New bounds for $m_r(2,37)$ and $t_r(2,37)$ in PG(2,37)

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ABSTRACT

In this paper we find the nearest complete (n,r)-arcs to the maximum bound and the minimum sizes for $m_r(2,37)$ where $2 \le r \le 36$ and $t_r(2,37)$ where $2 \le r \le 37$ .Also we show that not exists (1236,34)-arc and some other arcs in PG(2,37).

***Keywords**: complete (n,r)-arcs, maximum bound $m_r(2,37)$, minimum bound $t_r(2,q)$, PG(2,37)*

حدود جديدة لـ $m_r(2, 37)$ و $t_r(2,37)$ في PG(2,37)

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الملخص

في هذا البحث تم إيجاد الاقواس-(n,r) التامة الاقرب الى القيد الاعلى $m_r(2,37)$ حيث $2 \le r \le 36$ والاقواس ذات الحجم الصغير القريبة من $t_r(2,q)$ عندما $2 \le r \le 37$.كذلك اثبتنا عدم وجود القوس-(1236,34) وبعض الاقواس الاخرى في الحقل PG(2,37)

***الكلمات المفتاحية:** الاقواس-(n,r) التامة ، القيد الاعلى $m_r(2,37)$ ، القيد الاصغر $t_r(2,q)$ ، الحقل PG(2,37)*

1. Introduction

Let PG(2,q) be a finite projective plane Π of order q, where q= p^h , $h \ge 1$ this plane consist of q^2+q+1 lines and the same number of points, q+1 points on every line and q+1 lines passing through every point.

An (n,r)–arc K in the projective plane Π is a set of n points such that some r ,but no r+1 of them are collinear. An (n,r)-arc K is complete if there is no (n+1,r)-arc containing it. A line L of the plane containing precisely i points of K, called an i-secant. Let T_i (also called τ_i) be denote the total number of i-secants to K in PG(n,q),all the T_i together are called the spectrum of the (n,r)-arc.

A t-fold blocking set B in a projective or affine plane, is a set of points such that each line contains at least t points of B and some line contains exactly t points of B.A (n,r)-arc of a projective plane is the complement of a t-fold blocking set with r + t = q + 1. [8]

In 1947 Bose [4] proved that $m_2(2, q)$=q+1 for q odd, and $m_2(2, q)$ = q+2 for q even. Barlotti [3] and also S. Ball [1] proved that $m_r(2, q)$ = (r-1)q+1 for q odd prime and r = (q + 1)/2, r = (q + 3)/2.

2. The projective plane PG(2,37)

Let $f(x)=x^3-3x^2-2x-4$ be an irreducible monic polynomial over GF(37) then companion matrix T of f(x)

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 2 & 4 \end{pmatrix}$$

is cyclic projectivity on PG(2,37).

Let p_0 be the point $U_0=(1,0,0)$ then $p_i=p_0T^i$,i=0,…,1406,are the 1407 points of PG(2,37).(see Table(1.1))

Table(1.1) Points of PG(2,37)

i	P_i		
0	1	0	0
1	0	1	0
2	0	0	1
3	1	19	20
⋮		⋮	
1405	1	36	15
1406	1	3	36

Let L_1 be the line at infinity (z=0) which contains the points:-

0,1,9,29,152,156,182,193,262,300,323,401,404,419,425,489,539,605,621,679,689,714, 754,851,923,945,972,1034,1195,1229,1231,1248,1262,1308,1321,1353,1360and 1365 .

then $L_i=L_1T^{i-1}$, i=1,…,1407,are the lines of PG(2,37), the 1407 lines L_i are given by the rows in (Table(2.1)).

