MGP and EGP Rings

Raida D. Mahammod **Shahla M. Khalil**

raida.1961@uomosul.edu.iq *moayadshahla@gmail.com*

College of Computer Sciences and Mathematics
University of Mosul, Iraq

Received on: 12/3/2008 **Accepted on: 12/6/2008**

ABSTRACT

The purpose of this paper is to study the rings in which every maximal (essential) right ideal is a left GP- ideal. Such rings will be called right MGP- rings (EGP- rings). We give the basic properties of such rings and their connection with strongly π- regular rings, fully left idempotent rings, and S - weakly regular rings.

Keywords: right MGP- rings,right EGP- rings, strongly π- regular rings, fully left idempotent rings, S - weakly regular rings.

الحلقات من النمط– MGP والحلقات من النمط EGP

رائدة داؤد محمود **شهلة مؤيد خليل**

كلية علوم الحاسوب والرياضيات

جامعة الموصل

تاريخ استلام البحث:2008/6/12 **تاريخ قبول البحث:2008/3/12**

الملخص

في هذا البحث تم دراسة الحلقات التي يكون فيها كل مثالي أيمن أعظم (أساسي) هو من النمط –GP الأيسر. هذه الحلقات تسمى حلقات من النمط –MGP (–EGP) يمنى. أعطينا الخواص الأساسية لهذه الحلقات وعلاقاتها مع حلقات اخرى مثل الحلقات المنتظمة بقوة من النمط – π ، والحلقات المتحايدة اليسرى الكاملة، وكذلك مع الحلقات المنتظمة الضعيفة من النمط. – S

الكلمات المفتاحية: حلقات من النمط –MGP، الحلقات من النمط EGP، الحلقات المنتظمة بقوة من النمط – π ، والحلقات المتحايدة اليسرى الكاملة، الحلقات المنتظمة الضعيفة من النمط. – S

1. Introduction:

In this paper, all rings are assumed to be associative ring with identity. An ideal I of a ring R is said to be right (left) pure ideal if for every $a \in I$ there exists $b \in I$ such that $a = ab$ $(a = ba)$. This concept was introduced by Fieldhouse [3],[4].

Recall that:

1- R is called reduced if R has no non-zero nilpotent element.
2- According to Cohn [2], a ring R is called reversible if $ab = 0$ implies $ba = 0$, for $a, b \in R$. It is easy to see that R is reversible if and only if right (left) annihilator of a in R is two sided ideal [2].
3- A ring R is said to be a right (left) Kasch ring if every maximal right (left) ideal is a right (left) annihilator [8].

4- According to [5], a ring R is called strongly π- regular if for every $x \in R$, there exists a positive integer n such that $x^n R = x^{2n} R$.
5- A ring R is called right (left) quasi-duo if every maximal right (left) ideal of R is two sided ideal [1].
6- A ring R is said to be ERT if every essential right ideal is two-sided [1].

2- Maximal Generalized Pure Rings (MGP- rings)

In this section, some basic properties of MGP- rings are given. Also the relations between such rings and strongly π- regular, weakly π- regular rings are given.

Following [10], an ideal I of a ring R is said to be right (left) GP- ideal if for every $a \in I$, there exists $b \in I$ and a positive integer n such that $a^n = a^n b$ $(a^n = ba^n)$.

Definition 2.1.[10]: A ring R is called a right (left) MGP- rings if and only if every maximal right (left) ideal is left (right) GP- ideal.

Theorem 2.2: Let R be a ring with $l(a^n) \subseteq r(a)$, for any $a \in R$ and a positive integer n. If R is a right MGP- ring, then the Jacobson radical of R is zero $(J(R) = (0))$.

