Unsteady Flow near the Junction Zone of Three Liquids

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ABSTRACT

In this paper we consider the motion near the junction zone of three approximately plane liquid films of semi–infinite extent in two dimensional polar coordinate system with negligible inertia. We use Stokes equation to describe such flow. The pressure in the region of large curvature is less than that on the surface of bulk fluid and this pressure gradient ensures that this problem is unsteady state case. The equation that governs such flow is solved analytically, the shape and the thickness are determined for some liquids.

Keywords: Liquid films, polar coordinate system, Stokes equation.

الجريان غير المستقر قرب منطقة الالتقاء لثلاثة أغشية سائلة

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الملخص

في هذا البحث تمت دراسة الحركة قرب منطقة الالتقاء لثلاثة أغشية سائلة مستوية شبه ممتدة في النظام القطبي الثنائي البعد بانعدام القصور الذاتي. استخدمت معادلة ستوكس لوصف هكذا جريان. إن الضغط في المنطقة الأكبر تقعراً يكون اصغر من الضغط خارج سطح المائع وهذا الانحدار في الضغط يؤكد بان المسالة تعتمد على الزمن. إن المعادلة التي تحكم هذا الجريان تم حلها تحليلياً وقد تم إيجاد شكل وسمك الغشاء لبعض السوائل.

الكلمات المفتاحية: الاغشية السائلة، النظام القطبي، معادلة ستوكس.

Introduction:

Thin liquid films appear in many contexts such as the cooling of gas turbine blade tips, rocket engines. Apart from these direct cooling applications of thin liquid layers, thin films form a crucial element in many other applications such as in industrial coating and spinning processes. Homsy (2000), studied the slow motion of a thin viscous film flowing over a topographical feature under the action of external forces, using the lubrication approximation and he obtained an equation of the free surface in time and space. Breward, D.R. and Darton R.C. (2000) investigate the flow of a liquid from the lamellae to the plateau border and the drainage flow that occur within the border. Bowen, M. and King, J.R. (2001) consider the asymptotic behavior of thin film equation in bounded domains. Schwartz, L.W. and Brien, S.B (2002) gave the theory and a mathematical modeling of thin film flow which reproduce many of the features of this process include the shape of the film thickness profiles and the large differences in drainage time scales for low and high surfactant. Leshansky A. and Rubinstein B. (2004) investigate the non– linear rupture of thin liquid films on solid surfaces.

The main object of this paper is to study the mechanics of the junction zone of three plane films as shown in Figure (1). An effect of surface tension is necessarily to cause a continual thickening of the films in the junction zone, with the additional liquid being supplied symmetrically by all the three films.

The hydrostatic pressure of the junction zone (called border which is a region of large curvature) is less than that on the surface of bulk fluid, further up the liquid, the pressure is much higher than that at the border and so the pressure gradient inside the

liquid film which forces the flow of liquid towards the border and this pressure gradient ensures that there must be some flow and this cannot represent a steady state situation.
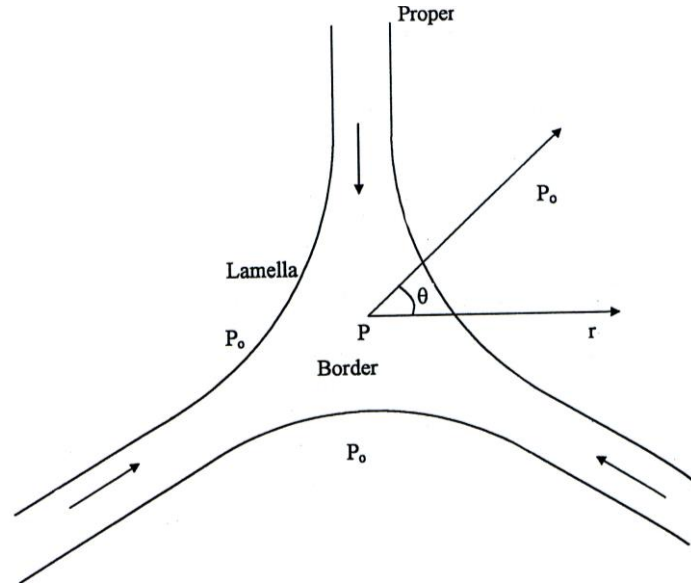

Figure (1): Motion near the junction zone of three fluids.