Table(2.1) Lines of PG(2,37)

Lin	Points for each line
Lin 1	0,1,9,29,152,156,182,193,262,300,323,401,404,419,425,489,539,605,621,679,689,714,754,851,923,945,972,1034,1195,1229,1231,1248,1262,1308,1321,1353,1360, 1365
Lin 2	1,2,10,30,153,157,183,194,263,301,324,402,405,420,426,490,540,606,622,680,690,715,755,852,924,946,973,1035,1196,1230,1232,1249,1263,1309,1322,1354,1361 , 1366
. . .	. . .
Lin 140	1406,0,8,28,151,155,181,192,261,299,322,400,403,418,424,488,538,604,620,678,688,713,753,850,922,944,971,1033,1194,1228,1230,1247,1261,1307,1320,1352,1359,1364

Theorem 2.1 [8]**:**

$$\mathbf{m}_2(2,\mathbf{q}) = \begin{cases} \mathbf{q}+2 & \textbf{for} \quad \mathbf{q} \quad \textbf{even} \\ \mathbf{q}+1 & \textbf{for} \quad \mathbf{q} \quad \textbf{odd} \end{cases}$$

where $m_r(2,q)$ be the maximum value such that an (n,r)-arc exists in PG(2,q).

3. The maximum value $m_r(2,q)$:

It is clear that a lot of researchers have done some work like [10] , so the latest data related with the plane PG(2,37) is updated from theorems and lemmas appeared in [10],[11],[2] and [6] .

Theorem 3.1[1]: Let $f \in GF(q)[x]$ be fully reducible, and suppose that $f(x) = x^q v(x) + w(x)$, where v and w have no common factor. Let m < q be the maximum of the degrees of v and w. Let e be maximal such that f (and hence v and w) is a pe-th power. Then one of the following holds:

1. e = h and m = 0;
2. $e \geq h/2$ and $m \geq p^e$;

3. e < h/2 and m $\geq pe\left|(p^{h-e}+1)/(p^e+1)\right|$;

4. e = 0, m = 1 and f(x) = a(x^q- x).

Note that in particular when q is prime and m > 1, then m $\geq$ (q + 1)/2.

Theorem 3.2[1]: Let B be a t-fold blocking set in PG(2, p), p > 3 prime.

1. If t < p/2 then |B| $\geq$ (t + 0.5)(p + 1).
2. If t > p/2 then |B| $\geq$ (t + 1)p.

In 1965 (Barlotti [3]) provided examples of t-fold blocking sets with (t + 1)q points when q is odd and t = (q - 1)/2 and t = (q + 1)/2. So if q = p > 3 is a prime then the bound in Theorem 3.2 is work. It is also true when t = p - 1 by the complement of a conic[1]

The theorem above will be used here to determine the existence of exact values and improved bounds on the maximum sizes of (n,r)-arcs in the Desarguesian projective plane of order 37.

Lemma 3.3 There exists no (1236,34)-arc in PG(2,37).

Proof : We will use the same manner that [1] have use it ,Finding a maximum (k,34)-arc is equivalent to finding the minimum 4-fold blocking set by considering complements. Theorem 3.2 says that since 37 is prime a 4-fold blocking set must have at least 171 points. If there were a 4-fold blocking set with exactly 171 points then the degree of α in the proof of Theorem 3.1 must be exactly equal to m + 1 which is equal to 20 in this case. This implies that each point of the 4-fold blocking set is on exactly eighteen 4-secants since the degree of α is equal to the number of non 4-secants through a point. The total number of 4-secants must therefore be 18*171/4 which is not an integer. Hence a 171 point 4-fold blocking set does not exist and equivalently a (1236,34)-arc does not exist.

Corollary 3.4: The following (n,r)-arcs does not exist: (1198,33)-arc;(1122,31)-arc, (1084,30)-arc;(1008,28)-arc;(970,27)-arc;(932,26)-arc;(894,25)-arc;(856,24)-arc; (818,23)-arc;(780,22)-arc;(742,21)-arc.

Proof: By the same technique used in Lemma 3.3 we prove the nonexistence .

Example3.5: Here it is an example for complete (46,3)-arc which is near to $m_3(2,37)$ which it's abstract appear in table(3.4) ; here n= 46 and r= 3**.**
arc points: { 9 0 2 3 4 5 45 413 305 908 1119 1004 772 75 103 485 502 654 269 1066 1396 260 219 822 1373 77 612 1127 1359 558 194 212 1009 1056 1020 1159 658 491 1241 821 910 1393 156 126 608 1067 }**,** have the spectrum $\tau_0 = 474$, $\tau_1 = 338$, $\tau_2 = 375$, $\tau_3 = 220$.

