A}|}{\|\overrightarrow{BC}\|^{q-1}} \cong \frac{\left(\frac{\beta}{\alpha}\right)^{q-1} - 1 |x'y^{(q)} - x^{(q)}y'|}{\left|\frac{\beta}{\alpha} - 1\right|^{q-1} q!(x'^2 + y'^2)^{\frac{q+1}{2}}}.$$

Since  $\alpha$  and  $\beta$  are arbitrarily chosen, so we may assume that  $\frac{\beta}{\alpha}$  is infinitesimal to get the required result.

### Corollary 3.5

The generalized curvature obtained in the previous theorem is limited and  $\left(\frac{|\tan \hat{A}|}{\|\overrightarrow{BC}\|^{q-1}}\right) \neq \mathbf{0}$ .

#### Proof:

Expand the curve  $\gamma$  using Taylor development up to the order  $q$  at each point  $B = \gamma(t_o + \alpha)$  and  $C = \gamma(t_o + \beta)$ , and use the coordinate form of the tangent vector and norm vector to get

$$\gamma(t_o + \alpha) = \gamma(t_o) + \alpha|\gamma'(t_o)|\vec{T} + \frac{\alpha^2}{2}\kappa\vec{N} + O(\alpha^2),$$

$$\text{and } \gamma(t_o + \beta) = \gamma(t_o) + \beta|\gamma'(t_o)|\vec{T} + \frac{\beta^2}{2}\kappa\vec{N} + O(\beta^2).$$

Then using the definition of big O we find that there exists  $M \in \mathbf{N}$  such that

$$\left| \frac{|\gamma(t_o + \alpha) - \gamma(t_o) - \alpha|\gamma'(t_o)|\vec{T} + \frac{\alpha^2}{2}\kappa\vec{N}|}{\alpha^2} \right| < M$$

That is  $\frac{|\kappa|}{2} < M$ ; therefore the curvature  $\kappa$  of  $\gamma$  is bounded and

so  $\frac{|\tan \hat{A}|}{\|\overrightarrow{BC}\|}$  is also bounded. Now we prove that  $\frac{|\tan \hat{A}|}{\|\overrightarrow{BC}\|^{q-1}}$  is limited.

Since  $\frac{|\tan \hat{A}|}{\|\overrightarrow{BC}\|^{q-1}} = \frac{|\tan \hat{A}|}{\|\overrightarrow{BC}\| \|\overrightarrow{BC}\|^{q-2}} = \frac{\kappa}{\|\overrightarrow{BC}\|^{q-2}}$ , then we have the

following two cases:

If  $\|\overrightarrow{BC}\|^{q-2}$  is not infinitesimal, then the result is obtained at once.

If  $\|\overrightarrow{BC}\|^{q-2}$  is infinitesimal, use Archimedean property to get that  $\frac{\kappa}{\|\overrightarrow{BC}\|^{q-2}}$  is limited.

The proof of  ${}^o\left(\frac{|\tan \hat{A}|}{\|\overrightarrow{BC}\|^{q-1}}\right) \neq \mathbf{0}$  is by contradiction. Since the only

standard infinitesimal is zero, then  $\frac{|\tan \hat{A}|}{\|\overrightarrow{BC}\|^{q-1}} = \mathbf{0}$  or  $\frac{|\tan \hat{A}|}{\|\overrightarrow{BC}\|^{q-1}} \cong \mathbf{0}$ .

Thus either  ${}^o(\tan \hat{A}) = \mathbf{0}$  or  $\|\overrightarrow{BC}\|$  is unlimited. In the first case, we get that  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$  are collinear which is a contradiction. The second case is impossible.

### Theorem 3.6

Let  $\gamma$  be a standard curve of order  $C^n$  and A be a standard singular point of order  $p-1$  on  $\gamma$ ; and let B and C be two points internally close to the point A, then

$${}^o\left(\frac{|\tan \hat{A}|}{\|\overrightarrow{BC}\|^{q-1}}\right) = \frac{(p!)^q_p |x^{(p)} y^{(q)} - x^{(q)} y^{(p)}|}{q! (x^{(p)2} + y^{(p)2})^{\frac{q+p}{2p}}} = \frac{(p!)^q_p |\gamma^{(p)} \times \gamma^{(q)}|}{q! \|\gamma^{(p)}\|_p^{q+1}},$$

where  $q$  is the order of the first vector derivative of  $\gamma$  not collinear with  $\gamma^{(p)}$ .

**Proof:**

Expanding the curve  $\gamma$  using Taylor development up to the order  $q$  at points  $B = \gamma(t_o + \alpha)$  and  $C = \gamma(t_o + \beta)$ , we get

$$\|\overrightarrow{BC}\| = |\beta - \alpha| \left[ \left( x'(t_o) + \frac{\beta + \alpha}{2} x''(t_o) + \frac{\sum_{k=1}^q \alpha^{q-k} \beta^{k-1}}{q!} x^{(q)} + \delta_3 (\beta - \alpha)^{q-1} \right)^2 + \left( x'(t_o) + \frac{\beta + \alpha}{2} x''(t_o) + \frac{\sum_{k=1}^q \alpha^{q-k} \beta^{k-1}}{q!} x^{(q)} + \delta_3 (\beta - \alpha)^{q-1} \right)^2 \right]^{\frac{1}{2}}$$