Formulation and governing equations:

We consider the flow of viscous liquid within the film in two dimensions and we suppose that there is no inertia the uses the stokes equations to describe the fluid motion in junction zone of three films in vector form as

$$\rho \frac{Dq}{Dt} + \nabla P = \mu \, \Delta q \qquad \qquad \ldots(1)$$

and

$$\text{Div } q = 0 \qquad \qquad \ldots(2)$$

where q, P, ρ and μ are the velocity, pressure, density and viscosity of fluid respectively, ∇ and Δ are the gradient and the laplacian operator.

The solution of equation (1) is equivalent to the solution of the following biharmonic equation

$$\Delta^2 \psi = \frac{1}{\upsilon} \frac{\partial}{\partial t} \left(\Delta \psi \right) \qquad \qquad \ldots(3)$$

where the stream function is related to the velocity components by

$$u_r = -\frac{1}{r} \frac{\partial \psi}{\partial \theta}, \; u_\theta = \frac{\partial \psi}{\partial r}, \; \text{and where } \upsilon = \frac{\mu}{\rho}.$$

Equation (3) can be written in polar coordinates and after simplifications to give:

$$\psi_{rrr} + \frac{2}{r} \psi_{rrr} - \frac{1}{r^2} \psi_{rr} + \frac{1}{r^3} \psi_r - \frac{2}{r^3} \psi_{r\theta\theta} + \frac{4}{r^4} \psi_{\theta\theta} + \frac{2}{r^2} \psi_{rr\theta\theta} + \frac{1}{r^4} \psi_{\theta\theta\theta\theta}$$

$$= \frac{1}{\upsilon} \frac{\partial}{\partial t} \left(\psi_{rr} + \frac{1}{r} \psi_r + \frac{1}{r^2} \psi_{\theta\theta} \right) \qquad \qquad \ldots(4)$$

Suppose that the stream function $\psi(r, \theta, t)$ is defined by:

$$\psi(r,\theta,t) = g(\eta)h(\theta), \quad h(\theta) = \sin 3k\theta \qquad \qquad \dots(5)$$

and

$$\eta = r(\upsilon t)^{-\frac{1}{2}} \qquad \qquad \dots(6)$$

Now, using (5) and (6) into equation (4), and after simplifications, we get

$$r^4(\upsilon t)^{-2}g^{(4)} + 2r^3(\upsilon t)^{-\frac{3}{2}}g^{(3)} - r^2(\upsilon t)^{-1}g'' + r(\upsilon t)^{-\frac{1}{2}}g' +$$

$$18k^2(\upsilon t)^{-\frac{1}{2}}rg' - 36k^2g - 18k^2r^2(\upsilon t)^{-1}g'' + 81k^4g = \frac{r^5}{2}(\upsilon t)^{-\frac{5}{2}}g^{(3)} -$$

$$\frac{-r^4}{2}(\upsilon t)^{-2}g'' - r^4(\upsilon t)^{-2}g'' - \frac{r^3}{2}(\upsilon t)^{-\frac{3}{2}}g' + \frac{9}{2}k^2r^3(\upsilon t)^{-\frac{3}{2}}g' \qquad \dots(7)$$

By using equation (6), equation (7) gives

$$\eta^4g^{(4)} + 2\eta^3g^{(3)} - (1+18k^2)\eta^2g''' + (1+18k^2)\eta g' + 9k^2(9k^2-4)g$$

$$= \frac{\eta^2}{2}\left[\eta^3g^3 + 3\eta^2g'' + (1-9k^2)g'\right] \qquad \dots(8)$$

Indeed, there may be no solution defined at $\eta = 0$ other than the trivial solution $g = 0$, so we seek a solution defined near $\eta = 0$ and the solution turn to be valid at $\eta = 0$, that is we seek a solution represented by a series which has the form

$$g(\eta) = \sum_{n=0}^{\infty} C_n \eta^{n+r} \qquad \qquad \dots(9)$$

By substituting (9) and their derivatives into equation (8) after simplifications, we get

$$\sum_{n=0} \left[(n+r^2) - 9k^2\right]\left[(n+r-2)^2 - 9k^2\right]C_n\eta^{n+r} =$$

$$-\frac{1}{2}\sum_{n=2}(n+r-2)\left[(n+r-2)^2 - 9k^2\right]C_{n-2}\eta^{n+r} \qquad \dots(10)$$