Proof: Let $0 \neq a \in J(R)$. If $aR + r(a) \neq R$, then there exists a maximal right ideal M containing $aR + r(a)$. Since R is right MGP- ring, then M is left GP- ideal, so there exists $b \in M$ and a positive integer n such that $a^n = ba^n$, $(1-b)a^n = 0$ hence $(1-b) \in l(a^n) \subseteq r(a) \subseteq M$. So $1 \in M$, a contradiction. Therefore $aR + r(a) = R$, and so $ar + d = 1$ for some $r \in R$ and $d \in r(a)$, this implies that $a = a^2 r$. Since $a \in J(R)$, then there exists an invertible element v in R such that $(1-ar)v = 1$, so $(a - a^2 r)v = a$ yields $a = 0$. This proves that $J(R) = 0$. #

Theorem 2.3: Let M be a maximal right ideal of R. Then R is MGP- ring if and only if for every $a \in M$, $M + l(a^n) = R$ for some positive integer n.

Proof: Let M be a maximal right ideal of R and $a \in M$, since R is MGP- ring, then M is a left GP- ideal so there exists $b \in M$ such that $a^n = ba^n$ for some positive integer n. This implies that $(1-b) \in l(a^n)$. Therefore $R = M + l(a^n)$.
Conversely, assume that $M + l(a^n) = R$ for every $a \in M$ and a positive integer n, then $t + s = 1$ for some $t \in M$ and $s \in l(a^n)$. So $ta^n + sa^n = a^n$ and this implies $a^n = ta^n$. Whence R is MGP- ring. #

Lemma 2.4.[6]: Every strongly π- regular is π- regular ring.

Recall that, R is said to be W.R.D (weakly right duo), if for each $a \in R$ there exists a positive integer n such that $a^n R = Ra^n R$.

Theorem 2.5.[7]: Let R be W.R.D ring. Then the following are equivalent:
1- R is a π- regular.
2- Every ideal of R is a left GP- ideal.

Theorem 2.6: If R is a right Kasch ring and reversible, then the following statements are equivalent:
1- R is a strongly π- regular ring.
2- R is a right MGP- ring.

Proof: $(1) \Rightarrow (2)$:

Assume that R is strongly π- regular ring then by Lemma 2.4 and Theorem 2.5, R is a right MGP- ring.

$(2) \Rightarrow (1)$:

Assume that R is a right MGP- ring. Let M be any maximal right ideal of R. Since R is a right Kasch ring, then $M = r(a^n)$ for some $a \in R$ and a positive integer n. For any $x \in M$, we have $a^n x = 0$, and so $a^n Rx = 0$. This implies that $Rx \subseteq r(a^n) = M$, which proves that M is a two sided ideal of R. We claim that $b^n R + r(b^n) = R$. If not, there is a maximal right ideal N of R such that $b^n R + r(b^n) \subseteq N$. Since R is MGP- ring, then N is a left GP- ideal and $b^n \in N$. Then there exists $y \in N$ such that $b^n = yb^n$. Hence $(1-y)b^n = 0$ and so $(1-y) \in l(b^n) = r(b^n) \subseteq N$ (because R is reversible). Thus $1 \in N$, a contradiction. Therefore $b^n R + r(b^n) = R$. In particular $b^n u + v = 1$ for some $u \in R$ and $v \in r(b^n)$, so $b^n = b^{2n} u$. Thus R is strongly π- regular. #

Theorem 2.7: Let R be a right Kasch ring and reversible. If R is a right MGP- ring. Then for each completely prime ideal P of R, $P = \bigcup_{x \notin P} r(x)$.

Proof: By Theorem 2.6, R is a strongly π- regular ring. Let $Q = \bigcup_{x \notin P} r(x)$ and show that $Q = P$. If $x \in Q$, then $x \in r(y)$ for some $y \notin P$, thus $yx = 0 \in P$ and hence $x \in P$. Therefore $Q \subseteq P$.