Now by applying Lemma 3.3 and from [6] we can copy and then use these three tables for exact values for $m_r(2,q)$ and lower bounds and upper bounds for $m_r(2,q)$ respectively

Table(3.1) Exact Values for $m_r(2,q)$

r	$q = p^h$, p prime	$m_r(2,q)$
2	q odd	$q+1$
2^e	2^h	$(2^e-1)q+2^e$
$\sqrt{q}+1$	q square, $q \geq 25$	$q\sqrt{q}+1$

$(q+1)/2$	q *odd prime*	$(q^2-q)/2+1$
$(q+3)/2$	q *odd prime*	$(q^2+q)/2+1$
q	q	q^2
$q-1$	q *square*, $q>4$	$q^2-q-2\sqrt{q}-1$
$q-2$	q *odd square*, $q>121$	$q^2-2q-3\sqrt{q}-2$
$q+1-t$, $t>1$	$q=p^{2e}, p>3$, $t<\min(q^{1/6}, q^{1/4}/2)$	$q^2+q+1-t(q+\sqrt{q}+1)$
$q+1-t$, $t>1$	$q=p^{2e}, p=2,3$, $t<\min(2^{-1/3}q^{1/6}, q^{1/4}/2)$	$q^2+q+1-t(q+\sqrt{q}+1)$

Table(3.2) Lower bound for $m_r(2,q)$

r	q	$m_r(2,q)\geq$
$r=3$	q *exceptional*	$q+\lfloor 2\sqrt{q}\rfloor$
$r=\sqrt{q}+1$	q *square*	$(r-1)q+1$
$r=(q+1)/2$	q *odd*	$(r-1)q+1$
$r=(q+3)/2$	q *odd*	$(r-1)q+1$
$r=q-1$	q	$(r-1)q+1$
$r=q-2$	q *even*	$(r-1)q+2$
r	q *square*	$(r-1)q+r-\sqrt{q}+\sqrt{q}(r-q)$
$r=q-\sqrt{q}$	q *square*	$(r-1)q+\sqrt{q}$
$(q-r)\mid q$	q	$(r-1)q+q-r$

Table(3.3) Upper bound for $m_r(2,q)$

r	Conditions	$m_r(2,q)\leq$
r	*for all* q	$(r-1)q+r$
r	q *odd*	$(r-1)q+r-2$
r	q *odd*, $r\mid q$	$(r-1)q+r/2$

r	$4 \le r < q,\ r \nmid q$	$(r-1)q + r - 3$
r	$9 \le r < q,\ r \nmid q$	$(r-1)q + r - 4$
r	$q\ prime,\ r \ge (q+3)/2$	$(r-1)q + r - (q+1)/2$
r	$r \le 2q/3$	$(r-1)q + q - r$
r	$r \ge q/2, (r,q) > 1,\ K\ has\ a\ skew\ line$	$(r-1)q + q - r$
r	$r > 2q/3$	$(r-1)q + (q+r-r/q)/2 - \sqrt{(q+r-r/q)^2/4 - (r^2-r)}$
r	no skew line to K	$rq - \sqrt{(q+1-r)q}$
r	no skew line to K	$m_r(2,q) \le (r-1)q + \lfloor r^2/q \rfloor$
r	no skew line to K, $\sqrt{q} \mid r$	$m_r(2,q) < (r-1)q + \lfloor r^2/q \rfloor$
r	$(r,q) = 1,\ r < \sqrt{q} + 1$	$(r-1)q + 1$
r	$(r,q) = 1,\ 6 \le r \le \sqrt{2q} + 1 \le q - 1$	$(r-1)q + 1$
r	$(r,q) = p^e,\ K\ has\ a\ skew\ line$	$(r-1)q + p^e$
r	$q\ \ prime,\ r \le (q+1)/2$	$(r-1)q + 1$

In this paper we improve the bounds in[2],[5],[6],[7],[10] and [11] together with the three tables mentioned above, so we put new numbers instead of the unknown values for some complete arcs which their lengths are near to $m_r(2,37)$ as appeared here in table (3.4):