Now for $n = 0$, equation (10), gives

$$\left(r^2 - 9k^2\right)\left[(r-2)^2 - 9k^2\right]C_\circ = 0 \qquad \qquad \dots(11)$$

Suppose that $C_\circ \neq 0$, equation (11) gives

$$r_1 = 3k, \quad r_2 = -3k, \quad r_3 = 2+3k, \quad r_4 = 2-3k$$

For $n = 1$, equation (10), gives

$$\left\{\left[r^2 - (1+9k^2)\right]^2 - 36k^2\right\}C_1 = 0 \qquad \qquad \dots(12)$$

Since the Coefficients of C_1 are non-zero for each of the roots r_1, r_2, r_3 and r_4 it follows that $C_1 = 0$ in each case.

Now for $n \geq 2$, equation (10) gives

$$\left[(n+r)^2 - 9k^2\right]C_n = -\frac{1}{2}(n+r-2)C_{n-2} \qquad \qquad \dots(13)$$

or

$$C_n = -\frac{(n+r-2)}{2\left[(n+r)^2 - 9k^2\right]}C_{n-2} \qquad \qquad \text{...(14)}$$

Now, to find the Frobenius series corresponding to the roots $r_1 = 3k$, and $r_3 = 2+3k$, we substitute in (14) to obtain the recursion formula. For $r = 3k$, and for convenience we write A_n instead of C_n to get

$$A_n = -\frac{(n+3k-2)}{2n(n+6k)}A_{n-2} \ , \quad n \geq 2 \qquad \qquad \text{...(15)}$$

and thus, we have

$$A_2 = -\frac{3k}{8(1+3k)}A_\circ$$

$$A_4 = -\frac{3k}{128(1+3k)}A_\circ$$

and so on-on substituting in (9), we have

$$g = A_\circ \eta^{3k}\left[1 - \frac{3k}{8(1+3k)}\eta^2 + \frac{3k}{128(1+3k)}\eta^4 - \frac{k(4+3k)}{3072(k+1)(3k+1)}\eta^2 \mp \cdots\right] \qquad \text{...(16)}$$

Now for $r = 2+3k$, we may write B_n instead of C_n, we get

$$B_n = -\frac{(n+3k)}{2(n+2)(n+2+6k)}B_{n-2} \ , \quad n \geq 2 \qquad \qquad \text{...(17)}$$

Thus, we have

$$B_2 = -\frac{1}{16}B_\circ \ , \quad B_4 = \frac{(4+3k)}{1152(1+k)}B_\circ \ , \quad B_6 = \frac{(2+k)}{12288(1+k)}B_\circ$$

and so on. On substituting in (9), we get the second solution which has the form

$$g = B_\circ \eta^{2+3k}\left[1 - \frac{1}{16}\eta^2 + \frac{(4+3k)}{1152(1+k)}\eta^4 - \frac{(2+k)}{12288(1+k)}\eta^6 \mp \cdots\right] \qquad \text{...(18)}$$

Similarly, we can find the solutions g_3 and g_4 at $r = -3k$ and $r = 2-3k$ respectively to obtain

$$g = C_\circ \eta^{-3k}\left[1 - \frac{3k}{8(1-3k)}\eta^2 - \frac{3k}{128(1-3k)}\eta^4 + \frac{k(4+k)}{3072(1+k)(1-3k)}\eta^6 \mp \cdots\right] \qquad \text{...(19)}$$

and

$$g = D_\circ \eta^{2-3k}\left[1 - \frac{1}{16}\eta^2 + \frac{(4-3k)}{1152(1-k)}\eta^4 - \frac{(2-k)}{12288(1-k)}\eta^6 \mp \cdots\right] \qquad \text{...(20)}$$