On the other hand, by the Lemma 2.4, P is π- regular ideal, thus for each $x \in P$, there exists $u \in P$ and a positive integer n such that $x^n = x^n u x^n$, which implies that $x^n(1-ux^n) = 0 \in P$. Then $x^n \in l(1-ux^n) = r(1-ux^n)$ (because R is reversible ring). Since $(1-ux^n) \notin P$ for otherwise $1 \in P$, which is impossible. Thus $x^n \in Q$, so that $P \subseteq Q$, whence $Q = P$. #

Following [9], a ring R is called right (left) weakly π- regular if, for every $x \in R$ there exists a positive integer n such that $x^n \in x^n R x^n R$ $(x^n \in R x^n R x^n)$. R is weakly π- regular if it is both right and left weakly π- regular.

The next result gives us a sufficient condition for MGP- ring to be weakly π- regular ring.

Theorem 2.8: Let R be a semi prime ring with each non-zero right ideal contains a non-zero two sided ideal. If R is an MGP- ring, then it is weakly π- regular ring.

Proof: Assume that $0 \neq a \in R$ such that $a^2 = 0$, then by assumption there is a non-zero two sided ideal of R with $I \subseteq aR$. We claim that $l(a) \cap I \neq (0)$, for if $Ia = 0$ then $I \subseteq l(a)$ and we are done. If $Ia \neq 0$, then $Ia \subseteq I \cap l(a) \neq (0)$ Now, $(I \cap l(a))^2 \subseteq l(a)I \subseteq l(a)aR = (0)$. Since R is semi prime ring, then $I \cap l(a) = (0)$ which is a contradiction, consequently R is reduced. We show that $Rx^n R + r(x^n) = R$ for any $x \in R$ and a positive integer n. Suppose that there exists $y \in R$ such that $Ry^n R + r(y^n) \neq R$. Then there exists a maximal right ideal M of R containing

$Ry^nR + r(y^n)$. Since R is a right MGP- ring, then M is a left GP- ideal. So there exists $a \in M$ and a positive integer m such that $(y^n)^m = a(y^n)^m$ implies $(1-a) \in l((y^n)^m) = r((y^n)^m) = r(y^n) \subseteq M$. Hence $(1-a) \in M$ and so $1 \in M$ implies that $M = R$ which is a contradiction. Therefore $Rx^nR + r(x^n) = R$ for any $x \in R$. In particular $cx^nd + u = 1$ for some $u \in r(x^n)$ and $c,d \in R$, hence $x^n = x^ncx^nd$. So R is a right weakly π- regular ring. Since R is reduced it is also can be easily verified that R is weakly π- regular ring. #

Theorem 2.9.[6]: Let R be duo ring. Then R is π- regular if and only if R is strongly π- regular.

Lemma 2.10.[13]:If R is left or right quasi- duo and $J(R) = (0)$ .Then R is reduced ring.

Proposition 2.11.[6]: Let R be W.R.D. Then the following statement are equivalent:
1- R is a weakly π- regular ring.
2- R is a strongly π- regular ring.

The following theorem extends Theorem 2.6.

Proposition 2.12: Let R be a right duo ring. Then the following statements are equivalent:
1- R is a strongly π- regular ring.
2- R is a π- regular ring.
3- R is a right weakly π- regular ring.
4- R is a right MGP- ring and $J(R)$ is nil.

Proof: $(1) \Rightarrow (2)$: Follows from Theorem 2.9.
$(2) \Rightarrow (3)$: Obvious.
$(3) \Rightarrow (4)$: Follows from Proposition 2.11 and Theorem 2.5. Thus every maximal right ideal is GP- ideal. Now, we show that $J(R)$ is nil.
Let $r \in J(R)$. Then $r^nR = r^nRr^nR$ for some positive integer n, and so $r^n(1-s) = 0$ for some $s \in Rr^nR$. But since $r \in J(R)$, $(1-s)$ is invertible and thus we have $r^n = 0$, showing that $J(R)$ is nil.
$(4) \Rightarrow (1)$: Assume (4), then from Theorems 2.5 and 2.9, R is strongly π- regular. #

Lemma 2.13.[10]: Let R be a reduced ring. Then every GP- ideal is pure ideal.

By the Proposition 2.12, Lemmas 2.10 and 2.13, we have the following theorem.