Table(3.4) Our improvement (written in bold face)

r	2	3	4	5	6	7	8
$m_r(2,37)$	38	**46**-75	**75**-112	**105**-149	**135**-186	**167**-223	**201**-260
r	9	10	11	12	13	14	15
$m_r(2,37)$	**234**-297	**271**-334	**303**-371	**339**-408	**374**-445	**409**-482	**445**-519
r	16	17	18	19	20	21	22
$m_r(2,37)$	**481**-556	**516**-593	**554**-630	667	704	**665**-741	**700**-779
r	23	24	25	26	27	28	29
$m_r(2,37)$	**737**-817	**779**-855	**816**-893	**853**-931	**894**-969	**930**-1007	**972**-1045
r	30	31	32	33	34	35	36
$m_r(2,37)$	**1009**-1083	**1050**-1121	**1089**-1159	**1128**-1197	**1169**-1235	**1212**-1273	**1259**-1311
r	37						
$m_r(2,37)$	1369						

For full details of any result here , the reader can e-mailing the author .

4. The minimum value $t_r(2,q)$

Let $t_r(2,q)$ be the minimum value for n such that an (n,r)-arc is complete in PG(2,q) [8].

Despite the maximum value is studied from a lot of researchers (because of relating with Code applications), the minimum value is studied from few researchers like (Leo storme [6], J.W.P.Hirschfeld, [9], S.Ball [2] and G.Keri [10]).

4.1 Complete (n,r)-arcs in PG(2,37) of length near to $t_r(2,q)$

There is no known value for $t_r(2,37)$.We use a computer program to get values that are very near to $t_r(2,37)$ to be the first approximating new values for it, we will sign it as $t_r^*(2,37)$.

Example 4.2: Here it is an example for complete (58,4)-arc which is near to $t_4(2,37)$ which it's abstract appear in table (4.1); here n = 58 and r = 4
arc points: {152 0 2 3 4 15 40 1348 553 631 1280 1304 884 1102 999 951 209 325 1238 568 1209 164 1076 675 64 1096 391 823 78 1007 1144 429 286 604 20 531 588 1341 1264 1388 458 350 144 536 1353 646 946 292 679 1132 294 398 332 200 363 1307 448 585 }, have the spectrum $\tau_0 = 275$, $\tau_1 = 495$, $\tau_2 = 348$, $\tau_3 = 143$, $\tau_4 = 146$.

By computer search we could find a collection of nontrivial complete (n,r)-arcs where r=2,3,…,37 .Tabled below with the spectrum for each one :-

Table(4.1) new small complete (n,r)-arcs with spectrum

n	r	Spectrum : Number of line secants { T_i : i=0,...,r}
16	2	919 368 120
34	3	543 569 162 133
58	4	275 495 348 143 146
83	5	132 347 399 258 120 151
112	6	57 212 315 322 210 138 153
138	7	33 110 239 300 266 205 111 143
166	8	16 58 155 240 274 241 177 111 135
194	9	4 30 104 164 235 271 211 160 91 137
226	10	2 14 41 123 172 240 239 184 157 93 142
254	11	1 6 22 59 130 221 227 202 193 121 84 141
285	12	0 2 12 33 81 141 204 218 211 170 119 74 142
316	13	0 0 5 20 40 90 168 188 218 190 171 104 73 140
346	14	0 0 2 10 25 45 124 150 200 202 200 140 109 71 129
379	15	0 0 0 3 14 31 69 109 164 189 187 186 151 121 54 129
417	16	0 0 0 1 4 13 38 77 104 156 166 197 192 145 112 57 145
450	17	0 0 0 1 1 7 17 42 71 117 154 196 170 180 145 100 67 139
486	18	0 0 0 0 0 1 7 20 39 76 131 172 170 172 161 145 88 80 145
518	19	0 0 0 0 0 0 1 14 21 43 80 138 169 176 181 149 117 108 76 134
556	20	0 0 0 0 0 0 1 3 8 30 41 73 139 171 184 161 142 137 96 77 144
592	21	0 0 0 0 0 0 0 1 3 17 17 58 83 120 157 191 154 151 144 106 57 148
630	22	0 0 0 0 0 0 0 0 2 5 11 27 57 81 113 158 159 183 158 131 110 69 143
665	23	0 0 0 0 0 0 0 0 0 3 6 13 37 44 88 112 143 170 183 172 125 103 75 133