Now for $k = 1$, the general solution is

$$g(\eta) = A_\circ \eta^3\left(1 - \frac{3}{2}\eta^2 + \frac{3}{2^9}\eta^4 - \frac{7}{3.2^{13}}\eta^6 \mp \cdots\right) + B_\circ \eta^5\left(1 - \frac{1}{2^4}\eta^2 + \frac{7}{3^2 2^8}\eta^4 - \frac{1}{2^{13}}\eta^6 \mp \cdots\right)$$

$$+ C_\circ \eta^{-1}\left(1 - \frac{1}{2^4}\eta^2 + \frac{1}{3^2 2^8}\eta^4 - \frac{5}{3.2^{14}}\eta^6 \mp \cdots\right) + D_\circ \eta^{-3}\left(1 - \frac{3}{2^4}\eta^2 + \frac{3}{2^8}\eta^4 - \frac{1}{2^{12}}\eta^6 \mp \cdots\right) \text{...(21)}$$

The constants in (21) can be determined by using the initial and boundary condition. The boundedness at the origin requires that the constants D_0 and C_0 must be vanished, that is $D_0 = C_0 = 0$, and the solution now is given by

$$g(\eta) = A_0 \sum_{n=0} \frac{3(-1)^n (2n+1)!}{2^{4n-1}(n!)^2(n+3)!} \eta^{2n+3} + B_0 \sum_{n=0} \frac{(-1)^n (2n+3)!}{2^{4n-2}[(n+1)!]^2(n+4)!} \eta^{2n+5} \qquad ...(22)$$

Boundary conditions:

At the free surface $F(\eta, \theta(\eta)) = 0$, we have the following boundary conditions. The tangential stress $\tau = 0$, that is

$$n_i t_i \sigma_{ij} = 0 \qquad ...(23)$$

and the normal stress condition

$$n_i n_i \sigma_{ij} = -\sigma k \qquad ...(24)$$

where σ is the surface tension and k is the curvature, the unit tangent and unit normal vectors at the free surface is give by

$$\vec{t} = \left[1 + r^2 \left(\frac{d\theta}{dr}\right)^2\right]^{-\frac{1}{2}} \left(\hat{r} + r\frac{d\theta}{dr}\hat{\theta}\right) \qquad ...(25)$$

and

$$\vec{n} = \left[1 + r^2 \left(\frac{d\theta}{dr}\right)^2\right]^{-\frac{1}{2}} \left(-\hat{r}r\frac{d\theta}{dr} + \hat{\theta}\right) \qquad ...(26)$$

where $(\hat{r}, \hat{\theta})$ denote the natural orthonormal basis of the coordinate system, the curvature k is given by

$$k = \left[1 + r^2 \left(\frac{d\theta}{dr}\right)^2\right]^{-\frac{3}{2}} \left(2\frac{d\theta}{dr} + r\frac{d^2\theta}{dr^2} + r\left(\frac{d\theta}{dr}\right)^3\right) \qquad ...(27)$$

The stress tensor σ in the fluid in given by

$$\left.\begin{aligned}
\sigma_{rr} &= -p + 2\mu\left(\frac{\partial u_r}{\partial r} - \frac{u_r}{r}\right) \\
\sigma_{r\theta} &= \mu\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_r}{r}\right) \\
\sigma_{\rho\theta} &= -p + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{u_r}{r}\right)
\end{aligned}\right\} \qquad ...(28)$$

where u_r and u_θ are the velocity components in the directions of $\hat{r}$ and $\hat{\theta}$ respectively.

Hence the boundary conditions (23) and (24) after simplifications become

$$-r\frac{\partial\theta}{\partial r}\sigma_{rr} + \left[1 - r^2\left(\frac{d\theta}{dr}\right)^2\right]\sigma_{r\theta} + r\frac{d\theta}{dr}\sigma_{\theta\theta} = 0 \qquad ...(29)$$

and

$$r^2\left(\frac{d\theta}{dr}\right)^2 \sigma_{rr} - 2r\frac{d\theta}{dr}\sigma_{r\theta} + \sigma_{\theta\theta} = -\sigma_k\left[1 + r^2\left(\frac{d\theta}{dr}\right)^2\right] \qquad ...(30)$$

The velocity components u_r and u_θ are related to the stream function $\psi(r, \theta, t) = \psi(\eta, \theta)$ by

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \text{ and } u_\theta = -\frac{\partial \psi}{\partial \theta} \qquad \ldots(31)$$