Theorem 2.14: Let R be a right quasi-duo and $J(R) = 0$. Then the following statements are equivalent:
1- R is a strongly π- regular ring.
2- R is a π- regular ring.
3- R is a right (left) weakly π- regular ring.
4- R is a weakly π- regular ring.
5- R is a right MGP- ring.
6- Every maximal right ideal is a left pure.

3- EGP- rings

In this section, we introduce a new essential GP- ideals which are called EGP- rings. We give some of their basic properties, as well as a connection between EGP- rings and $S-$ weakly regular rings, strongly regular rings.

Definition 3.1: A ring R is said to be right EGP- rings, if every essential right ideal of R is a left GP- ideals.

Recall that [11], a ring R is a right (left) $S-$ weakly regular if for each $a \in R$, $a \in aRa^2R$ $(a \in Ra^2Ra)$. A ring R is called $S-$ weakly regular ring if it is both right and left $S-$ weakly regular ring.

We start this section by recalling the following propositions:

Proposition 3.2.[12]: A ring R is $S-$ weakly regular ring if and only if R is a reduced weakly regular ring.

Proposition 3.3.[10]: Let R be a duo ring. Then R is regular if and only if every ideal I of R is left pure.

Now, the following result is given:

Theorem 3.4: Let R be a ring with $aR = Ra$. Then R is a reduced, right EGP- ring if and only if R is an $S-$ weakly regular ring.

Proof: Assume that R is a reduced right EGP- ring. Let $a \in R$ and $I = Ra^2R + r(a)$. We claim that I is an essential right ideal of R. Suppose this is not true, then there exists a non-zero ideal J of R such that $I \cap J = (0)$. Then $(Ra^nR)J \subseteq IJ \subseteq I \cap J = (0)$. Since $a^2R \subseteq Ra^2R$, then $a^2R \cap J = (0)$. But $(a^2R)J \subseteq a^2R \cap J = (0)$ implies $J = (0)$, a contradiction; hence I is an essential right ideal. Since R is right EGP- ring, then I is a left GP- ideal, for every $a \in I$ there exists $b \in I$ and a positive integer n such that $a^n = ba^n$. Since $b \in I$ and $I = Ra^2R + r(a)$, then $b = ca^2d + h$ for some $c, d \in R$ and $h \in r(a)$ hence $a^n = ba^n = ca^2da^n + ha^n$. Since R is reduced then $r(a) = l(a) = l(a^n)$. So $a^n = ca^2da^n + 0$ implies that $(1 - ca^2d)a^n = 0$ and $(1 - ca^2d) \in l(a^n) = r(a^n) = r(a)$ hence $a = aca^2d$. Therefore R is an $S-$ weakly regular ring.

Conversely, assume that R is an $S-$ weakly regular ring. Then by Proposition 3.2, R is reduced weakly regular ring. Let $a \in R$ and $aR = aRaR = aRRa = aRa$ since $(aR = Ra)$. So R is regular and by Proposition 3.3, R is EGP- ring. #

Lemma 3.5.[9]: Let R be a weakly regular ring. Then $J(R) = (0)$.

However, we have the following proposition:

Proposition 3.6: Let R be a right duo ring. Then the following statements are equivalent:

1- R is a strongly regular ring.
2- R is a regular ring.
3- R is a right weakly regular ring.
4- R is an $S-$ weakly regular ring.
5- R is a right EGP- ring and $l(a^n) \subseteq r(a)$ for every $a \in R$ and a positive integer n.

Proof: $(1) \Rightarrow (2) \Rightarrow (3)$: They are obvious.