702	24	0 0 0 0 0 0 0 0 0 0 3 5 20 31 45 86 111 152 153 185 180 132 105 59 140
740	25	0 0 0 0 0 0 0 0 0 0 0 3 9 16 23 51 76 127 130 190 165 172 134 103 70 138
775	26	0 0 0 0 0 0 0 0 0 0 0 0 2 8 16 26 48 87 121 151 173 176 160 124 120 64 131
812	27	0 0 0 0 0 0 0 0 0 0 0 0 0 2 11 15 23 45 86 117 163 173 161 163 154 107 57 130
855	28	0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 4 18 24 43 72 122 133 178 180 174 146 102 69 141
896	29	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 8 9 19 46 54 120 148 166 196 179 131 105 83 142
937	30	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 2 4 6 19 40 55 111 147 161 206 183 134 102 97 140
976	31	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 9 18 36 52 102 144 179 212 166 139 136 69 144
1020	32	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 3 4 13 32 41 93 137 171 211 189 149 136 86 142
1065	33	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 2 5 6 18 32 88 115 195 195 187 180 151 83 149
1114	34	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 3 7 10 30 62 93 151 204 232 189 168 106 151
1162	35	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 2 2 10 21 29 82 152 191 229 222 190 119 158
1218	36	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 2 0 9 24 49 96 173 216 256 254 149 179
1284	37	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 6 14 46 84 180 277 297 271 231

For full details of any result here , the reader can e-mailing the author .

So, this paper will contain the following table for the first time for $t_r^*(2,37)$

Table(4.2) $t_r^*(2,37)$

r	2	3	4	5	6	7	8
$t_r^*(2,37)$	16	34	58	83	112	138	166
r	9	10	11	12	13	14	15
$t_r^*(2,37)$	194	226	254	285	316	346	379
r	16	17	18	19	20	1	22
$t_r^*(2,37)$	417	450	486	518	556	592	630
r	23	24	25	26	27	28	9
$t_r^*(2,37)$	665	702	740	775	812	855	896
r	30	31	32	33	34	35	6
$t_r^*(2,37)$	937	976	1020	1065	1114	1162	1218
r	37						
$t_r^*(2,37)$	1284						

REFERENCE

[1]. Ball,S.,(1996)," *Multiple blocking sets and arcs in finite planes*", J. London Math. Soc. 54, 427–435.

[2]. Ball,S. and J.W.P.Hirschfeld (2005),"*Bounds on (n,r)-arcs and their application to linear codes*",Finite Fields Appl. 11,pp.326-336.

[3]. Barlotti,A.,(1965),"*Some topics in finite geometrical structures*", Institute of Statistics, University of Carollina, mimeo series, 439.

[4]. Bose,R.,(1947),"*Mathematical theory of the symmetrical factorial design*", Sankyha, 8, 1947, 107–166.

[5]. Chao,J. M. and Kaneta, H. (1996), *"A complete 24-arc in PG(2,29) with the automorphism group PSL(2,7)"*, Rendiconti di Matematica, Serie VII Volume 16 , Roma, 537-544

[6]. Colbourn, Charles J. and Dinitz, Jeffrey H.(2007)"***The CRC Handbook of Combinatorial Designs***", Author Preparation Version 26 ,a chapter done by Leo Storme (Active e-mail: ls@cage.ugent.be).

[7]. Hirschfeld ,J.W.P(2001), *"Complete arcs"*, Discrete Math., North-Holland Mathematics Studies 123, North-Holland,Amsterdam,243-250

[8]. Hirschfeld, J.W.P, (1979),"***Projective Geometries over Finite Fields***", Oxford University Press, Oxford.

[9]. Hirschfeld ,J.W.P. and Storme ,L. (2001), *" The packing problem in statistics, coding theory and finite projective spaces"*, update 2001, in: Finite Geometries, Developments in Mathematics 3, Kluwer , 201–246.

[10]. Keri,G.,*"On the number of large complete arcs in PG(2,q), $23 \le q \le 32$"*, (Active e-mail: keri@sztaki.hu)

[11]. Muhammad, H.H.(2006*),"The Maximum Size of (n,r)-arcs in the Projective Plane PG(2,q) "* M.Sc. Thesis, University of Mosul-Iraq.