Now the stream tensor given by (28) and by using the condition (29) and (30) is then become

$$\left.
\begin{aligned}
\sigma_{rr} &= -p + 2\mu\left(\frac{1}{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\psi}{\partial\theta}\right)\\[2mm]
\sigma_{r\theta} &= \mu\left(-\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right)\\[2mm]
\sigma_{\theta\theta} &= -p + 2\mu\left(-\frac{1}{r}\frac{\partial\psi}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial\psi}{\partial\theta}\right)
\end{aligned}
\right\} \qquad \ldots(32)$$

By substituting (32) into (29) and (30), we have

$$4r\frac{d\theta}{dr}\left(-\frac{1}{r}\frac{\partial\psi}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial\psi}{\partial\theta}\right) + \left[1 - r^2\left(\frac{d\theta}{dr}\right)^2\right]\left(-\frac{\partial^2\psi}{\partial^2 r} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right) = 0 \qquad \ldots(33)$$

and

$$r^2\left(\frac{d\theta}{dr}\right)^2\left(\frac{1}{r}\frac{\partial^2\psi}{\partial\theta} - \frac{1}{r^2}\frac{\partial\psi}{\partial\theta}\right) - r\frac{d\theta}{dr}\left(-\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right)$$

$$+ \left(-\frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial\psi}{\partial\theta}\right) = \frac{1}{2\mu}\left[1 + r^2\left(\frac{\partial\theta}{\partial r}\right)^2\right](\rho - \sigma k) \qquad \ldots(34)$$

Now by using the transformations (5) and (6) in (33) and after some simplifications, we get

$$4\eta\frac{d\theta}{d\eta}\left(-\frac{1}{\eta}g'h' + \frac{1}{\eta^2}gh'\right) + \left[1 - \eta^2\left(\frac{d\theta}{d\eta}\right)^2\right]\left(-g''h + \frac{1}{\eta}g'h + \frac{1}{\eta^2}gh''\right) = 0 \qquad \ldots(35)$$

The leading term in the general solution in equation (22) is given by
$$g(\eta) = A\eta^3 \qquad \ldots(36)$$

By substitute (36) and their derivatives in equation (35), we get

$$\eta^2\left(\frac{d\theta}{d\eta}\right)^2\sin 3\theta - 2\eta\cos 3\theta\frac{d\theta}{d\eta} - \sin 3\theta = 0 \qquad \ldots(37)$$

Equation (37), gives.

$$\frac{d\theta}{d\eta} = \frac{2\eta\cos 3\theta \pm \sqrt{4\eta^2\cos 3\theta + 4\eta^2\sin 3\theta}}{2\eta^2\sin 3\theta} \quad \text{or}$$

$$\frac{d\eta}{\eta} = \frac{\sin 3\theta}{\cos 3\theta \pm 1}d\theta \qquad \ldots(38)$$

Integrating (38), we get

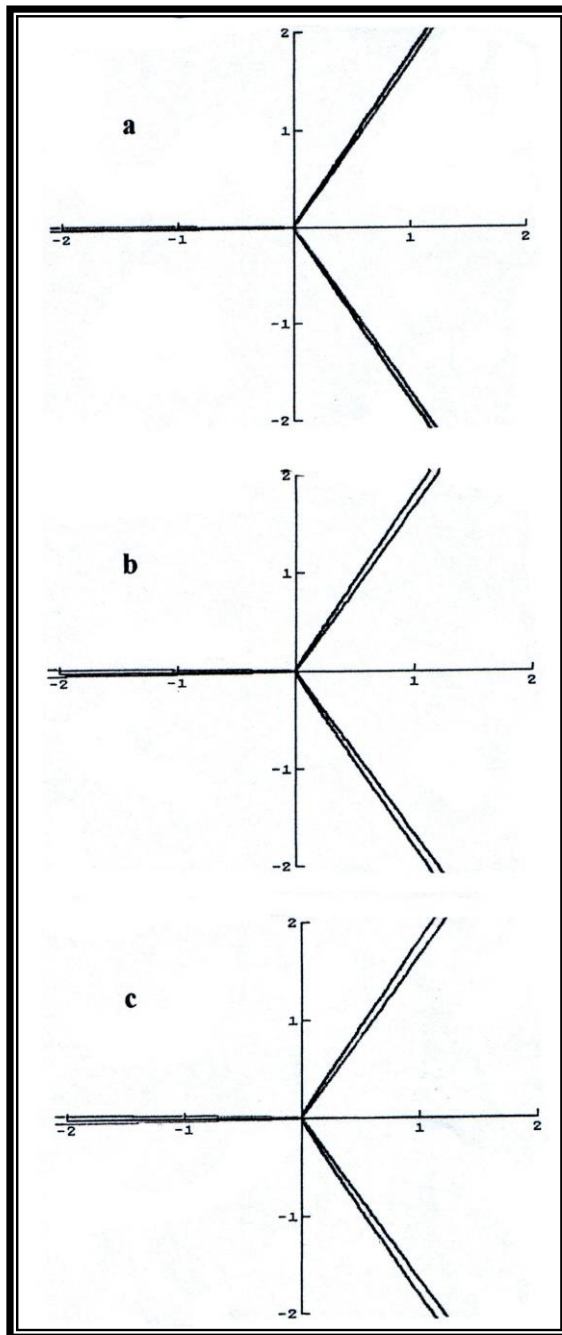$$\ln\eta = -\frac{1}{3}\ln(\cos 3\theta \pm 1) + \ln C \quad \text{or}$$