$(3) \Rightarrow (4)$: Assume (3), then by Lemmas 3.5, 2.10 and Proposition 3.2, R is an $S-$ weakly regular ring.
$(4) \Rightarrow (5)$: It follows from Theorem 3.4.
$(5) \Rightarrow (1)$: Assume that R is right EGP- ring. For any $a \in T$, let $T = aR + r(a)$ be a right ideal, by a similar method of the proof used in Theorem 3.4, T is an essential ideal. Since R is a right EGP- ring, then T is a left GP- ideal. For every $a \in T$, there exists $b \in T$ and a positive integer n such that $a^n = ba^n$, which implies $(1-b) \in l(a^n) \subseteq r(a) \subseteq T$, so $1 \in T$ and $T = R$. Therefore $aR + r(a) = R$.
In particular $ar + d = 1$ for some $r \in R$ and $d \in r(a)$. Then $a = a^2 r$. Thus R is strongly regular ring. #

Definition 3.7.[8]: A ring R is called fully right (left) idempotent if every right (left) ideal of R is idempotent.

Theorem 3.8: If R is ERT, then the following condition are equivalent:
1- R is a fully left idempotent ring.
2- R is a right EGP- ring.

Proof: $(1) \Rightarrow (2)$: Assume (1), and let E be an essential right ideal of R then it is an ideal of R (since R is ERT). Since R is a fully left idempotent ring, then for any $x \in E$, $Rx = (Rx)^2$ which implies that $x = ax$ for some $a \in RxR \subseteq E$. Therefore $x \in Ex$ for each $x \in E$. So E is left pure implies E is a left GP- ideal. Thus R is EGP- ring.

$(2) \Rightarrow (1)$: Assume (2), for any $a \in R$, set $L = Ra^n R + l(Ra^n R)$, and let K be a complement right ideal of R such that $L \oplus K$ is an essential right ideal of R. Now, $KRa^n R \subseteq K \cap Ra^n R \subseteq K \cap L = (0)$ implies that $K \subseteq l(Ra^n R)$. Whence $K \subseteq K \cap L = (0)$. This shows that L is an essential right ideal of R which is an ideal of R, by hypothesis R is a right EGP- ring. Therefore L is left GP- ideal and $a \in L$ implies that $a^n = da^n$ for some $d \in L$ and a positive integer n. If $d = u + v$, $u \in Ra^n R$ and $v \in l(Ra^n R)$, then $a^n = ua^n + va^n = ua^n \in Ra^n Ra^n$ which implies that $a^n \in (Ra^n)^2$; whence $Ra^n = (Ra^n)^2$. Therefore R is a fully left idempotent ring. #

REFERENCES

[1] Brown, S.H. (1973), "Rings over which every simple module is rationally complete", Canad. J. Math. 25, pp. 693-701.

[2] Cohn, P.M. (1999), "Reversible rings", Bull. London Math. Soc. 31, pp.641-648.

[3] Fieldhouse, D.J. (1969), "Pure theories", Math. Ann. 184, pp. 1-18.

[4] Fieldhouse, D.J. (1970), "Pure simple and indecomposable rings", Con. Math. Ball. 13, pp. 71-78.

[5] Hirano, Y. (1978), "Some studies on strongly π-regular rings", Math. J. Okayama Univ. 20, pp. 141-144

[6] Mahammad, M.R. (2000), "On π-regular rings", Ph.D. Thesis, University of Mosul.

[7] Mahmood, R.D. and Khalil, S.M. (2008), "ON GP-IDEALS", No. 34. Under publication.

[8] Ming, R.Y.C. (1986), "On semi prime and reduced ring", Riv. Math. Univ. Parama(4) 12, pp.167-175.

[9] Ramamurthi, V.S. (1973), "Weakly regular rings", Canda Math. Bull. 16, pp. 317-321.

[10] Shuker, N.H. and Mahmood, R.D. (2000), "On Generalization of pure ideal", J. Edu. And Sci, Vol. (43), pp. 86-90.

[11] Vasantha Kandasamy, W.B. and Gupta, V. (1993), "S-weakly regular group rings", Avch. Math. (BRNO) Tomus 29, pp.39-41.

[12] Younis, A.M. (2003), "On S-weakly regular ring", M. Sc. Thesis, Mosul University.

[13] Yu, H. P. (1995), "On quasi-duo", Glasgow Math. J. 37, pp. 21-31.