Since  $A$  is a singular point of order  $p-1$  we get  $\gamma' = \gamma'' = \dots = \gamma^{(p-1)} = \mathbf{0}$

and  $\gamma^{(p)} \neq \mathbf{0}$  then  $\|\overrightarrow{BC}\| \cong \frac{|\beta^p - \alpha^p|}{p!} (x^{(p)^2} + y^{(p)^2})^{\frac{1}{2}}$

$$\text{Therefore } \|\overrightarrow{BC}\|_p^{q-1} \cong \frac{|\beta^p - \alpha^p|^{\frac{q-1}{p}}}{p!} (x'^2 + y'^2)^{\frac{q-p}{2p}}.$$

Thus

$$\frac{|\tan \hat{A}|}{\|\overrightarrow{BC}\|_p^{q-1}} = \frac{\left( \sum_{k=1}^q \frac{\alpha^{k-1} x^{(k)}}{k!} + \delta_1 \alpha^{q-1} \right) \left( \sum_{k=1}^q \frac{\beta^{k-1} y^{(k)}}{k!} + \delta_2 \beta^{q-1} \right) - \left( \sum_{k=1}^q \frac{\beta^{k-1} x^{(k)}}{k!} + \delta_2 \beta^{q-1} \right) \left( \sum_{k=1}^q \frac{\alpha^{k-1} y^{(k)}}{k!} + \delta_1 \alpha^{q-1} \right)}{p! \left( \sum_{k=1}^q \frac{\alpha^{k-1} x^{(k)}}{k!} + \delta_1 \alpha^{q-1} \right) \left( \sum_{k=1}^q \frac{\beta^{k-1} y^{(k)}}{k!} + \delta_2 \beta^{q-1} \right) + \left( \sum_{k=1}^q \frac{\beta^{k-1} x^{(k)}}{k!} + \delta_2 \beta^{q-1} \right) \left( \sum_{k=1}^q \frac{\alpha^{k-1} y^{(k)}}{k!} + \delta_1 \alpha^{q-1} \right)} (x^{(p)^2} + y^{(p)^2})^{\frac{q-p}{2p}}$$

Therefore

$$\frac{|\tan \hat{A}|}{\|\overline{BC}\|^{\frac{q}{p}-1}} = \frac{\left( \frac{\alpha^{p-1} \beta^{q-1} x^{(p)} y^{(q)}}{p!q!} + \frac{\alpha^{q-1} \beta^{p-1} x^{(q)} y^{(p)}}{p!q!} + i.s \right) - \left( \frac{\alpha^{q-1} \beta^{p-1} x^{(p)} y^{(q)}}{p!q!} + \frac{\alpha^{p-1} \beta^{q-1} x^{(q)} y^{(p)}}{p!q!} + i.s \right)}{\frac{|\beta^p - \alpha^p|^{\frac{q}{p}-1}}{p!} \left( \left( \frac{\alpha^{p-1} \beta^{q-1}}{p!p!} x^{(p)} x^{(p)} + i.s \right) + \left( \frac{\alpha^{p-1} \beta^{q-1}}{p!p!} y^{(p)} y^{(p)} + i.s \right) \right) \left( x^{(p)^2} + y^{(p)^2} \right)^{\frac{q-p}{2p}}}$$

... (3.6.1)

$$\left( \frac{|\tan \hat{A}|}{\|\overline{BC}\|^{\frac{q}{p}-1}} \right) = \frac{(p!)^{\frac{q}{p}}}{q!} \cdot \frac{\left( \frac{\beta}{\alpha} \right)^{q-1} - 1}{\left( \left( \frac{\beta}{\alpha} \right)^p - 1 \right)^{\frac{q-1}{p}}} \cdot \frac{(p!)^{\frac{q}{p}} |x^{(p)} y^{(q)} - x^{(q)} y^{(p)}|}{q! \left( x^{(p)^2} + y^{(p)^2} \right)^{\frac{q+p}{2p}}}$$

... (3.6.2)

Since  $\alpha$  and  $\beta$  are arbitrarily chosen, so we may assume that  $\frac{\beta}{\alpha}$  is infinitesimal to get the required result.

Now, since  $A$  is only regular, then  $\gamma' \neq \mathbf{0}$  and since the first vector derivative which is not collinear with  $\gamma'$  is  $\gamma^{(q)}$ , then we find that all the terms containing  $\gamma' \gamma'', \gamma' \gamma''', \dots, \gamma' \gamma^{(q-1)}$  on the numerator and denominator of the right hand side of equation (3.6.1) vanish.

$$\text{Therefore } \frac{|\tan \hat{A}|}{\|\overline{BC}\|^{q-1}} = \frac{\left( \frac{\beta}{\alpha} \right)^{q-1} - 1 |x' y^{(q)} - x^{(q)} y'|}{\left| \frac{\beta}{\alpha} - 1 \right|^{q-1} q! (x'^2 + y'^2)^{\frac{q+1}{2}}}$$

Similarly, from equation (3.6.2), we get:

$$\left( \frac{|\tan \hat{A}|}{\|\overline{BC}\|^{\frac{q}{p}-1}} \right) = \frac{(p!)^{\frac{q}{p}} |x^{(p)} y^{(q)} - x^{(q)} y^{(p)}|}{q! \left( x^{(p)^2} + y^{(p)^2} \right)^{\frac{q+p}{2p}}} = \frac{(p!)^{\frac{q}{p}} |\gamma^{(p)} \times \gamma^{(q)}|}{q! \|\gamma^{(p)}\|^{\frac{q+1}{p}}}$$

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