$$\eta = C(\cos 3\theta \pm 1)^{-\frac{1}{3}} \qquad \ldots(39)$$

By using (6) equation (39), given

$$r = C\sqrt{\upsilon t}\,(\cos 3\theta \pm 1)^{-\frac{1}{3}} \qquad\qquad \ldots(40)$$

where $\upsilon = \dfrac{\mu}{\rho}$ and equation (40) represent the shape of the thin liquid film in the neighborhood of the origin.

Some of the solution curves for equation (40) are presented for different liquid namely water, glycerin and mercury and for different values of time $t = (0.1,\ 0.5,\ 1)$ as shown in figures (2), (3), (4), and (5).
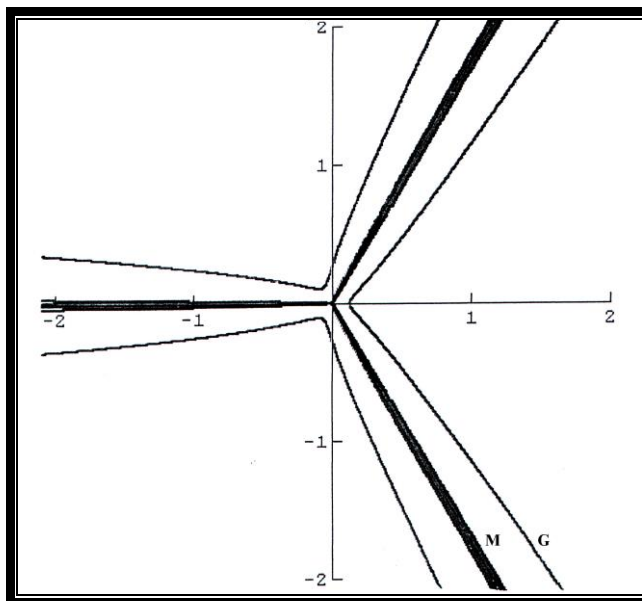

Figure (2): The thickness and the shape of the free surface for the mercury for different values of time. a) $t = 0.1$, b) $t = 0.5$, c) $t = 1$.
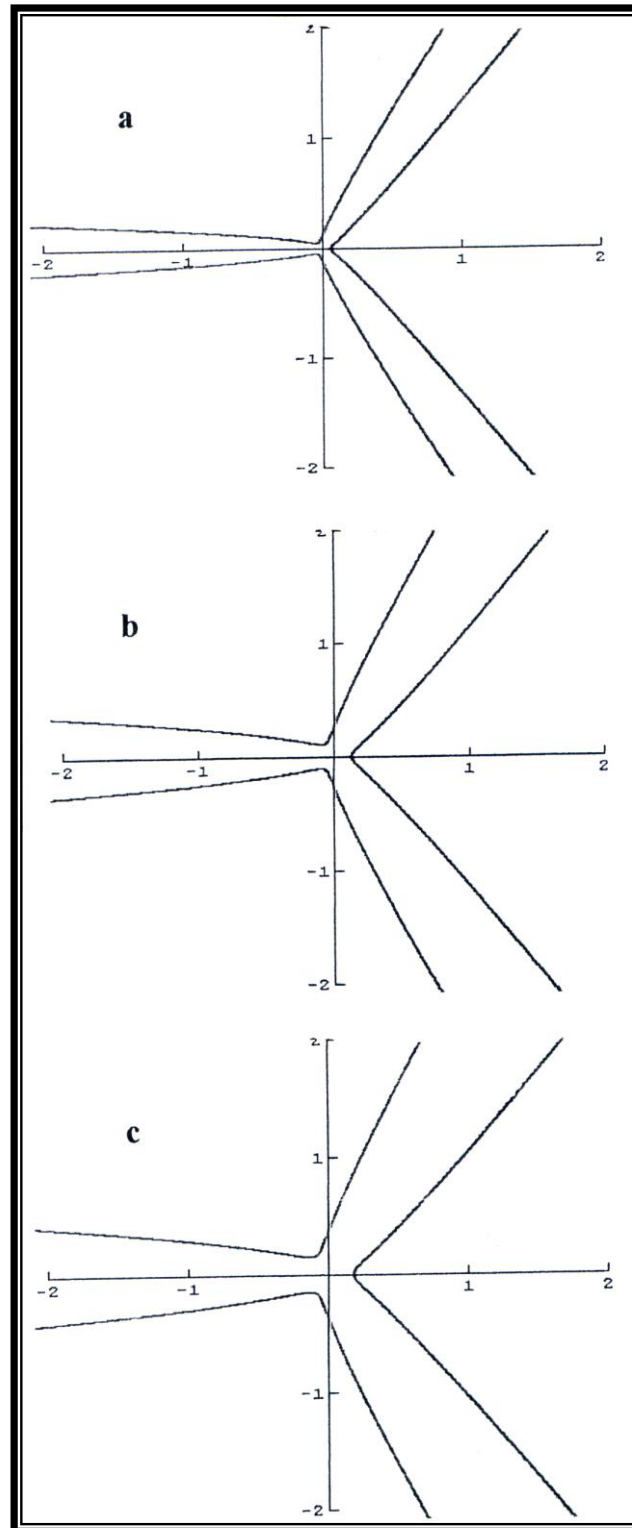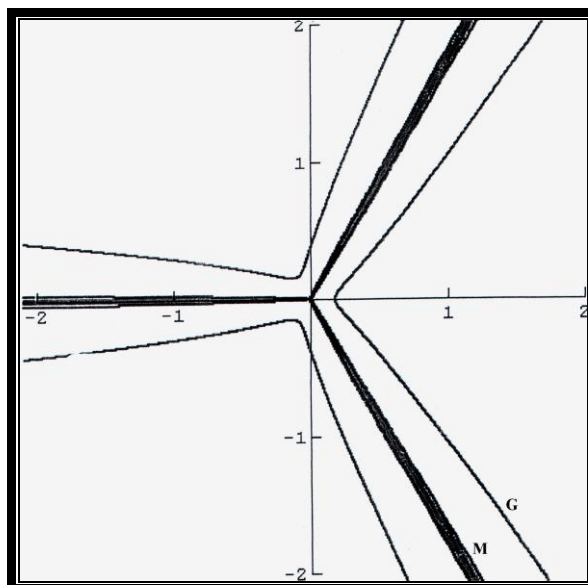
Figure (3): The thickness and the shape of the free surface for the glycerin for different values of time. a) $t = 0.1$, b) $t = 0.5$, c) $t = 1$.

Figure (4): The thickness and the shape of the free surface for different liquids at time $t = 0.1$

W: Water, M: Mercury, G: Glycerin

Figure (5): The thickness and the shape of the free surface for different liquids at time $t = 0.5$.

W: Water, M: Mercury, G: Glycerin

Conclusions:

The flow of a liquid in the junction zone of these approximately plane liquid films is essentially unsteady initial value problem and the solution controls the life span of foam. The shape of three surfaces is determined for some liquids, namely for water, mercury and glycerin and it is seen from figure (2) and (3) that at time increases, the thickness of the liquid film also increases and when the supply of liquid is exhausted, that component ruptures. Furthermore, the thickness of liquid film in water is less than that of glycerin and the reason may be related to viscosity of the liquid.

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